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THE BULLETIN OF SYMBOLIC LOGIC
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THE BULLETIN OF SYMBOLIC LOGIC

RICHARD A SHORE, Managing Editor

At the 1993 Annual meeting of the *Association for Symbolic Logic*, the Council of the association voted to establish a new journal to be called *The Bulletin of Symbolic Logic*. The intended goal of the Council was to produce a journal that would be both accessible and of interest to as wide an audience as possible, with the stated purpose of keeping the logic community abreast of important developments in all parts of our discipline. The first issue was to appear in March of 1995 and you now have it in your hands.

In accordance with the Council resolution, we intend to publish primarily two types of papers. The first section of *The Bulletin*, *Articles*, will usually be devoted to works of an expository or survey nature. These papers will generally present topics of broad interest in a way that should be accessible to a large majority of the members of the Association. Topics will be drawn from all areas of logic including mathematical or philosophical logic, logic in computer science or linguistics, the history or philosophy of logic, logic education and applications of logic to other fields. One view of a role that this section of *The Bulletin* will play is as an ongoing handbook of logic. Thus while most *Articles* will be of an introductory nature, we also see a role for expositions of more technical or specialized topics. In all cases, however, such *Articles* will be expected to pay attention to the history of the area and provide explanations for the general reader of its importance, concerns and achievements. We also expect to publish *Articles* based on major addresses delivered to the Association such as the Gödel Lectures and the Retiring Presidential Addresses when they seem appropriate for *The Bulletin*.

This issue, for example, contains two *Articles*. The first, "The mathematical work of S. C. Kleene" by Joseph Shoenfield, is a presentation of the basic work of a major figure in Logic who also played an important role in the history of the Association as both president of the association and editor of *The Journal of Symbolic Logic*. We commissioned this article last year in commemoration of Kleene's death. The second is "Platonism and mathematical intuition in Kurt Gödel's thought" by Charles Parsons and is based on his 1994 Retiring Presidential Address. It attempts to characterize both Gödel's well known realism and his conception of mathematical intuition which is described as belonging to what others might call a theory of reason.

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The second issue of *The Bulletin* should contain three *Articles*. The first will be "Inner models and large cardinals" by Ronald Jensen. It is a brief but wonderful survey of these areas of set theory as well as their relations to Axioms of Determinacy that is based on his 1990 Gödel Lecture. In addition to the major recent achievements in these areas, the article touches on both history and philosophy. The other two *Articles* will be by Angus Macintyre and Dirk van Dalen. Macintyre's piece will be based on his 1993 Gödel Lecture, "The logic of real and p-adic analysis: achievements and challenges" and his description of the work in this area by Alex Wilkie for which Wilkie shared the Karp prize awarded at this same meeting. Van Dalen's will be on Herman Weyl's views on constructivity. Using, in addition to the relevant publications, the correspondence between Weyl and Brouwer, it will relate Weyl's intuitionistic views to his earlier opinions and the work of Brouwer.

The second section of *The Bulletin*, *Communications*, will contain announcements of important new results and ideas in all aspects of logic. The *Communications* may be short papers in their final form or preliminary announcements (extended abstracts) of longer, full papers which will be published elsewhere. In any case, they should, in addition to a description of the new results or ideas, include enough history, background and explanation to make the significance of the work apparent to a wide audience. We will have a very fast track for refereeing and publishing the communications. Moreover, we expect them to appear within six months of the submission of final versions of the papers. As the standards for both results and exposition will be high and time to publication will be short, the *Association* expects *The Bulletin* to become *the* vehicle for keeping the entire community abreast of important developments in all areas of logic. This issue, for example, contains the *Communication* "$\mathrm{HOD}^{L(\mathbb{R})}$ is a core model" by John Steel. It is an announcement of major advances in the study of inner models, determinacy and large cardinals that will be described in the *Article* by Jensen described above.

The Bulletin, will also be taking over the publication of meeting reports, abstracts and announcements from the *Journal* as well as many of the notices that have appeared in the *Newsletter* of the *Association*. In addition, we will from time to time publish other items of general interest to the logic community. For example, in 1993 the *Council* adopted the "Guidelines for Logic Education" prepared by a committee chaired by Jon Barwise with a resolution to disseminate them in an appropriate fashion. In accordance with a later decision of the *Council*, we have published them as the first item in this issue of *The Bulletin*.

We encourage the members of the *Association* and other readers of *The Bulletin* to help us fulfill the goals of the *Council* and make *The Bulletin* into the outstanding, accessible and interesting journal we hope it will soon become. Survey and expository papers and suggestions of topics for such *Articles* should be submitted to Andreas R. Blass. Other possible *Articles* as well as *Communications* can be submitted to any of the other editors: Alexander S. Kechris, Daniel Lascar, Charles D. Parsons, Andrew M. Pitts or Richard A. Shore. Postal and email addresses for all the editors are listed on the inside back cover.

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THE BULLETIN OF SYMBOLIC LOGIC
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GUIDELINES FOR LOGIC EDUCATION

THE ASL COMMITEE ON LOGIC AND EDUCATION

Introduction. Yesterday's education does not meet the needs of tomorrow's world. In particular, the increasingly technical demands placed on people by the information revolution makes it all the more important that people understand basic logical principles of reasoning.

The Association for Symbolic Logic (ASL) is a worldwide organization that has been devoted to the study of logic since 1936. After careful study, the ASL has adopted the following guidelines and recommendations for logic instruction, with the aim to insure that all—scientists, humanists, the general population and professional logicians—can acquire from logic the tools they need.

This document draws out in broad strokes the material in the field of logic that should be incorporated into the educational system at various levels—for young children, adolescents, and college students. It does not go into the question of what should be available at the graduate level of various disciplines; that issue is too special to the traditions of particular institutions and research interest of their respective faculty members.

Primary and secondary education. Everyone needs to be able to tell, at some intuitive level, the difference between a valid argument and an invalid one. One needs to be able to give simple valid arguments, and to spot logical fallacies in others. Just how much one needs to know depends, of course, on

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As a result of continuing informal discussions, the *Association for Symbolic Logic* established an *ad hoc* Committee on Logic and Education in 1991, charged with drafting a set of specific recommendations on logic education The committee circulated a draft proposal in 1992 and a final set of recommendations in 1993. These recommendations were discussed at a special meeting of members at the 1993 ASL Annual Meeting. They were adopted by the Council, along with the mandate that the recommendations be widely distributed in an attempt to have a positive impact on logic education throughout the world.

The members of the Committee on Logic and Education who prepared this document were Jon Barwise, Chair, James Baumgartner, Martin Davis, Claudia Henrion, David Marker, Wilfried Sieg, Albert Visser, and Peter Woodruff.

Comments and questions about this report may be addressed to the Association for Symbolic Logic, Department of Mathematics, University of Illinois, 1409 West Green St. Urbana, IL 61801, email: `asl@symcom.math.uiuc.edu`.

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many factors. A mathematician, for example, presumably needs these skills honed to a finer edge than someone in a manual vocation.

Recommendation: *Promote and facilitate logical (i.e., analytical) reasoning at an early age.*

The notion of a correct proof and the method of debunking fallacious proofs by means of counterexamples should be introduced as early as possible. It is not necessary, or even advisable, to introduce specific courses in logic. Rather, the recognition of valid and invalid arguments should be an integral part of education in the sciences (mathematical, physical, biological, and social), and the humanities quite generally. After all, part of what it means to be literate involves the ability to distinguish valid reasoning from invalid reasoning.

Strategy: Ages 5–9: Integrate logical matters on "good" and "bad" arguments into other material in a completely informal way with effective, "inquisitive" teaching techniques.

Strategy: Ages 10–13: Emphasize heuristic strategies for problem solving in the spirit of Polya's *How to Solve It*. Take the first steps towards recognizing the form of statements, formulating corresponding rules, and using interpretations. Give some word problems that have a distinctively logical component to their solution.

Strategy: Ages 14–17: Teach the explicit use of logical notions and techniques to give proofs, counterexamples, etc. Mathematics courses are a natural place for the inclusion of such material.

Beginning post-secondary education. Recommendation: *All post-secondary institutions should offer at least one introductory course which teaches the basic notions of logic. All students should be encouraged to take such a course. These courses should include the following:*

- *The informal notion of "logically correct argument".*
- *Informal strategies for producing logically correct arguments and counterexamples to fallacious arguments.*
- *The propositional calculus as an example of a formal language, formal proofs, and the formalization of natural language arguments.*
- *A discussion of the relationship of proof, truth, and counterexamples, including a discussion of the Soundness Theorem.*
- *The predicate calculus extension of propositional logic.*
- *At least an informal discussion of the Completeness Theorem, if time permits.*

Comments: Courses satisfying the above criteria are successfully taught in several kinds of settings, by different kinds of faculty, and addressed to the needs of different kinds of students. Philosophy departments often present this sort of course for a general audience. Mathematics and computer science departments often present a more technical version.

Here are some tips to help make these courses successful.

1. Most important, make sure the instructor is interested in and well grounded in logic.
2. Spend some time on amusing logic problems and puzzles of the sort Raymond Smullyan has made famous. Consider discussing some topics from this history of logic.
3. Spend some time on real-life examples of sound and unsound reasoning. (In the U.S. at least, this could include the kind of logic problems commonly found on GRE and LSAT exams.)
4. If formal rules are taught, present them as a mathematical model of informal reasoning methods.
5. Treat some applications that have been made of ideas in logic, say in computer science, in detail.
6. Consider using some of the available courseware in logic. Logic provides just the sort of material for which computer aided instruction can be used to good effect. And there are several good programs for teaching logic available.

Advanced post-secondary education. Like other areas of intellectual inquiry, logic now has certain "core" material, material which everyone who claims any proficiency in the subject simply must know, as well as a great deal of additional material. This "core" should form the basis of a logic course available at all institutions that go beyond the first two years of post-secondary education.

Recommendation: *Institutions of higher learning should, in addition, offer a course or (sequence of courses) which cover the following logic-related topics.*

- *Elementary facts about sets (up through basic facts about binary relations, the diagonal method, the proof that uncountable sets exist, and the basic properties of countable sets).*
- *Basic facts about inductive definitions and proofs by induction, of the kind that permeate logic.*
- *Propositional and Predicate Calculus (The formalization of informal argument, the axiomatic method in mathematics and science).*
- *Semantics (truth and validity, definability, the Soundness Theorem, the notion of consistency, the Gödel Completeness Theorem).*
- *An introduction to model theory (at least the Compactness Theorem for countable languages with an application or two).*
- *The Gödel Incompleteness Theorems, their philosophical and foundational consequences.*

The course's format, the instructor's field, and the interests of the students and instructor will all influence the tone, the presentation, the emphasis, and the choice of additional topics. The basic concerns and results of logic listed here, however, are relevant and applicable to many areas of science and scholarship, and should be considered within the core of logic.

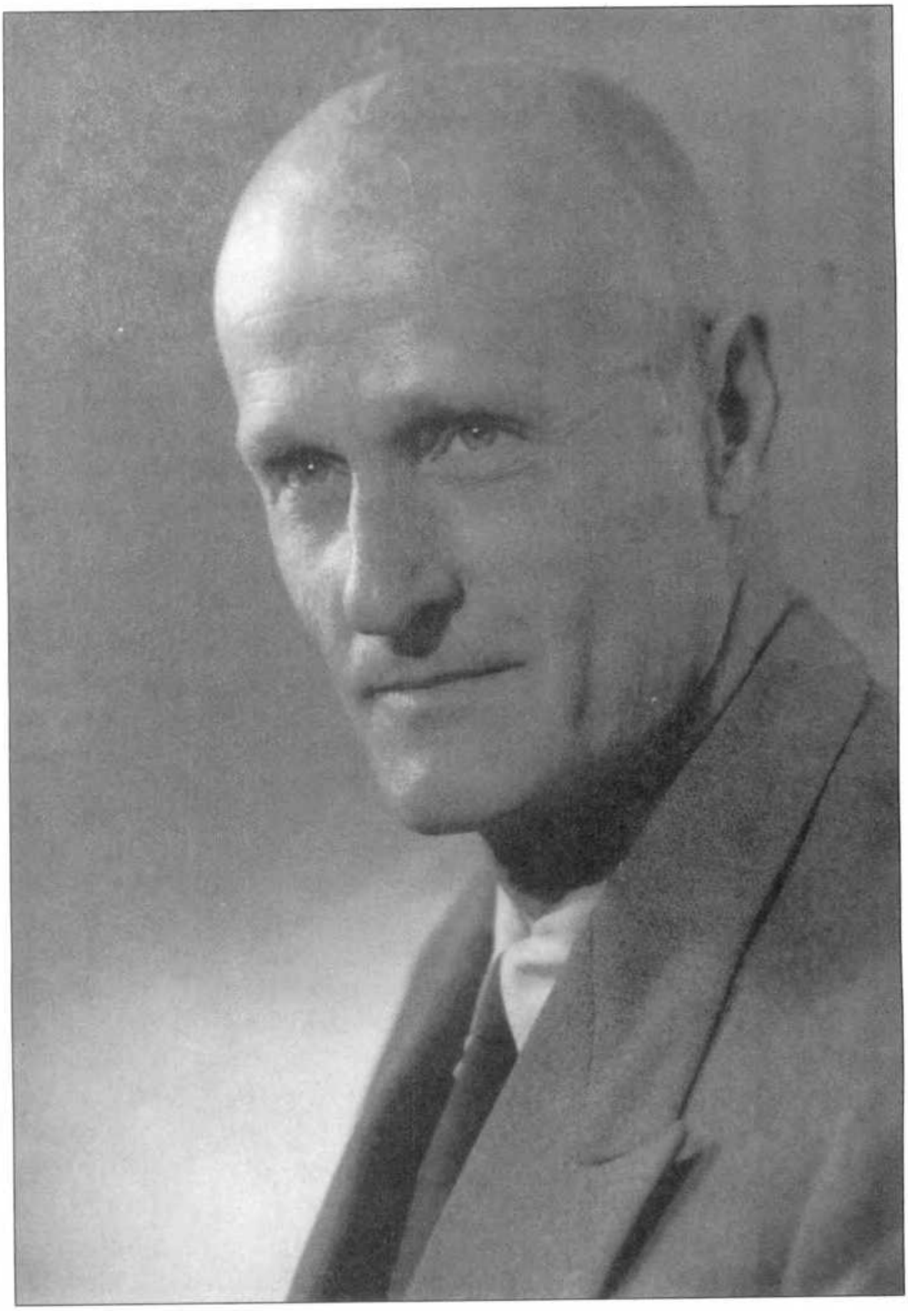
Whether this core material should be covered in one course or a sequence of two or more courses would depend on many parameters: the backgrounds and abilities of students, the length of the course, and the depth one wanted to go into the various topics, for example. These are matters which will have to be settled at the local level.

Beyond this core material, there is additional material which should be made available to all students.

Recommendation: *Institutions of higher learning should also offer courses which include the following material within their scope.*

- *An introduction to proof theory (Natural Deduction, the Gentzen Hauptsatz, Herbrand's Theorem, for example).*
- *Some additional model theory (e.g., the Löwenheim-Skolem theorems for countable languages, the decidability of the theory of dense linear orderings, the non-expressibility of various mathematical notions in first-order logic, non-standard models of arithmetic).*
- *Some additional set theory (some cardinal and ordinal arithmetic, a discussion of the axiom of choice).*
- *An introduction to computability theory (some machine model of effective computability, Church's Thesis; absolutely unsolvable problems; the undecidability of validity).*
- *An introduction to other kinds of logic. Just which would depend on the interests of the faculty in question. Some examples include intuitionistic logic, higher-order logic, modal logic, temporal logic, infinitary logic, and substructural logics.*
- *An introduction to uses of logic in computer science (e.g., unification and the resolution method and their connections to* PROLOG, *and the λ-calculus and its connections to* LISP *in particular and computation in general).*

Unlike the previous recommendation, the order in this list carries no significance. Nor does this recommendation suggest that all these topics would fit in the courses recommended earlier or any other single course, or that new logic courses should be established to cover them. There are many ways these topics could be covered in a combination of courses in various departments. Still, any student who is preparing for further study or work in mathematics, computer science, philosophy, cognitive science, or linguistics should be exposed to most of them.



Stephen Cole Kleene in 1969, at the age of 60.

THE BULLETIN OF SYMBOLIC LOGIC
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THE MATHEMATICAL WORK OF S. C. KLEENE

J. R. SHOENFIELD

§1. The origins of recursion theory. In dedicating a book to Steve Kleene, I referred to him as the person who made recursion theory into a theory. Recursion theory was begun by Kleene's teacher at Princeton, Alonzo Church, who first defined the class of recursive functions; first maintained that this class was the class of computable functions (a claim which has come to be known as Church's Thesis); and first used this fact to solve negatively some classical problems on the existence of algorithms. However, it was Kleene who, in his thesis and in his subsequent attempts to convince himself of Church's Thesis, developed a general theory of the behavior of the recursive functions. He continued to develop this theory and extend it to new situations throughout his mathematical career. Indeed, all of the research which he did had a close relationship to recursive functions.

Church's Thesis arose in an accidental way. In his investigations of a system of logic which he had invented, Church became interested in a class of functions which he called the λ-definable functions. Initially, Church knew that the successor function and the addition function were λ-definable, but not much else. During 1932, Kleene gradually showed[1] that this class of functions was quite extensive; and these results became an important part of his thesis *1935a* (completed in June of 1933).

It was almost immediate from the definition that every λ-definable function was computable, i.e., that for each such function there was an algorithm by which, given an argument to the function, we could compute the corresponding value. As Kleene's investigations continued, Church came to believe that all computable functions were λ-definable. Kleene himself was doubtful; at first he tried to disprove it by a use of the diagonal method.

In 1934, Gödel, already the world's most famous logician, came to the Institute for Advanced Study. Church proposed his thesis to Gödel, who was very skeptical. A little later, Gödel defined another class of functions,

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Received September 22, 1994.

We are grateful to the University of Wisconsin for the picture on the opposite page.

[1] Kleene writes that he achieved his first success in this direction, the proof that the predecessor function was λ-definable, while having some wisdom teeth pulled. How many wisdom teeth? In one place he says two, in another, four. I am afraid that this number is not computable at this stage.

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each of which was clearly computable; but he was by no means convinced at the time that all computable functions were in this class. Eventually, Church and Kleene *1936b* proved that Gödel's class was identical to the class of λ-definable functions; the members of this class came to be known as the *recursive* functions.

Kleene was finally convinced of Church's Thesis when he discovered the Recursion Theorem (discussed below). About this time, Alan Turing, unaware of the work being done at Princeton, had defined a class of functions which he claimed was the class of computable functions. In his definition, the class of functions was the class of all functions which could be computed by a particular kind of computing machine. He gave arguments that his machines could compute any computable function; these arguments were very convincing to Gödel and others. Eventually Turing read of Church's work and came to Princeton; while there, he published a proof [13] that his class of functions was equal to the class of recursive functions. From this point, Church's Thesis was accepted by almost all logicians.

One result of all this was the virtual disappearance of λ-definability from recursion theory. Kleene found, as he had remarked, that λ-definability did not have much sex appeal for logicians; so he switched to a variant of Gödel's definition. More recently, λ-definability has been revived and studied by computer scientists. A good introduction to the modern theory of λ-definability is [3].

Church realized at an early stage that his thesis made it possible to prove mathematically that finding algorithms to solve certain problems was impossible. He combined this idea and techniques used by Gödel in proving the Incompleteness Theorem[2] to show that there was no algorithm for deciding, given a sentence of the predicate calculus, if it was a theorem. A bit later, Turing showed that there was no algorithm for deciding, given an input for his machine, if the machine would eventually stop if started with this input. Since then, many difficult and interesting results on the nonexistence of algorithms, such as the unsolvability of Hilbert's tenth problem and of the word problem for groups, have been proved; and these results are one of the most important justifications of recursion theory.

§2. Basic recursion theory. Kleene began the development of basic recursion theory in *1936a*, and extended and improved it in *1952b*. We are going to sketch the principal features of this development. From now on, everything we do is attributable to Kleene unless otherwise indicated.

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[2] Kleene has pointed out that if Gödel had not already proved his Incompleteness Theorem, Church might well have derived it from his results The reader interested in such a derivation of the Incompleteness Theorem can consult my book ***Mathematical Logic***. Of course, this takes nothing away from Gödel's supreme achievement in discovering the Incompleteness Theorem.

First we consider which functions to allow. Our functions will take natural numbers (i.e., nonnegative integers) as arguments and values. They can be n-ary for any n. An n-ary function need not include among its arguments all n-tuples of numbers; if it does, it is called a *total* function. (In classical recursion theory, including all of Kleene's work, *function* means *total function* and what we have called a function is called a *partial function*. Partial functions were not even used in recursion theory until Kleene introduced them in *1938*. However, they have replaced total functions in most of recursion theory, which explains our change of terminology.) A total unary function is called a *real*. (This terminology from set theory is not yet popular in recursion theory, but it should be.)

We also want to consider relations; specifically, n-ary relations between natural numbers. (Thus a unary relation is just a set of numbers.) In recursion theory, it is convenient to identify a relation R with its *representing function*, which is the total function F defined by $F(x_1, \ldots, x_n) = 0$ if $R(x_1, \ldots, x_n)$ and $F(x_1, \ldots, x_n) = 1$ otherwise. We shall understand *number* to mean *natural number* and *set* to mean *set of numbers*. (Other sets will be referred to as classes.) We use x, y, and z for numbers; F, G, and H for functions; P, Q, and R for relations; and A, B, and C for sets.

Next we need a definition of recursive. We have already mentioned three definitions, due to Church, Gödel, and Turing; we shall use a fourth, due to Kleene. We first list some simple properties which we want the class of recursive functions to have. We shall write $\vec{x}$ for $x_1, \ldots, x_n$ when it is not important to indicate n.

I. The functions Z, Sc, and I defined by $Z(x) = 0, Sc(x) = x + 1$, and $I(x) = x$ are recursive.

II. If G is recursive and F is defined by[3] $F(x_1, \ldots, x_n) = G(x_{i_1}, \ldots, x_{i_k})$ where $1 \leq i_j \leq n$ for all j, then F is recursive.

III. If G and H are recursive and F is defined by $F(\vec{x}) = G(H(\vec{x}), \vec{x})$, then F is recursive.

IV. If G and H are recursive and F is defined inductively by $F(0, \vec{x}) = G(\vec{x}), F(y + 1, \vec{x}) = H(F(y, \vec{x}), y, \vec{x})$, then F is recursive.[4]

V. If G is recursive and F is defined by $F(\vec{x}) = \mu y(G(y, \vec{x}) = 0)$, then F is recursive. (Here $\mu y(G(y, \vec{x}) = 0)$ is equal to z if $G(z, \vec{x}) = 0$ and $G(y, \vec{x})$ is defined and different from 0 for every $y < z$. If there is no such z, $\mu y(G(y, \vec{x}) = 0)$ is undefined. In the important case when G is total, we may read μy as *the least y such that*.)

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[3] When we write an equality between expressions which may be undefined, we mean that either both sides are undefined, or both sides are defined and have the same value.

[4] As Kleene pointed out in *1936c*, we can use an idea of Gödel to leave out IV, provided we add the functions $+, \cdot$ and $=$ to I. While this is useful for some purposes, it would make the following development more complicated.

We say a class of functions $\mathcal{A}$ is *recursively closed* if I–V are true when 'is recursive' is replaced by 'is in $\mathcal{A}$'. A *recursive* function is a function which belongs to every recursively closed class. Since the intersection of any number of recursively closed classes is recursively closed, the class of recursive functions is the smallest recursively closed class; so I–V are correct as written.

A function F is *computable* if there is an algorithm which, if applied to the inputs $\vec{x}$, will give the output $F(\vec{x})$ if $F(\vec{x})$ is defined and will give no output otherwise. It is easy to see that the class of computable functions is recursively closed; so every recursive function is computable. The converse is by no means evident; in fact, there is no reason to believe it at this stage of the development.

From I–V we can prove that many functions and relations are recursive, e.g., all constant total functions, the functions $+$ and $\cdot$, and the relations $=, <, >, \leq$ and $\geq$. We can also prove results of the same nature as I–V, of which we mention two.[5]

VI. If P and Q are recursive relations, then $\neg P, P \vee Q, P \,\&\, Q, P \rightarrow Q, P \leftrightarrow Q, P_1$, and P_2 are recursive, where P_1 and P_2 are defined by $P_1(z, \vec{x}) \leftrightarrow (\forall y < z)P(y, z, \vec{x})$ and $P_2(z, \vec{x}) \leftrightarrow (\exists y < z)P(y, z, \vec{x})$.

VII. Let $R_1, \ldots, R_k$ be recursive relations such that for all $\vec{x}$, exactly one of $R_1(\vec{x}), \ldots, R_k(\vec{x})$ holds. If $G_1, \ldots, G_k$ are recursive and total, and F is defined by

$$F(\vec{x}) = G_i(\vec{x}) \text{ if } R_i(\vec{x})$$

for $i = 1, \ldots, k$, then F is recursive.

These consequences are fairly easy to prove; we note only that the P_1 and P_2 in VI should be defined by induction on z.

How shall we treat functions whose arguments and values are objects other than numbers? One method (first introduced by Gödel in his proof of the Incompleteness Theorem) is *coding*. To each of these objects we assign a number, called the *code* of that object, in such a way that different objects have different codes. If we have a function with these objects as arguments or values, we replace each such argument or value by its code. (The term 'code' was not used by Kleene, but it has become standard in recent years.)

Here is an important example of coding. Let the objects be all finite sequences of natural numbers. Let $p_0, p_1, \ldots$ be the primes in increasing order. Then the code of the sequence $x_1, \ldots, x_n$ is $\prod_{i=1}^{n} p_i^{x_i+1}$, which we designate by $\langle x_1, \ldots, x_n \rangle$. A useful property of this coding is that $x_i < \langle x_1, \ldots, x_n \rangle$; we call this the *sequence code inequality*.

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[5] The fact that these results could be derived from I–V (or even I–IV) was noted by Gödel in his proof of the Incompleteness Theorem.

In this and other codings, the exact choice of codes is not important. What is important is that certain functions and relations associated with the coding are recursive. For example, $\langle x_1, \ldots, x_n \rangle$ is a recursive function of $x_1, \ldots, x_n$. The set of codes of sequences is recursive. There are recursive total functions lh and $(x)_i$ such that if $x = \langle x_0, \ldots, x_{n-1} \rangle$, then $lh(x) = n$ and $(x)_i = x_i$ for $i < n$. Note that the statement that every member of the sequence with code x satisfies P can be written as $(\forall i < lh(x))P((x)_i)$.

One of the uses of codes of sequences is to replace n-ary functions by unary functions. If F is n-ary, we define the *contraction* $\langle F \rangle$ of F by $\langle F \rangle(x) = F((x)_0, \ldots, (x)_{n-1})$. Then $F(x_1, \ldots, x_n) = \langle F \rangle(\langle x_1, \ldots, x_n \rangle)$. These equations show that F is recursive iff $\langle F \rangle$ is recursive.

Another use of codes of sequences is to give an extension of IV. In a definition under IV, the value of $F(y, \vec{x})$ depends on the value of $F(y - 1, \vec{x})$. In a more general form of definition by induction (sometimes called definition by *course-of-values induction*), the value of $F(y, \vec{x})$ depends on the value of $F(z, \vec{x})$ for all $z < y$. To code these values by a single number, we define $\overline{F}(y, \vec{x}) = \langle F(0, \vec{x}), \ldots, F(y-1, \vec{x}) \rangle$. Then a definition by course-of-values induction has the form $F(y, \vec{x}) = G(\overline{F}(y, \vec{x}), y, \vec{x})$. We want to show that if G is recursive, then F is recursive. Since $F(y, \vec{x}) = (\overline{F}(Sc(y), \vec{x}))_y$, it is enough to prove that $\overline{F}$ is recursive. But it is easy to see that $\overline{F}$ has an inductive definition which comes under IV.

We now assign numbers, called *indices*, to recursive functions. We assign the index $\langle 1, 0 \rangle$ to Z, $\langle 1, 1 \rangle$ to S, and $\langle 1, 2 \rangle$ to I. If F and G are as in II and g is an index of G, then $\langle 2, g, \langle n, i_1, \ldots, i_p \rangle \rangle$ is an index of F. If F, G, H are as in III and g, h are indices of G, H, then $\langle 3, g, h \rangle$ is an index of F. If F, G, H are as in IV and g, h are indices of G, H, then $\langle 4, g, h \rangle$ is an index of F. If F, G are as in V and g is an index of G, then $\langle 5, g \rangle$ is an index of F. Clearly no number is an index of two recursive functions, and every recursive function has at least one index. It is easy to prove that the set of indices is recursive by giving a definition by course-of-values induction, using the sequence code inequality.

We let $\{f\}(x_1, \ldots, x_n)$ be $F(x_1, \ldots, x_n)$ where f is an index of F; if f is not the index of an n-ary function, then $\{f\}(x_1, \ldots, x_n)$ is undefined. By a *formula*, we mean an equation of the form $\{f\}(x_1, \ldots, x_n) = y$ where $f, x_1, \ldots, x_n, y$ are particular numbers and f is the code of an n-ary function. The code of this formula is $\langle f, \langle x_1, \ldots, x_n \rangle, y \rangle$. The set of codes of formulas is recursive.

We shall show that the true formulas are the theorems of a formal system. The axioms of the formal system are the true formulas $\{f\}(\vec{x}) = y$ where f is the code of a function in I. There are four rules of inference, one for each of II–V; we give the last two and leave the other two to the reader. The rule corresponding to IV says: $\{\langle 4, g, h \rangle\}(0, \vec{x}) = z$ is inferable from $\{g\}(\vec{x}) = z$, and $\{\langle 4,g,h\rangle\}(y+1,\vec{x}) = z$ is inferable from $\{\langle 4,g,h\rangle\}(y,\vec{x}) \doteq v$ and $\{h\}(v,y,\vec{x}) = z$. The rule corresponding to V says: $\{\langle 5,g\rangle\}(\vec{x}) = z$ is inferable from $\{g\}(z,\vec{x}) = 0$ and $\{g\}(y,\vec{x}) = v_y$ for all $y < z$, provided that each $v_y \neq 0$. As usual, a *proof* is a nonempty finite sequence of formulas, each of which is inferable from earlier formulas in the sequence (where an axiom is said to be inferable from the empty set); and a *theorem* is a formula which is the last formula in some proof.

Since formulas inferable from true formulas are true, every theorem is true. It is easy to prove that every true formula $\{f\}(\vec{x}) = y$ is a theorem by induction on f, using the sequence code inequality.

The code of a proof is the code of the sequence consisting of the codes of the formulas in the proof. The set of codes of proofs is recursive. The proof of this requires writing out all the rules of inference; it is tedious but straightforward.

We let $T_n(f,x_1,\dots,x_n,y)$ mean that y is the code of a proof, and that the last formula of the proof has $\{f\}(x_1,\dots,x_n)$ as its left side; this is the famous Kleene T-predicate. Each T_n is recursive. It is easy to define a recursive real U such that if y is the code of a proof, then $U(y)$ is the number which appears on the right side of the last formula in the proof. We then have

$$\{f\}(x_1,\dots,x_n) = U(\mu y T_n(f,x_1,\dots,x_n,y)).$$

This important equality is called the Normal Form Theorem.

If F is $(n+1)$-ary, we define $F_{(y)}$ by $F_{(y)}(\vec{x}) = F(y,\vec{x})$. We say that F *enumerates* a class of functions if that class has $F_{(0)}, F_{(1)}, \dots$ as its members. Thus if $F(f,x_1,\dots,x_n) = \{f\}(x_1,\dots,x_n)$, then F enumerates the class of recursive n-ary functions. It is an immediate consequence of the Normal Form Theorem that this F is recursive. This result is called the Enumeration Theorem. (Note that it is trivial that $\{f\}(x_1,\dots,x_n)$ is a recursive function of $x_1,\dots,x_n$ for each f; but this is a much weaker statement than the Enumeration Theorem.)

If F is recursive, then each $F_{(y)}$ is recursive; for $F_{(y)}(\vec{x}) = F(G(\vec{x}),\vec{x})$ where G is the constant function with value y. By examining the assignment of indices to recursive functions, we can define a total recursive function S such that if f is an index of F, then $S(f,y)$ is an index of $F_{(y)}$. We can then define S_m inductively so that $S_m(f,y_1,\dots,y_m)$ is an index of the function $F_{(y_1)\cdots(y_m)}$.

We now show that in defining a recursive function F, we may use an index of F.

RECURSION THEOREM. *If G is recursive, there is a recursive function F with an index f such that $F(\vec{x}) = G(f,\vec{x})$ for all $\vec{x}$.*

To prove this, define a recursive function H by $H(y,\vec{x}) = G(S(y,y),\vec{x})$, and let h be an index of H. Let $F = H_{(h)}$ and $f = S(h,h)$. Then F is recursive; f is an index of F; and $F(\vec{x}) = H(h,\vec{x}) = G(S(h,h),\vec{x}) = G(f,\vec{x})$.

The Recursion Theorem has many uses in recursion theory. We shall show how it can be used to prove that certain functions are recursive. If G and H are total, then $F(0,x) = G(x)$, $F(y+1,x) = H(F(y,2x),y,x)$ is a legitimate definition of $F(y,x)$ by induction on y; but it does not come under IV. We want to show that if G and H are recursive, then F is recursive. Define K by

$$\begin{aligned} K(f,0,x) &= G(x), \\ K(f,y+1,x) &= H(\{f\}(y,2x),y,x). \end{aligned}$$

By VII and the Enumeration Theorem, K is recursive. Hence there is a recursive F' with an index f such that $F'(y,x) = K(f,y,x)$. This F' satisfies the equations for F; so $F' = F$ and hence F is recursive.

Now let us sketch how we prove that our definition of recursive is equivalent to other definitions. Suppose $\mathcal{A}$ is the class of functions given by the alternative definition. First we prove that $\mathcal{A}$ is recursively closed. Then every recursive function is in $\mathcal{A}$. Now let F be in $\mathcal{A}$. Generally, the alternative definition will provide a method of computing $F(\vec{x})$. After suitably coding the objects occurring in the computation, we can show by the above methods that the output of the computation is a recursive function of the inputs. But this means that F is recursive.

What kind of evidence does the theory give for Church's Thesis? We can hope that, given an algorithm for computing F, we can prove that F is recursive by the above methods. Many experiments with particular algorithms have confirmed this. We have already observed that Turing's definition of recursiveness is itself evidence for Church's Thesis. Possibly the best evidence is how well Church's Thesis fits in with the theory. If we replace recursive by computable in results in recursion theory, we often obtain a statement which is evident, or at least more evident than the original result.[6]

We conclude this section with a sample proof of nonexistence of an algorithm. First, we define a real F which is not recursive. We do this by the diagonal method; that is, we make F differ from the recursive function

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[6] Kleene *1952b*, Chapter XII gives many further arguments for Church's Thesis. I must admit that in reading that chapter, I am sometimes reminded of Anatole France's version of the Dreyfuss affair in ***Penguin Island***. General Panther fills his office from floor to ceiling with proofs of Dreyfuss' guilt. The Minister of War complains he is spoiling the simplicity of the case. Of course it is good to have proofs, he says, but perhaps it is better to have none at all.

$\{f\}$ at f. Thus we define $F(f) = \{f\}(f) + 1$ if $\{f\}(f)$ is defined and $F(f) = 0$ otherwise. It follows that the set of f such that $\{f\}(f)$ is defined is not recursive; for otherwise, the Enumeration Theorem and VII would show that this F is recursive. Hence if G is the recursive function defined by $G(f) = \{f\}(f)$, then Church's Thesis shows that there is no algorithm by which, given f, we can decide if $G(f)$ is defined.

§3. **The arithmetical hierarchy.** Kleene left Princeton in 1936 for the University of Wisconsin, where he spent the rest of his career except for visits to other schools and service in the Navy during World War II. One of the first matters he studied there was the effect of putting quantifiers in front of a recursive relation. These results appeared in *1943* and later in *1952b*.

A relation R is *arithmetical* if it has a definition of the form

$$R(\vec{x}) \leftrightarrow Qy_1 \dots Qy_k P(y_1, \dots, y_k, \vec{x})$$

where P is recursive and each Qy_i is either $\exists y_i$ or $\forall y_i$. We call $Qy_1 \dots Qy_k$ the *prefix* and $P(y_1, \dots, y_k, \vec{x})$ the *matrix* of this definition.

We shall show that if in the prefix two adjacent quantifiers are of the same kind (i.e., both $\exists$ or both $\forall$), then they may be replaced by a single such quantifier. Suppose, for example, the definition is $R(x) \leftrightarrow \exists w \exists y \forall z P(w, y, z, x)$. As u varies through all numbers, $(u)_0$, $(u)_1$ varies through all pairs w, y of numbers. Hence we may rewrite our definition as $R(x) \leftrightarrow \exists u\ \forall z\ P((u)_0, (u)_1, z, x)$. Of course, the matrix has changed; but it is still recursive (since $(u)_i$ is a recursive function of u and i).

It follows that every arithmetical relation has a definition in which the prefix does not have two successive quantifiers of the same kind. Then if the prefix has k quantifiers and begins with $\exists$, the relation is Σ^0_k; if the prefix has k quantifiers and begins with $\forall$, the relation is Π^0_k.

If we put a quantifier on x in front of an expression not containing x, we do not change its meaning. From this, we conclude that every recursive relation is Σ^0_k and Π^0_k for all k, and that every Σ^0_k or Π^0_k relation is Σ^0_n and Π^0_n for every $n > k$.

Using the rules for obtaining prenex forms in logic, we obtain closure properties of these relations similar to those given by VI. Thus if P and Q are Σ^0_k, then so are $P \vee Q$ and $P \,\&\, Q$, and the P_1 and P_2 of VI; and $\neg P$ is Π^0_k. If $P_3(\vec{x}) \leftrightarrow \exists y P(y, \vec{x})$ and $P_4(\vec{x}) \leftrightarrow \forall y P(y, \vec{x})$, then P_3 is Σ^0_k and P_4 is Π^0_{k+1}. All of the results also hold with Π and Σ (and $\forall$ and $\exists$) interchanged. These results generally enable us to find the classification of any arithmetical relation we have defined.

The domain of an n-ary function is a class of n-tuples of numbers, i.e., an n-ary relation. We say that a relation is *recursively enumerable* (abbreviated RE) if it is the domain of a recursive function. We say that a relation R is *semicomputable* if there is an algorithm which, applied to $\vec{x}$, gives an output iff $R(\vec{x})$. It is easy to show from Church's Thesis that a relation is RE if it is semicomputable. For this and other reasons, RE relations are of special interest in Recursion Theory.[7]

A relation is Σ_1^0 iff it is RE. For if $P(\vec{x}) \leftrightarrow \exists y R(y, \vec{x})$ with R recursive, then P is the domain of the recursive function $\mu y R(y, \vec{x})$; while if P is the domain of the recursive function with index f, then $P(\vec{x}) \leftrightarrow \exists y T_n(f, \vec{x}, y)$ by the Normal Form Theorem. From this, we can prove that many relations are RE. For example, the set of codes of true formulas is RE; for x is such a code iff there exists a y such that y is the code of a proof and x is the last number in the sequence with code y.

We write W_f for the domain of $\{f\}$, and say that f is an RE-*index* of W_f. Thus every RE relation has at least one RE-index. By the Normal Form Theorem, $W_f(x_1, \ldots, x_n) \leftrightarrow \exists y T_n(f, x_1, \ldots, x_n, y)$. It follows that $W_f(x_1, \ldots, x_n)$ is an RE relation of $f, x_1, \ldots, x_n$. Since this relation enumerates the class of n-ary RE relations, this result is called the RE Enumeration Theorem. It is easy to see that $W_{S(f,y)} = (W_f)_{(y)}$ and similarly for S_m. From this, we may obtain a Recursion Theorem for defining RE relations.

The classification of the arithmetical relations into Σ_k^0 and Π_k^0 relations is called the arithmetical *hierarchy*. We have yet to see that it is nontrivial, i.e., that Σ_k^0 is different from Σ_n^0 if $k \neq n$. The key to this is the following important result.

ARITHMETICAL ENUMERATION THEOREM. *For each n and k, there is a Σ_k^0 $(n+1)$-ary relation R which enumerates the class of Σ_k^0 n-ary relations; and similarly with Π for Σ.*

The proof is by induction on k. The Σ_1^0 case is just the RE Enumeration Theorem. If R is an enumerating relation for the Σ_k^0 n-ary relations, then $\neg R$ is an enumerating relation for the Π_k^0 n-ary relations. If R is an enumerating relation for the Π_k^0 $(n+1)$-ary relations, we may obtain an enumerating relation for the Σ_{k+1}^0 n-ary relations by putting an existential quantifier in front of R.

Now let R be the enumerating relation for the Π_k^0 unary relations. Define Q by $Q(x) \leftrightarrow \neg R(x, x)$; then Q is Σ_k^0. Since $Q(x) \leftrightarrow \neg R_{(x)}(x)$, $Q \neq R_{(x)}$; so Q is not Π_k^0, and hence not recursive or Σ_n^0 or Π_n^0 for any $n < k$. Similarly, we can find a set which is Π_k^0, but not Σ_k^0, and hence not recursive or Σ_n^0 or Π_n^0 for any $n < k$. In particular, we can find an RE set which is not recursive.

A relation R is Δ_k^0 if it is both Σ_k^0 and Π_k^0. A simple but important result says that a relation is Δ_1^0 iff it is recursive. We already know that recursive relations are Δ_1^0. Suppose that $R(\vec{x}) \leftrightarrow \exists y P(y, \vec{x}) \leftrightarrow \forall y Q(y, \vec{x})$ with P and

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[7] RE relations were discovered and studied in the earliest days of recursion theory by Church, Kleene, and Rosser.

Q recursive. Then

$$R(\vec{x}) \leftrightarrow P(\mu y(P(y,\vec{x}) \vee \neg Q(y,\vec{x})),\vec{x});$$

so R is recursive.

To get a similar result for Δ^0_{k+1}, we need the concept of relative recursiveness, which was invented by Turing and has played a large role since. Let Φ be a class of *total* functions. A function is recursive *in* Φ or *relative* to Φ if it belongs to every recursively closed class which includes Φ. Almost all of recursion theory remains valid when *recursive* is replaced by *recursive in* Φ. (In some cases, we must restrict ourselves to finite Φ. This is not a serious difficulty, since it is easy to see that a function is recursive in Φ iff it is recursive in a finite subset of Φ.)

As an example, we can define arithmetical in Φ, Σ^0_k in Φ, etc., and prove all of the above results for these concepts (although the Enumeration Theorem requires that Φ be finite). We also have a new result: a relation is Σ^0_1 in a Π^0_k relation iff it is Σ^0_{k+1}. For if P is Σ^0_{k+1}, then $P(\vec{x}) \leftrightarrow \exists y Q(y,\vec{x})$ where Q is Π^0_k; so P is Σ^0_1 in Q. The converse requires writing down a definition of P and then using the above ideas to show that P is Σ^0_{k+1}.

By applying this result to both R and $\neg R$, we see that R is Δ^0_1 in a Π^0_k relation iff it is Δ^0_{k+1}. From this and the relativization of our result on Δ^0_1, we get:

POST'S THEOREM. *A relation is* Δ^0_{k+1} *iff it is recursive in a* Π^0_k *relation.*

When Kleene was about to publish this result, he discovered an abstract by Emil Post containing the same result. He immediately gave it the name Post's Theorem.[8]

There has been much further work on the above ideas, particularly in the field of RE sets. The reader who wants an introduction to the latter subject cannot do better than consult Post's paper [9]; it has been an inspiration to recursion theorists throughout the half century since it was written.

§4. **Degrees.** We now come to a topic in recursion theory on which Kleene did little work; he wrote only one joint paper on the subject. However, because of the importance which the subject has achieved and the fundamental character of this paper, it is worth some discussion.

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[8] Although careful to not appropriate others' work, Kleene worried little about people appropriating his own. In the fifties, a recursion theorist who should have known better rediscovered the Recursion Theorem (in a slightly different form) and called it the Fixed Point Theorem. In a lecture in which he had used the Recursion Theorem, Kleene merely remarked, "I see some people are now calling this the Fixed Point Theorem."

The concept is simple. If F and G are total[9] functions, let $F \leq G$ mean that F is recursive in G. The relation $\leq$ has some of the properties of a partial ordering; it is reflexive and transitive, but not symmetric. Researchers in partial orderings long ago told us what to do in such a case. We define $F \simeq G$ to mean that $F \leq G \,\&\, G \leq F$. Then $\simeq$ is an equivalence relation. The equivalence class of F is called the *degree* of F, and is designated by $\operatorname{dg} F$. We now define $\operatorname{dg} F \leq \operatorname{dg} G$ to mean that $F \leq G$. The properties of $\leq$ mentioned above guarantee that this definition depends only on $\operatorname{dg} F$ and $\operatorname{dg} G$, not on F and G, and that the resulting $\leq$ is a partial ordering of the degrees.[10]

Every degree contains a relation; for F is equivalent to the graph of F, which is the set of all $(n+1)$-tuples $(x_1, \ldots, x_n, F(x_1, \ldots, x_n))$. Since the contraction of F is equivalent to F, every degree contains a real and every degree contains a set.

What can we say about the partially ordered set of degrees? The class of recursive total functions is clearly a degree, which is the smallest degree; it is designated by **0**. Every two element set (and hence every finite set) of degrees have a least upper bound; for the least upper bound of $\operatorname{dg} F$ and $\operatorname{dg} G$ is $\operatorname{dg} H$, where $H(x) = \langle F(x), G(x) \rangle$. Many of the results of the early days of degree theory consisted of showing that this partially ordered set lacks other nice properties.

Post discovered all of this and some deeper results; but he did not publish them. Kleene heard of the results and wrote to Post, saying that he should certainly publish. Post compromised by writing out some things in a very disorganized form and suggesting that Kleene find a graduate student to work out the details. Kleene tried, but the graduate student was not successful; so Kleene (after making his own contribution by proving that two element sets need not have greatest lower bounds) completed the paper and published it as *1954*.

The first important result of *1954* is that the partially ordered set of degrees is not linearly ordered. In terms of functions, this means that there are reals F and G such that neither is recursive in the other. We shall sketch the proof of this, since it is fundamental for all the degree theory which followed.

We want to ensure that $F \neq \{f\}^G$ and $G \neq \{f\}^F$ for all f (where $\{f\}^G$ is the analogue of $\{f\}$ for recursion in G). We shall construct F and G in infinitely many steps, at each of which we define finitely many values of F

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[9] It is possible to also consider degrees of partial functions; but there are important differences between the two theories. Most of the research on degrees have been on degrees of total functions.

[10] In proving results about degrees, equivalence classes are difficult to deal with; so we usually pick one function in the equivalence class and deal with it. Then why bother with the equivalence classes at all? Because the theory of partially ordered sets is well-developed and suggests many problems about degrees, which have been the heart of the theory

and G. At the end of the fth step, we will have insured that $F \neq \{f\}^G$ and $G \neq \{f\}^F$, and that $F(f)$ and $G(f)$ are defined. At the fth step, pick a number x such that $F(x)$ is not yet defined. If there is no way to complete the definition of G so that $\{f\}^G(x)$ is defined, then we ensure $F \neq \{f\}^G$ by setting $F(x) = 0$. Suppose we can complete the definition of G so that $\{f\}^G(x) = y$. It is easy to see from the definition of *recursive in* G that the truth of the equation $\{f\}^G(x) = y$ depends only on the values of G at finitely many arguments. Hence at step f, we may insure that $\{f\}^G(x) = y$; and we may then insure that $F \neq \{f\}^G$ by defining $F(x) = y + 1$. By defining further values of F and G, we may also insure that $G \neq \{f\}^F$ and that $F(f)$ and $G(f)$ are defined.

Before publishing this paper, Kleene showed it to his graduate student Clifford Spector, who solved some of the problems which Kleene and Post left open. In particular, he proved in [12] that there were minimal degrees, i.e., degrees $\boldsymbol{a} > \mathbf{0}$ such that there is no degree $\boldsymbol{b}$ with $\mathbf{0} < \boldsymbol{b} < \boldsymbol{a}$. However, neither Kleene nor Spector did any further work in degrees.

We cannot even summarize all the work in degrees that has taken place since then; so we will only mention the most important single development. Call a degree RE if it contains a RE relation. The degree $\mathbf{0}'$ of a binary RE relation which enumerates all RE sets is easily seen to be the largest RE degree. Since there is an RE relation which is not recursive, $\mathbf{0}' \neq \mathbf{0}$; so there are at least two RE degrees. Post had long been interested in the problem of whether there were additional RE degrees. It would certainly suffice to show that there are incomparable RE degrees; but the Kleene-Post proof will not do, since the F and G constructed there are not RE. For this, we should need the construction to be effective; but there is no effective way of deciding if we can complete the definition of G in the way required.

The additional idea needed to make the construction effective is the Priority Method, which was discovered independently by the undergraduate Friedberg[11] in the US and the undergraduate Muchnik in Russia not long after the publication of the Kleene-Post paper. (See [10] for references and for much else about priority.) In the forty years since then, this method has been intensively studied, but we are still very far from discovering all of its possibilities.

§5. Real arguments and the analytical hierarchy. Suppose we take the functions we have been considering up to now and use them as arguments or values for new functions. How shall we extend recursion theory to this

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[11] Friedberg, who was at Harvard, learned of Post's problem in a course given by Hartley Rogers at MIT. This course seems to have attracted some of the best young minds in Boston. David Mumford, who has since made his name well-known in other fields, is still known to some elder logicians as the second best student in Rogers' recursion theory course.

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situation? Coding is of no use here; for the set of functions is uncountable and hence cannot be coded by numbers. Kleene gave much thought to this question from the time he returned from the war. When he had a satisfactory theory, and he and his graduate students had solved some of the resulting problems, he published his results in a series of three important papers (*1955a, 1955b, 1955c*).

First, we have to decide which functions to allow as arguments and values. There is no need to consider functions as values; for if F takes functions as values, it may be replaced by the F' defined by $F'(x, y) = (F(x))(y)$. We will allow only total functions as arguments. By replacing a total function by its contraction, we may assume that the arguments are reals.

We shall thus henceforth consider functions which take reals and numbers as arguments and numbers as values. We shall always put the real arguments first. A function is (m, n)-ary if it has m real arguments and n number arguments. We use α, β and γ as variables for reals.

How can we extend computability to such functions? The problem here is: how are we to be given a real argument α of our function? The obvious answer is that we are given an object which, when presented with any given number x, will give us $\alpha(x)$. Following Turing, we call such an object an *oracle* for α.

Let F be an (m, n)-ary function. An *associate* of F is a total function G of number variables such that if $G(\overline{\alpha}_1(z), \dots, \overline{\alpha}_m(z), x_1, \dots, x_n) = y + 1$, then $F(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n) = y$, and such that if $F(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n) = y$, then $G(\overline{\alpha}_1(z), \dots, \overline{\alpha}_m(z), x_1, \dots, x_n) = y + 1$ for all sufficiently large z. It is easy to see that F is computable iff it has a computable associate. We therefore define an (m, n)-ary function to be recursive if it has a recursive associate.[12]

Extending the results of §2 to our new functions is mostly an exercise in constructing associates. There are also a few new results; e.g., $\alpha(x)$ and $\overline{\alpha}(x)$ are recursive functions of α and x. (On the other hand, $\alpha = \beta$ is not a recursive function of α and β.) As an index of a recursive F, we use any index of an associate of F. For $m \neq 0$, the Normal Form Theorem now has the form

$$\{f\}(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n) = U(\mu y T_{m,n}(f, \overline{\alpha}_1(y), \dots, \overline{\alpha}_m(y), x_1, \dots, x_n)).$$

We now let $F_{(y)}$ be the function obtained from F by substituting y for the first number argument to F. We then have the Recursion Theorem as before.

In III, we substituted a recursive function for a number argument of a recursive function G. We now do the same for real arguments. Of course, we cannot substitute $H(\dots)$ for α, since the former is a number and the latter is a real; so we need some new notation. If A is an expression containing x which designates a number, then $\lambda x A$ designates the function whose value at any x is A. (This is the λ which gives the name to λ-definable functions.) Clearly $\lambda x A$ is a unary function, which is a real iff A is defined for every value of x. The proof of the following consists of a straightforward construction of an associate for F.

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[12] There are many satisfactory definitions of recursive in this case. The one that we use here comes from Kleene *1959b*.

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SUBSTITUTION THEOREM. *If G and H are recursive and H is total, then the F defined by $F(\vec{\alpha}, \vec{x}) = G(\lambda z H(\vec{\alpha}, \vec{x}, z), \vec{\alpha}, \vec{x})$ is recursive.*

What about relativization to a class Φ of total functions? If the functions in Φ have real arguments, there are difficulties; this will be considered further in §7. If Φ is a class of functions of number variables, we may define relative recursiveness as before. However, there is in this case another simple definition: F is recursive in Φ iff there are $G_1, \dots, G_j \in \Phi$ and a recursive F' such that $F(\vec{\alpha}, \vec{x}) = F'(\langle G_1 \rangle, \dots, \langle G_j \rangle, \vec{\alpha}, \vec{x})$ for all $\vec{\alpha}$ and $\vec{x}$.

Now let us turn to the generalization of the arithmetical hierarchy. Here, of course, we wish to allow both number quantifiers and real quantifiers. An *analytical* relation is one which has a definition consisting of number and real quantifiers followed by a recursive relation.

By a proof only slightly more complicated than the one we used in the arithmetical hierarchy, we show that every analytical relation has a definition in which the prefix does not have two successive quantifiers of the same kind, and there is exactly one number quantifier, which comes last. Then if the prefix has k real quantifiers and beings with $\exists$, the relation is Σ^1_k; if the prefix has k real quantifiers and begins with $\forall$, the relation is Π^1_k.

We can say a bit more about the matrix. Suppose that R is a $(0,1)$-ary Π^1_1 relation, so that $R(x) \leftrightarrow \forall\alpha\exists y P(\alpha, y, x)$ with P recursive. Let $F(\alpha, x) = \mu y P(\alpha, y, x)$ and let f be an index of F. Then by the Normal Form Theorem,

$$R(x) \leftrightarrow \forall\alpha(F(\alpha, x) \text{ is defined}) \leftrightarrow \forall\alpha\exists y T_{1,1}(f, \overline{\alpha}(y), x).$$

The general conclusion is: the definition may be chosen so that if α and y and the last two variables in the prefix, then they occur in the matrix only in the context $\overline{\alpha}(y)$.

As before, we can show that every arithmetical relation is Π^1_k and Σ^1_k for all k, and that every Σ^1_k or Π^1_k relation is Σ^1_n and Π^1_n for all $n > k$. We can also find closure properties similar to the arithmetical case. In particular, prefixing number quantifiers does not change the position of a relation in the analytical hierarchy. We can prove an Analytical Enumeration Theorem (again using the RE Enumeration Theorem) and use it to show that the hierarchy is nontrivial.

We shall establish a connection between Π^1_1 sets and ordinals. We first define a set O_σ for each ordinal σ by induction on σ as follows: $e \in O_\sigma$ if $W_e \subset \bigcup_{\tau<\sigma} O_\tau$. We let O be[13] the union of the O_σ. If $e \in O$, $|e|$ is the least σ such that $e \in O_\sigma$. An ordinal σ is *recursive* if $\sigma = |e|$ for some $e \in O$; any such e is then called a *notation* for σ.

Since O_σ is a nondecreasing function of σ, we have

$$e \in O_\sigma \leftrightarrow e \in O \,\&\, |e| \leq \sigma. \tag{1}$$

Hence, setting $O^*_\sigma = \bigcup_{\tau<\sigma} O_\tau$,

$$e \in O^*_\sigma \leftrightarrow e \in O \,\&\, |e| < \sigma. \tag{2}$$

LEMMA. For every e, $e \in O$ iff $W_e \subset O$; in this case, $|e|$ is the least ordinal greater than $|f|$ for all $f \in W_e$.

If $e \in O$, then $e \in O_{|e|}$; so $W_e \subset O^*_{|e|}$; so $W_e \subset O$ and $|f| < |e|$ for $f \in W_e$ by (2). If $\sigma < |e|$, $e \notin O_\sigma$; so there is an $f \in W_e$ such that $f \notin O^*_\sigma$ and hence $\sigma \leq |f|$ by (2). If $W_e \subset O$, then $W_e \subset O^*_\sigma$ for some σ; so $e \in O_\sigma \subset O$.

From the Lemma, it is easy to show that if σ is recursive, then $\sigma + 1$ is recursive. It follows that every finite ordinal is recursive. A little more work shows that most of the countable ordinals which occur in practice (e.g., ω and ε_0) are recursive.

A sequence $\{x_n\}$ is *O-descending* if $x_{n+1} \in W_{x_n}$ for all n. We show that $x \in O$ iff there is no O-descending sequence beginning with x. If $x \in O$ and $\{x_n\}$ is an O-descending sequence with $x_0 = x$, then $x_{n+1} \in O$ and $|x_{n+1}| < |x_n|$ for all n by induction on n and the Lemma; and this is impossible. If $x \notin O$, the Lemma shows that we can choose inductively a O-descending sequence $\{x_n\}$ such that $x_0 = x$ and $x_n \notin O$ for all n. Thus we have

$$x \in O \leftrightarrow \neg\exists\alpha(\alpha(0) = x \,\&\, \forall n(\alpha(Sc(n)) \in W_{\alpha(n)}));$$

so O is Π^1_1.

We shall show that O is, in a certain sense, the most complicated Π^1_1 set. We say that a set A is *many-one reducible* to a set B if there is a recursive real F such that $x \in A \leftrightarrow F(x) \in B$ for all x. This clearly implies that A is recursive in B.

THEOREM. *Every Π^1_1 set is many-one reducible to O.*

If A is Π^1_1, we have $x \in A \leftrightarrow \forall\alpha\exists y R(x, \overline{\alpha}(y))$ where R is recursive. Let $z \in O$. Let $w\char`^m$ be a recursive function of w and m such that if $w =$

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[13] Our O is a simplification of the O defined by Church and Kleene and used throughout Kleene's work. (Caution: Our $<_O$ is somewhat different from the $<_O$ of Church and Kleene.)

$\langle x_1, \dots, x_n\rangle$, then $w\hat{\ }m = \langle x_1, \dots, x_n, m\rangle$. Then $\exists m(\{f\}(x, w\hat{\ }m) = y)$ is an RE relation of x, f, w, y. Let q be an RE-index of this relation. Using the Recursion Theorem, define a recursive total F with index f by

$$\begin{aligned} F(x,w) &= z && \text{if } R(x,w), \\ &= S_3(q,x,f,w) && \text{otherwise.} \end{aligned}$$

We complete the proof by showing that $x \in A \leftrightarrow F(x, \langle\rangle) \in O$. Suppose that $x \notin A$, and choose α so that $\forall y \neg R(x, \overline{\alpha}(y))$. Then $F(x, \overline{\alpha}(y))$ is an RE-index of the set of $F(x, \overline{\alpha}(y)\hat{\ }m)$, which contains $F(x, \overline{\alpha}(y+1))$. Hence $\{F(x, \overline{\alpha}(n))\}$ is an O-descending sequence; so $F(x, \langle\rangle) \notin O$. Now suppose that $F(x, \langle\rangle) \notin O$. We will define $\alpha(n)$ by induction so that for all n, $F(x, \overline{\alpha}(n)) \notin O$. Then by the definition of F, $\forall n \neg R(x, \overline{\alpha}(n))$; so $x \notin A$. We have $F(x, \overline{\alpha}(0)) \notin O$. Suppose $\overline{\alpha}(n)$ has been defined and $F(x, \overline{\alpha}(n)) \notin O$. Then $F(x, \overline{\alpha}(n)) = S_3(q, x, f, w)$. By the Lemma, there is an m such that $F(x, \overline{\alpha}(n)\hat{\ }\langle m\rangle) \notin O$. Letting $\alpha(n) = m$, we have $F(x, \overline{\alpha}(n+1)) \notin O$.

COROLLARY. O is not Σ^1_1.

Every set many-one reducible to a Σ^1_1 set is Σ^1_1. Hence if O were Σ^1_1, then every Π^1_1 set would be Σ^1_1, which is not the case.

A relation is Δ^1_n if it is both Π^1_n and Σ^1_n. Recall that a relation is Δ^0_1 iff it is recursive. We shall give a similar (but much more difficult) characterization of Δ^1_1. This can be done for all relations; but, following Kleene, we shall consider only relations of number variables. In fact, we consider only sets, since n-ary relations can be replaced by their contractions.

We define the set H_σ by induction on σ as follows: $H_\sigma(x) \leftrightarrow (\exists\tau < \sigma)\neg W_x(H_\tau)$ (where W_x is the $(1,0)$-ary relation with RE-index x). A set is *hyperarithmetical*[14] if it is many-one reducible to H_σ for some recursive σ. We shall show that a set is Δ^1_1 iff it is hyperarithmetical.

We first obtain an inductive definition of H_σ which uses notations instead of ordinals. Let $f <_O e$ mean that there is a sequence $z_1, \dots, z_n$ with $n > 1$, $z_1 = f$, $z_n = e$, and $z_i \in W_{z_{i+1}}$ for $1 \le i < n$. Writing this out (using a code of the sequence), we see that the relation $<_O$ is Σ^0_1. If $e \in O \,\&\, f <_O e$, then $f \in O \,\&\, |f| < |e|$ by the Lemma. If $e \in O \,\&\, \sigma < |e|$ then there is an $f <_O e$ such that $|f| = \sigma$; this is easily proved by induction on $|e|$, using the Lemma. It follows that for $e \in O$,

$$H_{|e|}(x) \leftrightarrow \exists f(f <_O e \,\&\, \neg W_x(H_{|f|})). \tag{3}$$

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[14] The hyperarithmetical sets were discovered independently by Kleene, Martin Davis, and Mostowski However, it was Kleene who discovered the relationship of hyperarithmetical relations to the analytical hierarchy. I discovered the definition given here about 1962.

We write $f \leq_O e$ for $f <_O e \vee f = e$. For $e \in O$, define $P_e(f,x) \leftrightarrow f \leq_O e \,\&\, H_{|f|}(x)$. Fix $e \in O$. Suppose that $P = P_e$. Then

$$\forall f \forall x (P(f,x) \rightarrow f \leq_O e). \tag{4}$$

If $f \leq_O e$, $P_{(f)} = H_{|f|}$. If also $g <_O f$, then $g \leq_O e$, so $P_{(g)} = H_{|g|}$. Thus by (3),

$$\forall f \forall x (f \leq_O e \rightarrow (P(f,x) \leftrightarrow \exists g(g <_O f \,\&\, \neg W_x(P_{(g)})))). \tag{5}$$

Conversely, suppose that (4) and (5) hold. Using induction on $|f|$ and (5), we get $P_{(f)} = H_{|f|}$ for $f \leq_O e$. From this and (4), $P = P_e$.

Let $Q(\alpha, e)$ be the conjunction of $\forall f \forall x (\alpha(\langle f, x\rangle) = 0 \rightarrow f \leq_O e)$ and

$$\forall f \forall x (f \leq_O e \rightarrow (\alpha(\langle f,x\rangle) = 0 \leftrightarrow \exists g(g <_O f \,\&\, \neg W_x(\lambda x \alpha(\langle g,x\rangle)))))).$$

By the RE Enumeration Theorem and the Substitution Theorem, Q is arithmetical. The above shows that for $e \in O$, $Q(\alpha, e)$ iff $\forall f \forall x (\alpha(\langle f, x\rangle) = 0 \leftrightarrow P_e(f,x))$.

We can now prove that every hyperarithmetical set is Δ^1_1. It suffices to show that H_σ is Δ^1_1 for all recursive σ. Let e be a notation for σ. Then

$$H_\sigma(x) \leftrightarrow \forall \alpha (Q(\alpha, e) \rightarrow \alpha(\langle e, x\rangle) = 0) \leftrightarrow \exists \alpha (Q(\alpha, e) \,\&\, \alpha(\langle e, x\rangle) = 0).$$

Thus H_σ is Δ^1_1. In fact, we have proved that there is a Π^1_1 relation H_Π and a Σ^1_1 relation H_Σ such that $H_{|e|}(x) \leftrightarrow H_\Pi(e,x) \leftrightarrow H_\Sigma(e,x)$ for $e \in O$.

We next show that there is a recursive real J such that $x \in O^*_\sigma \leftrightarrow J(x) \in H_\sigma$ for all σ. We assume that J is defined and prove the equivalence by induction on σ. It is enough to prove that $x \in O_\tau \leftrightarrow \neg W_{J(x)}(H_\tau)$ for $\tau < \sigma$. Now $x \in O_\tau \leftrightarrow (\forall y \in W_x)(y \in O^*_\tau)$; so by the induction hypothesis, $x \in O_\tau \leftrightarrow (\forall y \in W_x)(J(y) \in H_\tau)$. Thus we have to show that $W_{J(x)}(H_\tau) \leftrightarrow (\exists y \in W_x)(J(y) \notin H_\tau)$. By the RE Enumeration Theorem, the right side is an RE relation of H_τ, x, j where j is an index of J. If q is an RE-index of this relation, we want to have $J(x) = S_2(q,x,j)$. This can be achieved by the Recursion Theorem.

It follows that for σ recursive, O^*_σ is hyperarithmetical. Now if σ is recursive, so is $\sigma+1$, and $O_\sigma = O^*_{\sigma+1}$ by (1) and (2). Hence O_σ is hyperarithmetical.

Boundedness Theorem. *If A is a Σ^1_1 subset of O, then $A \subset O_\sigma$ for some recursive σ.*

If not, it follows from (1) that every recursive σ is $< |x|$ for some $x \in A$. It follows by (2) that

$$\begin{aligned} e \in O &\leftrightarrow (\exists x \in A)(e \in O^*_{|x|}) \\ &\leftrightarrow (\exists x \in A)(J(e) \in H_{|x|}) \\ &\leftrightarrow \exists x (x \in A \,\&\, H_\Sigma(J(e), x)). \end{aligned}$$

Hence O is Σ^1_1, contradicting the Corollary above.

Now we shall prove that every Δ^1_1 set is hyperarithmetical. Let A be Δ^1_1. Since A is Π^1_1, there is a recursive real F such that $x \in A \leftrightarrow F(x) \in O$. If $B = \{F(x) : x \in A\}$, then $y \in B \leftrightarrow \exists x(x \in A \,\&\, y = F(x))$; so B is Σ^1_1. By the Boundedness Theorem, there is a recursive σ such that $B \subset O_\sigma$. It follows that $x \in A \leftrightarrow F(x) \in O_\sigma$; so A is many-one reducible to O_σ. Since O_σ is hyperarithmetical, A is hyperarithmetical. (This proof, which is different from Kleene's original proof, comes from the thesis of his student Spector [11].)

One might hope to extend this result in the same way that the characterization of Δ^0_1 was extended by Post's Theorem. The desired result would say that a set is Δ^1_{n+1} iff it is hyperarithmetical in a Π^1_n relation. However, this is false, as Addison and Kleene showed in *1957*.

More information about the H_σ is given by the next result.

THEOREM. *Let ρ be a recursive ordinal, and let A be Π^0_1 in H_ρ. Then there is a recursive real F such that $x \in A \leftrightarrow F(x) \in H_\sigma$ for all $\sigma > \rho$.*

Let e be a notation for ρ. If J is as above, then $\rho < \tau \leftrightarrow H_\tau(J(e))$ for all τ by (2). Hence $\rho = \tau \leftrightarrow H_{\tau+1}(J(e)) \,\&\, \neg H_\tau(J(e))$. From this and the definition of H_τ, $\rho = \tau \leftrightarrow \neg W_{J(e)}(H_\tau) \,\&\, \neg H_\tau(J(e))$.

There is an RE Q such that $x \in A \leftrightarrow \neg Q(H_\rho, x)$. Then for $\sigma > \rho$,

$$\begin{aligned} x \in A &\leftrightarrow (\exists \tau < \sigma)(\rho = \tau \,\&\, \neg Q(H_\tau, x)) \\ &\leftrightarrow (\exists \tau < \sigma)\neg(W_{J(e)}(H_\tau) \vee H_\tau(J(e)) \vee Q(H_\tau, x)). \end{aligned}$$

Hence if q is an RE-index of the relation $W_{J(e)}(\alpha) \vee \alpha(J(e)) = 0 \vee Q(\alpha, x)$ of α, x and $F(x) = S(q, x)$, then $x \in A \leftrightarrow F(x) \in H_\sigma$.

In particular, it follows that if $\rho < \sigma$, then H_ρ is many-one reducible to H_σ. On the other hand, H_σ is not recursive in H_ρ; for if it were, then every set Π^0_1 in H_ρ would be recursive in H_ρ, which is false. Using the theorem and Post's Theorem, it is not difficult to show that a set is Π^0_n iff it is many-one reducible to H_n. It follows that H_ω is an example of a hyperarithmetical relation which is not arithmetical.

There has been considerably more study of the hyperarithmetical sets. In particular, Kleene *1959c* showed that hyperarithmetical sets could be defined in other interesting ways.

In his thesis [1], Kleene's student Addison studied the analytic hierarchy relative to the set Φ of all reals, or, as it is usually called, the *projective* hierarchy. There is a natural topology on the set of all $\alpha_1, \dots, \alpha_m, x_1, \dots, x_n$ in which the open sets are the Σ^0_1 in Φ sets. Then Π^1_k *in* Φ and Σ^1_k *in* Φ can be given topological definitions. These concepts turn out to be identical with concepts studied in the 1920's by Lusin, Suslin, Sierpiński, and others. For example, the theorem that Δ^1_1 in Φ is the same as hyperarithmetical in Φ is a famous theorem of Suslin; so the equivalence of Δ^1_1 and hyperarithmetical is usually called the Suslin-Kleene Theorem.

§6. Realizability. As Yiannis Moschovakis observed in a brief talk on Kleene's work at an ASL meeting, Kleene rarely worked on problems formulated by others. In particular, he did little on the application of recursion theory to other fields. By far his most important work on applications was in the field of intuitionism, where the problems to be solved were not clear before he entered the field. We shall give only a brief summary of the large amount of work he did in this field between 1940 and 1969, since there is an excellent survey of this work by Kleene *1973*.

Kleene became interested in intuitionism by reading the works of Brouwer, Heyting, Gödel, Gentzen and others on the subject. His interest was a practical one; he felt that recursive functions must have a role in explaining the nature of a constructive proof, which is what intuitionism is all about. This consideration led him to the idea of realizability, which he developed in various forms over the years. The conclusions did not fulfill Kleene's greatest hope of giving a complete explanation of intuitionism in terms of recursive functions; but they did give many interesting results and much understanding.

Since Brouwer was quite prejudiced against formal systems, the formalization of intuitionism was left to his successors. By 1940, the formalization of the intuitionistic predicate calculus had been agreed upon. The notation is the same as the classical predicate calculus; but intuitionistically, none of the logical symbols $\neg$, $\vee$, $\&$, $\rightarrow$, $\forall$ and $\exists$ can be defined in terms of the others. The intuitionistically valid sentences are a proper subset of the classically valid sentences. An example of a classically valid but not intuitionistically valid sentence is $A \vee \neg A$; this is intuitionistically true only if we have a constructive method of deciding which of A and $\neg A$ is true.

Intuitionistic arithmetic is formed from the intuitionistic predicate calculus by adding the symbols 0 and Sc, the Peano axioms, and symbols for some basic functions together with the defining equations for these functions. (The exact choice of these functions is not too important. We shall suppose that they include the T_n.) In arithmetic, we can eliminate $\neg$ by defining $\neg A$ to mean $A \rightarrow 0 = 1$.

One explanation of intuitionism is this: to say a mathematical statement A is true is to say that one has a construction which demonstrates A. Kleene's idea was to take e *realizes* A to mean that the number e codes a construction which demonstrates A. After some false starts, he arrived in *1945* at the following definition (by induction on the length of the sentence A of arithmetic):

(a) If A is prime (or, as it is usually called nowadays, atomic), e realizes A if A is true;
(b) If A is $B \,\&\, C$, e realizes A if $(e)_0$ realizes B and $(e)_1$ realizes C;
(c) If A is $B \vee C$, e realizes A if either $(e)_0 = 0$ and $(e)_1$ realizes B or $(e)_0 \neq 0$ and $(e)_1$ realizes C;
(d) If A is $B \rightarrow C$, e realizes A if for each a, if a realizes B, then $\{e\}(a)$ is defined and realizes C;
(e) If A is $\forall x B(x)$, e realizes A if for each k, $\{e\}(k)$ is defined and realizes $B(k)$.
(f) If A is $\exists x B(x)$, e realizes A if $(e)_0$ realizes $B((e)_1)$.

We then say that A is *realizable* if some e realizes A. Note that $\neg A$ is realizable iff A is not realizable.

Kleene conjectured that if A was a theorem of intuitionistic arithmetic, then A was realizable. This conjecture was verified by Kleene's first Ph.D. student, David Nelson, in his thesis [6].

One consequence of this result was that certain sentences which could be proved (nonconstructively) in classical arithmetic were not provable in intuitionistic arithmetic. For example, let $A(x)$ be the formula $\exists z T_1(x, x, z)$. Then $\forall x(A(x) \vee \neg A(x))$ is provable in the classical predicate calculus. Suppose that it is provable in intuitionistic number theory and hence realized by some e. Then for each x, $\{e\}(x)$ is defined and realizes $A(x) \vee \neg A(x)$. It follows that $A(x)$ is realizable iff $(\{e\}(x))_0 = 0$. But $A(x)$ is realizable iff $T_1(x, x, z)$ is realizable for some z; and $T_1(x, x, z)$ is realizable iff it is true. Thus $\exists z T_1(x, x, z)$ iff $(\{e\}(x))_0 = 0$, and hence $\exists z T_1(x, x, z)$ is a recursive relation. Since $\exists z T_1(x, x, z)$ iff $\{x\}(x)$ is defined, this contradicts the result at the end of §2. A consequence of this is that the classically false sentence $\neg\forall x(A(x) \vee \neg A(x))$ is realizable.

Kleene saw that the usefulness of Nelson's result could be increased by modifying the notation of realizability by replacing *e realizes A* at some places by *e realizes A and A is a theorem.* (Specifically, replace $(e)_1$ *realizes B* and $(e)_1$ *realizes C* in clause (c), *a realizes B* in clause (d), and $(e)_0$ *realizes* $B((e)_1)$ in clause (f).) Nelson's result still goes through, and it now has the following easy corollaries:

(i) If $A \vee B$ is a theorem, then either A or B is a theorem.
(ii) If $\exists x A(x)$ is a theorem, there is a number k such that $A(k)$ is a theorem.
(iii) If $\forall x \exists y A(x, y)$ is a theorem, then there is a recursive real F such that $A(k, F(k))$ is a theorem for each number k.

The last of these results confirms a conjecture that Kleene made at the beginning of his consideration of intuitionism. Kleene later (in *1962a*) noticed that (i) and (ii) could be proved by a much simpler definition of *realizes* which did not involve recursive functions at all.

Beginning about 1950, Kleene considered the same problems for second order number theory, or, as it is often called, analysis. This work, together with related work of his student Vesley, is contained in *1965*.

To obtain intuitionistic analysis from intuitionistic arithmetic, we add real variables. (It is not necessary to add variables for n-ary functions, since the coding of finite sequences can be carried out in intuitionistic arithmetic.) This, of course, requires extending the intuitionistic predicate calculus to the new variables; but this is no problem. As further axioms, we first need for each numerical term $t(x)$ an axiom saying there is one and only one real α such that $\alpha(x) = t(x)$ for all x. There is a weak form of the axiom of choice expressible in this notation which is intuitionistically valid. We take as an axiom Brouwer's bar theorem, which says that certain types of transfinite induction are allowed.

The last axiom, which Kleene calls Brouwer's principle, is a strong form of the statement: if to each α a number $y(\alpha)$ is assigned, then for each α the value of $y(\alpha)$ depends only on a finite number of values of α. This is intuitionistically valid because $y(\alpha)$ must be obtained constructively from an oracle for α; so we can only consult the oracle finitely often. Unlike the other axioms, it is not classically valid, as we see by taking $y(\alpha) = 0$ if $\forall x(\alpha(x) = 0)$ and $y(\alpha) = 1$ otherwise.

The extension of realizability from arithmetic to analysis presents considerable difficulty. In arithmetic, we consider only sentences; formulas with free variables were treated by replacing each free variable by a name of a number. This does not work in analysis, since we do not have names for all reals. Thus we must now define realizability for a pair (A, Φ), where Φ is a function which assigns a meaning to each variable free in A. Now the meaning of a real variable may well be a nonrecursive real; so we can expect the construction for (A, Φ) to have some nonconstructive elements. This is best handled by allowing (A, Φ) to be realized by a real instead of a number.

We are thus led to define ε *Φ-realizes* A for ε a real and A a formula of analysis. The definition is virtually a copy of the definition for arithmetic. We take $(\varepsilon)_i$ to be the real defined by $(\varepsilon)_i(x) = (\varepsilon(x))_i$. The explanation of $\{\varepsilon\}(\alpha)$ (which occurs in the definition of ε *realizes* $A \to B$ and ε *realizes* $\forall\alpha A(\alpha)$) is that $\{\varepsilon\}(\alpha) = F(\alpha)$ where ε is an associate of F.[15]

With this definition, it is fairly straightforward to show that every theorem of intuitionistic analysis is realizable. One consequence is that intuitionistic analysis is seen classically to be consistent; this is not trivial, since intuitionistic analysis is not a subsystem of classical analysis. As in arithmetic, realizability could be used to show that certain classically provable results were not intuitionistically provable. Other interesting independence results could be obtained by restricting in some way the class of reals allowed to realize a formula. For example, one can show that the bar theorem is not provable from the remaining axioms. (This has some interest, since Brouwer gave various proofs of the bar theorem from results he thought to be simpler.)

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[15] If A is a sentence of arithmetic, realizability in the new sense is not always the same as realizability in the old sense. For example, the classically true but unrealizable sentence discussed above is now realized by a suitable nonrecursive ε.

Extending (i)–(iii) to analysis proved to be quite difficult. Kleene's student Joan Rand Moschovakis [5] was able to extend (i) and (ii), but only at the cost of replacing Brouwer's principle by a weaker axiom. Kleene finally found a proof of (i)–(iii) in *1969*. It required carrying out the definition of realizability within the language of analysis. This completed Kleene's work on realizability; but it has since been used by several investigators, usually to prove independence results.

§7. Objects of higher types. In §5, we added some of the functions of our theory as new arguments, thus obtaining new functions. This process can be repeated once more, or even infinitely many times. Kleene began thinking about the resulting theory about 1952. The principal results appeared in a two part paper (*1959a, 1963*) which contains some of Kleene's most profound work.

We first define the *objects of type t* by induction on t. An object of type 0 is a number; an object of type $t + 1$ is a mapping of the set of objects of type t into the set of numbers. Thus an object of type 1 is just a real. We use α^t, β^t, and γ^t as variables for objects of type t; we omit the superscript whenever possible.

The functions we now consider will have n arguments, each of a specified type, and will have numbers as values. In describing computability of such functions, a new problem arises. Again we are given an argument α^{t+1} by being given an oracle which, presented with a β^t, will give us the value $\alpha(\beta)$. The trouble is that the object β may be highly noncomputable; so we will have introduced noncomputable information into our computation.

One solution is to require the oracle to give the value $\alpha(\beta)$ after having been presented with only finitely many values of β. This means that the α must be of a special kind; its value at any β is determined by only finitely many values of β. The recursion theory of these objects was developed by Kreisel and Davis (independently), and Kleene eventually took up the question in *1959b*. As Kleene showed, this theory reduces to the theory in §5. (For references and more information about this recursion theory, see [7].)

Kleene was, however, interested in a theory which would really use the objects of higher type. His idea was that an oracle for α would give the value of $\alpha(\beta)$ if it were presented with an algorithms for computing β. (The algorithm could use any oracles which we already had on hand, including an oracle for α.) This idea could be mixed with any of the definitions of recursive we have mentioned (although in each case there are some complications). Kleene obtained his first results by using λ-definability; but then he saw how to generalize recursively closed sets. He eventually published papers (*1962b, 1962c, 1962d, 1962e*) showing that all these approaches were equivalent.

We are going to modify I–V to fit the present situation. We add to I three functions Pr, DC, and Ap defined by

$$Pr(0) = 0, Pr(x+1) = x,$$

$$DC(0, y, z) = y, DC(x+1, y, z) = z,$$

$$Ap(\alpha^1, x) = \alpha(x).$$

II and III remain the same, except that the variables may be of any types. For reasons we shall explain shortly, we omit the old IV and V. Our new IV corresponds to asking the oracle for an object of type ≥ 2 for a value of the object. (Oracles for reals are taken care of by the function Ap.)

IV. If G is recursive and F is defined by $F(\alpha^{t+2}, \vec{\beta}) = \alpha(\lambda\delta^t G(\alpha, \vec{\beta}, \delta))$, then F is recursive.

We assign indices to all these functions much as before. In particular, if F and G are as in IV and g is an index of G, then $\langle 4, g\rangle$ is an index of F. We again want to have the Enumeration Theorem: $\{f\}(\vec{\alpha})$ is a recursive function of $f, \vec{\alpha}$. Our new V says that this function is recursive for each choice of the types of the α_i.

V. If F is defined by $F(f, \vec{\alpha}) = \{f\}(\vec{\alpha})$, then F is recursive.

To the F of V we assign the index $\langle 5, \langle t_1, \dots, t_n\rangle\rangle$, where t_i is the type of α_i. Note that in V, F is defined in terms of all recursive functions together with their indices. Hence it is impossible to define the recursive functions by I–V as before. Instead, we will define the true formulas to be the theorems of a formal system, and then use this to define the recursive functions and their indices.

A formula is now an equation $\{f\}(\vec{\alpha}) = y$, where f is an index of a function and the types of α_i are such that $\vec{\alpha}$ is an appropriate argument for $\{f\}$. The formal system is very much like the formal system in §2; so we shall give only the rule of inference corresponding to IV. It says that $\{\langle 4, g\rangle\}(\alpha^{t+2}, \vec{\beta}) = \alpha(\gamma)$ is inferable from the set of formulas $\{g\}(\alpha, \vec{\beta}, \delta) = \gamma(\delta)$ for δ an object of type t. Since this rule has infinitely many hypotheses, we can no longer define proofs as before. Instead, we define a class T_σ of formulas for each ordinal σ by induction on σ as follows: a formula is in T_σ if it is inferable from formulas in $\bigcup_{\tau<\sigma} T_\tau$. Note that T_σ is a non-decreasing function of σ. A formula is a *theorem* if it belongs to T_σ for some σ.

A class T of formulas is *regular* if for each f and $\vec{\alpha}$, there is at most one y such that $\{f\}(\vec{\alpha}) = y$ is in T. We show that the class of theorems is regular. It will clearly suffice to prove that for each σ, T_σ is regular; and we do this by induction on σ. We suppose $f = \langle 4, g\rangle$; the proof is much the same for other values of f. Suppose that $\{f\}(\alpha, \vec{\beta}) = y$ and $\{f\}(\alpha, \vec{\beta}) = y'$ are in T_σ. Then there are γ and γ' such that $y = \alpha(\gamma)$, $y' = \alpha(\gamma')$, and for each δ, $\{g\}(\alpha, \vec{\beta}, \delta) = \gamma(\delta)$ and $\{g\}(\alpha, \vec{\beta}, \delta) = \gamma'(\delta)$ are in $\bigcup_{\tau<\sigma} T_\tau$. By the induction hypothesis, $\gamma(\delta) = \gamma'(\delta)$ for all δ; so $\gamma = \gamma'$; so $y = y'$.

We let $\{f\}(\vec{\alpha})$ be the unique number y such that $\{f\}(\vec{\alpha}) = y$ is a theorem; if no such y exists, $\{f\}(\vec{\alpha})$ is undefined. Then $\{f\}(\vec{\alpha}) = y$ is true iff it is a theorem. We say that f is an index of F if $F(\vec{\alpha}) = \{f\}(\vec{\alpha})$ for all $\vec{\alpha}$, and define a function to be recursive if it has an index. It is then clear that each of I–V is correct, and that the functions defined by I–V have the indices we have indicated.

If $\{f\}(\vec{\alpha}) = y$ is a true formula, the *ordinal* of $\{f\}(\vec{\alpha}) = y$ is the least σ such that $\{f\}(\vec{\alpha}) = y$ is in T_σ. Since y is uniquely determined by f and $\vec{\alpha}$, we also call σ the ordinal of $\{f\}(\vec{\alpha})$; so $\{f\}(\vec{\alpha})$ has an ordinal iff it is defined. If $\{f\}(\vec{\alpha}) = y$ has ordinal σ, then it is inferable from a subclass A of $\bigcup_{\tau<\sigma} T_\tau$; and an examination of the rules of inference shows that A is uniquely determined. The members of A are called the *predecessors* of $\{f\}(\vec{\alpha}) = y$, and the left sides of the members of A are called the *predecessors* of $\{f\}(\vec{\alpha})$. These predecessors then have ordinals $< \sigma$; this is useful in giving proofs by induction on σ.

The Enumeration Theorem is a consequence of our definition. Since we can define the S function much as before, we can prove the Recursion Theorem. This can be used to show that the old IV and V (extended to allow higher type arguments) hold in the present theory. The proof of IV is much like the application of Recursion Theorem to definitions by induction given in §2. (It is here that we need the functions *Pr* and *DC*.) To treat V, we use the Recursion Theorem to define a function H such that $H(y, \vec{\alpha}) = 0$ if $G(y, \vec{\alpha}) = 0$ and $H(y, \vec{\alpha}) = Sc(H(Sc(y), \vec{\alpha}))$ if $G(y, \vec{\alpha}) > 0$. We leave it to the reader to show that $\mu y(G(y, \vec{\alpha}) = 0) = H(0, \vec{\alpha})$. We can then develop the basic theory as in §2; but there is no Normal Form Theorem. Observe also that $\alpha^{t+1}(\beta^t)$ is a recursive function of α and β; this is easily proven by induction on t, using IV.

We would, of course, like to show that for arguments of type 0 and 1, the recursive functions given by the definition of this section are the same as those defined in §2 and §5. We shall consider only functions with arguments of type 0; when these are taken care of, the rest is simple. The above shows that the recursive functions of arguments of type 0 form a recursively closed class, and therefore include the recursive functions of §2. For the converse, one notes that when dealing only with functions of type 0 arguments, one can prove a Normal Form Theorem almost exactly as before. Of course, the T_n and U are different from the T_n and U in §2; but they are defined similarly and hence are recursive in the sense of §2. But then the Normal Form

Theorem shows that every function of type 0 variables which is recursive in the present sense is recursive in the sense of §2.

The Substitution Theorem as stated in §5 does not hold here, as the following example shows. Let A be a Π^0_1 set which is not Σ^0_1; say $x \in A \leftrightarrow \forall y R(x,y)$ with R recursive. Let $K(x,y) = \mu z R(x,y)$, so that $K(x,y) = 0$ if $R(x,y)$ and $K(x,y)$ is undefined otherwise. Then K is recursive and $\lambda y K(x,y)$ is a real iff $x \in A$. Define recursive G and H by $G(\alpha^{(2)}, x) = \alpha(\lambda y K(x,y))$, $H(x, \beta^{(1)}) = 0$, and let $F(x) = G(\lambda\beta H(x,\beta), x)$. Then $F(x)$ is defined iff $x \in A$. Since the domain of F is not RE, F is not recursive. There are various assumptions on G and H which make the Substitution Theorem true; the simplest is that G is total.

Substitution Theorem. *If G and H are recursive and total, and F is defined by $F(\vec{\alpha}) = G(\lambda\delta H(\vec{\alpha},\delta),\vec{\alpha})$, then F is recursive.*

If $i = \langle n, i_1, \dots, i_p\rangle$ where $1 \le i_j \le n$ for all j, we write $\pi_i(\alpha_1, \dots, \alpha_n)$ for $(\alpha_{i_1}, \dots, \alpha_{i_p})$. The Substitution Theorem is a consequence of the following.

Lemma. *For each $t \ge 1$, there is a recursive total function Z_t such that $\{Z_t(f,h,i)\}(\vec{\alpha}) = \{f\}(\pi_i(\lambda\delta^{t-1}\{h\}(\vec{\alpha},\delta),\vec{\alpha}))$ provided the right side is defined.*

The proof is by induction on t. We let $i = \langle n, i_1, \dots, i_p\rangle$ and write ε for $\lambda\delta\{h\}(\vec{\alpha},\delta)$. We assume that Z_t is defined and prove $\{Z_t(f,h,i)\}(\vec{\alpha}) = \{f\}(\pi_i(\varepsilon,\vec{\alpha}))$ by induction on the ordinal of $\{f\}(\pi_i(\varepsilon,\vec{\alpha}))$, dividing into cases according to the nature of the index f. In each case, we find that Z_t must satisfy an equality whose left side is $Z_t(f,h,i)$, and whose right side may contain Z_t or an index of Z_t. We can then find a recursive Z_t which satisfies the equalities by the Recursion Theorem; and it will be total.

We shall examine one important case. Suppose $f = \langle 4, g\rangle$ and $i_1 = 1$. Then

$$\begin{aligned}\{f\}(\pi_i(\varepsilon,\vec{\alpha})) &= \varepsilon(\lambda\gamma\{g\}(\pi_i(\varepsilon,\vec{\alpha}),\gamma)) \\ &= \{h\}(\vec{\alpha},\lambda\gamma\{g\}(\pi_i(\varepsilon,\vec{\alpha}),\gamma)) \\ &= \{h\}(\vec{\alpha},\lambda\gamma\{g\}(\pi_j(\varepsilon,\vec{\alpha},\gamma)))\end{aligned}$$

where j is a recursive function of i. Using II, we can find a recursive function h' of h such that $\{h'\}(\vec{\alpha},\gamma,\delta) = \{h\}(\vec{\alpha},\delta)$. Since $\{g\}(\pi_j(\varepsilon,\vec{\alpha},\gamma))$ is a predecessor of $\{f\}(\pi_i(\varepsilon,\vec{\alpha}))$, the induction hypothesis shows that $\{g\}(\pi_j(\varepsilon,\vec{\alpha},\gamma)) = \{Z_t(g,h',j)\}(\vec{\alpha},\gamma)$, and hence

$$\{f\}(\pi_i(\varepsilon,\vec{\alpha})) = \{h\}(\vec{\alpha},\lambda\gamma\{Z_t(g,h',j)\}(\vec{\alpha},\gamma)).$$

Since ε is of type t, γ is of type $t-2$. Hence by the induction hypothesis on t,

$$\{h\}(\vec{\alpha},\lambda\gamma\{Z_t(g,h',j)\}(\vec{\alpha},\gamma)) = \{Z_{t-1}(h, Z_t(g,h',j),k)\}(\vec{\alpha})$$

where k is a recursive function of i. Thus we want $Z_t(f, h, i) = Z_{t-1}(h, Z_t(g, h', j), k)$.

By induction on t, we define a mapping *up* of the set of type t objects into the set of type $t+1$ objects and a mapping *do* of the set of type $t+1$ objects into the set of type t objects such that $do(up(\alpha)) = \alpha$ for all α (and hence *up* is one-one and *do* is onto). For $t = 0$, we let $up(x) = \lambda yx$ and $do(\alpha) = \alpha(0)$. For $t > 0$, let $up(\alpha^{(t)}) = \lambda\beta^{(t)}\alpha(do(\beta))$ and $do(\alpha^{(t+1)}) = \lambda\beta^{(t-1)}\alpha(up(\beta))$. By the Substitution Theorem, we see that *up* and *do* can be used in definitions of total recursive functions.

Now we shall see how to assign codes to finite sequences of objects. For $t > 0$, we define a type t object $\langle\alpha_1, \dots, \alpha_n\rangle^t$ whenever $\alpha_1, \dots, \alpha_n$ all have type $\leq t$. Let β_i be the type t object obtained by applying *up* repeatedly to α_i. Then $\langle\alpha_1, \dots, \alpha_n\rangle^t = \lambda\gamma^{t-1}\langle\beta_1(\gamma), \dots, \beta_n(\gamma)\rangle$. For $j \leq t$, let $(\alpha^t)_i^j$ be the object of type j obtained from $\lambda\beta(\alpha(\beta))_i$ by repeatedly applying *do*. Then for α_i of type t_i and $t \geq t_i$ for all $i < n$, $((\alpha_0, \dots, \alpha_{n-1})^t)_i^{t_i} = \alpha_i$. All of these functions can be used in definitions of total recursive functions.

We can use this coding of finite sequences to define the contraction of a total function F as before. If the arguments of F have types $t_1, \dots, t_n$, then $\langle F\rangle$ is an object of type $1 + \max t_i$; and F is recursive iff $\langle F\rangle$ is recursive. We can then define relative recursion by using the alternate definition of §5: F is recursive in Φ if there are $G_1, \dots, G_n \in \Phi$ and a recursive F' such that

$$F(\vec{\alpha}) = F'(\langle G_1\rangle, \dots, \langle G_n\rangle, \vec{\alpha})$$

for all $\vec{\alpha}$.

Now let us turn to extensions of the analytical hierarchy. For $r \geq 1$, a relation is of *order* $r+1$ if it has a definition consisting of quantifiers of type $\leq r$ followed by a recursive relation. Much as before, we can suppose that in the prefix no two successive quantifiers are of the same kind, and that all of the quantifiers are of type r except the last, which is of type $r-1$. If the prefix has k quantifiers of type r and the first is existential { universal }, the relation is $\Sigma_k^r\{\Pi_k^r\}$. A relation is Δ_k^r if it is both Σ_k^r and Π_k^r.

As before, every relation of order r is Σ_k^r and Π_k^r for all k, and every Σ_k^r or Π_k^r relation is Σ_n^r and Π_n^r for all $n > k$. Again we have closure properties; these show that the position of a relation in the order $r+1$ hierarchy is not changed by prefixing quantifiers of type $< r$.

We again define a relation to be RE if it is the domain of a recursive function. Again Σ_1^0 relations are RE; but the converse is no longer true. The best we can prove is the following.

THEOREM. *Let $r \geq 1$, and let R be an RE relation having arguments of type $\leq r+1$. Then R is Π_1^1 if $r = 1$ and Δ_1^r if $r > 1$.*

Let us see how this theorem gives us the Enumeration Theorem for Σ^r_k and Π^r_k provided that all the arguments of the functions have type $\leq r+1$. (The Enumeration Theorem is false without this provision.) It is enough to treat the Π^r_1 case. Let all the variables in $\vec{\alpha}$ have type $\leq r+1$ and define

$$P(e,\vec{\alpha}) \leftrightarrow \forall\beta^r\exists\gamma^{r-1}W_e(\vec{\alpha},\beta,\gamma).$$

Now $W_e(\vec{\alpha},\beta,\gamma)$, as a relation of $e,\vec{\alpha},\beta,\gamma$, is the domain of the enumerating function $\{e\}(\vec{\alpha},\beta,\gamma)$; so it is RE and hence Π^r_1. By the closure conditions, P is Π^r_1; so each $P_{(e)}$ is Π^r_1. If Q is a Π^r_1 relation of $\vec{\alpha}$, then $Q(\vec{\alpha}) \leftrightarrow \forall\beta\exists\gamma R(\vec{\alpha},\beta,\gamma)$ with R recursive. Then R is Σ^0_1 and hence RE. If e is an RE index of R, then $Q = P_{(e)}$.

Now we outline the proof of the theorem. (We have incorporated some simplifications of Kleene's proof due to Platek [8].) Fix r, and let $\vec{\alpha}$ be a sequence of objects having types $\leq r+1$. An *$\vec{\alpha}$-formula* is a formula $\{f\}(\vec{\beta}) = y$ where every object in $\vec{\beta}$ is either in $\vec{\alpha}$ or of type $< r$. Of course, $\{f\}(\vec{\alpha}) = y$ is an $\vec{\alpha}$-formula; and it is easy to see that every predecessor of a true $\vec{\alpha}$-formula is an $\vec{\alpha}$-formula. We can code the $\vec{\alpha}$-formulas by type $r-1$ objects as follows. Let $\vec{\alpha}$ be $\alpha_0,\dots,\alpha_{n-1}$. If β_i is α_j, γ_i is j; if β_i is a number z, $\gamma_i = n+z$; otherwise, $\gamma_i = \beta_i$. Then the code of $\{f\}(\vec{\alpha}) = y$ is $\langle f,\langle\vec{\gamma}\rangle,y\rangle$. Let $Tr(\vec{\alpha},\beta)$ mean that β is the code of a true $\vec{\alpha}$-formula. We shall prove that Tr is Π^1_1 if $r=1$ and Δ^r_1 if $r>1$. Then if f is an RE-index of R,

$$R(\vec{\alpha}) \leftrightarrow \exists y Tr(\vec{\alpha},\langle f,\langle 0,\dots,n-1\rangle,y\rangle);$$

so R is Π^1_1 if $r=1$ and Δ^r_1 if $r>1$.

Let $Inf(\vec{\alpha},\beta^r,\gamma^{r-1})$ mean that β is the set of codes of a regular class A of $\vec{\alpha}$-formulas and that γ is the code of an $\vec{\alpha}$-formula inferable from formulas in A. Then Inf is of order r. To prove this, we write out a definition of Inf and check that all quantifiers are of type $\leq r-1$. The only problem arises when γ is the code of a formula $\{\langle 4,g\rangle\}(\alpha_i,\vec{\delta}) = x$ where α_i is of type $r+1$. We must say that $x = \alpha_i(\iota)$ where for each v, $\{g\}(\alpha_i,\vec{\delta},v) = \iota(v)$ is in A. Then ι is of type r and v of type $r-1$; so we may quantify over v but not over ι. This is all right, because regularity tells us that $\iota = \lambda\vec{v}\mu y(\beta(\dots)=0)$ where $\dots$ is the code of $\{g\}(\alpha_i,\vec{\delta},v) = y$.

A class A of $\vec{\alpha}$-formulas is *$\vec{\alpha}$-closed* if it is regular and every $\vec{\alpha}$-formula inferable from formulas in A is in A. The class of true $\vec{\alpha}$-formulas is clearly $\vec{\alpha}$-closed. Moreover, it is the smallest $\vec{\alpha}$-closed class. To see this, we must show that if A is $\vec{\alpha}$-closed, then every true $\vec{\alpha}$-formula a is in A; and we do this by induction on the ordinal of a. The predecessors of a are true $\vec{\alpha}$-formulas and hence in A; so $a \in A$ because A is $\vec{\alpha}$-closed.

Let $Cl(\vec{\alpha},\gamma^r)$ mean that γ is the set of codes of an $\vec{\alpha}$-closed class. Since Inf is of order r, Cl is of order r. By the above paragraph,

$$Tr(\vec{\alpha},\beta) \leftrightarrow \forall\gamma^r(Cl(\vec{\alpha},\gamma) \rightarrow \gamma(\beta)=0).$$

Hence *Tr* is Π^r_1.

It remains to show that *Tr* is Σ^r_1 if $r > 1$. We recall that if R is a binary relation on a class A, then R *well-founds* A if there is no sequence $\{a_n\}$ in A which is *R-descending*, i.e., such that $R(a_{n+1}, a_n)$ for all n. Suppose that a is a true $\vec{\alpha}$-formula; A is the regular class of true $\vec{\alpha}$-formulas; and $R(b, c)$ iff $b, c \in A$ and the ordinal of b is less than the ordinal of c. Then: (i) $a \in A$; (ii) R well-founds A; (iii) for all $b \in A$, b is inferable from formulas in $\{c \ : \ R(c, b)\}$. Conversely, if a is an $\vec{\alpha}$-formula, A a regular class of $\vec{\alpha}$-formulas, R a binary relation on A and (i)–(iii) hold, then a is true; for otherwise, we can use (i) and (iii) to construct an R-descending sequence of false formulas in A, contradicting (ii).

It follows that if A varies through sets of type $r-1$ objects and R through binary relations on type $r-1$ objects,

$$Tr(\vec{\alpha}, \beta) \leftrightarrow$$
$$\exists A \exists R[\beta \in A \,\&\, R \text{ well-founds } A \,\&\, (\forall \gamma \in A)(\mathit{Inf}(\vec{\alpha}, \{\delta \ : \ R(\delta, \gamma)\}, \gamma)].$$

Thus it suffices to show that 'R well-founds A' is a relation of A and $\langle R \rangle$ of order r when $r > 1$. For γ of type $r-1$, define $[\gamma]_i = \lambda \delta^{r-2} \gamma(\langle i, \delta \rangle)$. Then for each infinite sequence $\{\gamma_i\}$ of type $r-1$ objects, there is a γ such that $\gamma_i = [\gamma]_i$ for all i; it is defined by $\gamma(\delta) = \gamma_i((\delta)_1^{i-1})$ where $i = (\delta)_0^0$. Then

$$R \text{ well-founds } A \leftrightarrow \forall \gamma^{r-1} \neg \forall n([\gamma]_n \in A \,\&\, R([\gamma]_{Sc(n)}, [\gamma]_n)).$$

A particularly interesting type 2 object is the object $\mathbf{E}$ defined by $\mathbf{E}(\alpha) = 0$ if $\exists x(\alpha(x) = 0)$, $\mathbf{E}(\alpha) = 1$ otherwise. We can reasonably identify this object with the existential number quantifier. It is easy to see that every arithmetical relation is recursive in $\mathbf{E}$. More generally:

THEOREM. *A set is recursive in* **E** *iff it is hyperarithmetical.*

First, every set which is RE in $\mathbf{E}$ is Π^1_1. The proof of this is practically the same as the above proof with $r = 1$, $\vec{\alpha} = \mathbf{E}$. The only place where the definition of $\mathbf{E}$ is pertinent is in the proof that *Inf* is of order r,i.e., arithmetical. This is all right, since, $\mathbf{E}(\alpha) = y$ is clearly arithmetical.

If A is recursive in $\mathbf{E}$, then A and $\neg A$ are RE in $\mathbf{E}$ and hence Π^1_1; so A is Δ^1_1 and hence hyperarithmetical. To prove the converse, it sufficed to show that each H_σ is recursive in $\mathbf{E}$. For this, we define a recursive G so that if $e \in O$, $H_{|e|} = \lambda x G(\mathbf{E}, e, x)$. Assume that G is defined, and let us prove this by induction on $|e|$. By (3) of §5 and the induction hypothesis,

$$H_{|e|}(x) \leftrightarrow \exists f (f <_O e \,\&\, \neg W_x(\lambda y \{g\}(\mathbf{E}, f, y))$$

where g is an index of G. The right side is arithmetical and hence is a recursive relation R of $g, \mathbf{E}, e, x$. Thus we need to have $G(\alpha^2, e, x) = R(g, \alpha, e, x)$, which we can accomplish by the Recursion Theorem.

A great deal of work in recursion in higher types has been done since the basic papers; but we shall only mention one point which continued to interest Kleene. This was the failure of the Substitution Theorem (in the strong form of §5). It is clear that the difficulty is that the right side of the equation is necessarily undefined if $\lambda\delta H(\vec{\alpha},\delta)$ is not total. Thus Kleene suggested in *1963* that one should allow objects to be partial. An object of type $t+1$ is to be a partial mapping of the set of objects of type t into the set of numbers. However, if $t \neq 0$, we do not take all such mappings. We require that if $\alpha^{t+1}(\beta) = x$, and β' is an extension of β, then $\alpha(\beta') = x$. These objects are called *monotone* objects.

The first person to develop recursion theory using monotone objects was Platek in his thesis [8]. He also used another idea first suggested in *1963*; instead of building the Enumeration Theorem into the definition, he built in the First Recursion Theorem, which is a form of the Recursion Theorem which does not involve indices.[16] This means that the recursive functions could be introduced directly by the schemes as in §2, instead of the indirect method used here. Platek actually used a great many more types (e.g., types of n-ary functions) so that there was no longer any difference between objects and functions. The result was an extremely elegant definition of the class of recursive functions. More or less the same theory, formulated more along the lines of Kleene, was discovered by Kleene's student Kierstead [4].

Kleene felt that the proper way to select the objects was by consideration of how an oracle for the object should behave. To him, this justified monotoneness; for if the oracle for α decides that $\alpha(\beta) = x$ from being given information about β, and later new information about β is discovered, then the oracle should not change her decision. (Oracles do not make mistakes!) In a series of papers[17] (*1978, 1980, 1982, 1984*), Kleene considered more closely how oracles behave and arrived at the more restrictive class of *unimonotone* objects. Perhaps because Kleene never got around to developing the recursion theory of these objects, it is not clear that the much more complicated unimonotone objects have any advantage over the monotone objects.

§8. Miscellaneous. There are two other articles of Kleene which deserve at least a brief discussion here. In *1952a,* he considers a problem from logic: when is a first order theory finitely axiomatizable? (We assume that all first order theories have only finitely many nonlogical constants.) This problem has inspired a lot of work by model theorists, without receiving a fully satisfactory answer.

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[16] Kleene himself seemed quite pleased with the idea of using the First Recursion Theorem to define the class of recursive functions. In *1981b,* he defines the recursive functions of §5 in this manner.

[17] These articles show Kleene's sense of humor in a way that earlier articles never did. For example, his oracles are feminine. He said this was because it was the most recent practice about oracles; but I suspect it was his bow to the feminist movement. I might also mention the obscure dedication of *1978*, which is only explained in the final footnote.

Kleene considered a modified problem: when is a theory finitely axiomatizable using additional constants? In other words, when does a theory T have a finitely axiomatizable extension T' such that a sentence in the language of T is provable in T' iff it is provable in T?

We assume that the formulas of each theory have been assigned codes in a reasonable way. We may then apply concepts of recursion theory by replacing expressions by their codes. Now if T and T' are as above, then the set of theorems of T' is RE; the proof is just like the proof in §3 that the set of codes of true formulas is RE. It follows that the set of theorems of T is RE. Kleene proved that this necessary condition for finite axiomatizability using additional constants is also sufficient.

Suppose that the set of theorems of T is an RE set A. We can construct a finitely axiomatized subsystem S of arithmetic with a formula $\phi(x)$ such that for each number k, $\phi(k)$ is provable in S iff $k \in A$. (This construction essentially goes back to Church in his proof of the nonexistence of algorithm result mentioned in §1.) We get T' from S by introducing the symbols of T and the unary predicate symbol N; the axiom $\phi(x) \to N(x)$; and a finite number of axioms which insure that if k is the code of a sentence ψ, then $\psi \leftrightarrow N(k)$ is a theorem. It is then clear that every theorem of T is a theorem of T'. Now suppose that χ is a sentence in T which is a theorem of T'. If χ is not a theorem of T, we can find a model of T having the set of numbers as its universe in which χ is false. It will be enough to extend this to a model of T'. The symbols of S have obvious meanings which makes the axioms of S true. The meaning of N is chosen to make the axioms used to prove $\psi \leftrightarrow N(k)$ true. The axiom $\phi(k) \to N(k)$ is true because if $\phi(k)$ is true, then k is the code of a theorem ψ of T, and hence ψ and $\psi \leftrightarrow N(k)$ are true in the model.

At the time, Kleene was much interested in intuitionism; so he wanted to show that his result held also in intuitionistic systems. Here we have no completeness theorem; so the last part of the proof must show directly how to convert a proof of χ in T' into a proof in T. This can be done by using Gentzen proof theory; but it requires supplementing Gentzen's Haupsatz by a result saying that we can always use Gentzen's rules in a certain order. The proof of this result (which is of general interest in proof theory) is the other paper in *1952a*.

In the summer of 1951, Kleene went to work at RAND. At someone's suggestion, he read an article by McCulloch and Pitts, who were trying to build a model of neural activity based on a simple kind of machine. Kleene abstracted from their ideas the notion of a finite automaton, and proceeded to describe exactly what sets were defined by such automata.

Let Ω be a finite set of symbols. An Ω-*word* is a finite sequence of symbols in Ω. A *finite automaton* over Ω consists of the following things: (a) a finite set S of *states*; (b) a σ_0 in S, called the *initial* state; (c) a mapping T of $S \times \Omega$ into S, called the *transition function*; (d) a subset S_0 of S, called the set of *designated states*. We start the automaton by putting the machine in its initial state and putting a tape containing an Ω-word into the machine with its first symbol scanned. If at any time, the machine is in state σ and is scanning the symbol a, then it passes to the state $T(\sigma, a)$ and scans the next symbol if there is one; if the scanned symbol is the last symbol of the Ω-word, the machine stops. The machine *accepts* the Ω-word if it is in a designated state when it stops.

If A and B are Ω-words, $A \cdot B = \{ab \ : \ a \in A \,\&\, b \in B\}$. We define A^n inductively by $A^0 = \{\emptyset\}$ (where $\emptyset$ is the empty Ω-word), $A^{n+1} = A^n \cdot A$. Let $A^* = \bigcup_n A^n$. Let $\mathcal{R}$ be the smallest class of sets of Ω-words such that: (i) the empty set is in $\mathcal{R}$; (ii) $\{\emptyset\}$ is in $\mathcal{R}$; (iii) if a is a symbol, then $\{a\}$ is in $\mathcal{R}$; (iv) if A and B are in $\mathcal{R}$, then $A \cup B$ and $A \cdot B$ are in $\mathcal{R}$; (v) if A is in $\mathcal{R}$, then A^* is in $\mathcal{R}$. Kleene's main result was that a set of Ω-words is the set accepted by a finite automaton iff it is in $\mathcal{R}$.

After writing up this proof in *1956a,* Kleene gave no more attention to finite automaton. However, his theorem has proved fundamental in all further research in the subject.

Besides research, Kleene did a great deal of expository writing, some of it aimed at nonlogicians or even nonmathematicians. His writing is not notable for elegance or conciseness; but it is always clear and precise, and tells the readers what he really needs to know. His most important expository work is *1952b,* which, as we have indicated, is also an important research work. It arrived too late for my education; I learned most of my basic logic from the original paper-covered edition of Church's ***Introduction to Mathematical Logic***, which I still consider one of the gems of the literature of logic. However, the logicians who followed me usually learned their logic from *1952b*. Gerald Sacks and Michael Morley (among others) have said they read the book word by word during their graduate careers. In looking back at *1952b* in preparation for writing this article, I was surprised how much of it seems pertinent today.

Kleene was not what one would call a brilliant teacher; but he worked hard at teaching and always did a good job. His list of Ph.D. students cannot match those of Church, Tarski, or Sacks; but it is quite respectable nevertheless. Although he would never consider writing a thesis for a student, he was willing to spend large amounts of time in encouraging and advising his students. It seems fair to say that each of them wrote the best thesis of which he was capable.

Kleene was a great supporter of the ASL, and served a term as its president. He almost single-handedly ran the research section of the JSL for many years (while Church ran the reviews).

Of course, mathematics was the central interest of Kleene's professional life; but he had many other interests. I recall him taking part in noncurricular activities at meetings, from the swimming at the July 4th picnic at the 1957 Cornell meeting to rock climbing at the Kleene Symposium in Madison in 1978.

Kleene was, of course, respected by several generations of logicians; but most of them also looked on him as a friend. He always treated the logicians he met well, irrespective of their abilities or reputations. I first met him in 1956[18], when I was a NSF Postgraduate Fellow at the Institute with very little published; he treated me as kindly then as in our meetings in later years, when I was better known in logical circles.

Stephen Cole Kleene died in Madison, Wisconsin on January 25, 1994.

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[18] I recall being paired with him for the logic treasure hunt at John Addison's Thanksgiving party that year. John later confessed that he paired us because he wanted to know if I could keep up with the strides of Kleene, who was well over six feet and reasonably athletic. I must have managed, since we found the prize, which was a bottle of white wine. John then announced that the entire wine budget has been spent on that bottle, and suggested we contribute it to the Thanksgiving dinner. Those were lean years economically for mathematicians.

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PLATONISM AND MATHEMATICAL INTUITION IN KURT GÖDEL'S THOUGHT

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The best known and most widely discussed aspect of Kurt Gödel's philosophy of mathematics is undoubtedly his robust realism or platonism about mathematical objects and mathematical knowledge. This has scandalized many philosophers but probably has done so less in recent years than earlier. Bertrand Russell's report in his autobiography of one or more encounters with Gödel is well known:

> Gödel turned out to be an unadulterated Platonist, and apparently believed that an eternal "not" was laid up in heaven, where virtuous logicians might hope to meet it hereafter.[1]

On this Gödel commented:

> Concerning my "unadulterated" Platonism, it is no more unadulterated than Russell's own in 1921 when in the *Introduction to Mathematical Philosophy* ... he said, "Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features." At that time evidently Russell had met the "not"

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[1] Russell [19, page 356].

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> even in this world, but later on under the influence of Wittgenstein he chose to overlook it.[2]

One of the tasks I shall undertake here is to say something about what Gödel's platonism is and why he held it.

A feature of Gödel's view is the manner in which he connects it with a strong conception of mathematical intuition, strong in the sense that it appears to be a basic epistemological factor in knowledge of highly abstract mathematics, in particular higher set theory. Other defenders of intuition in the foundations of mathematics, such as Brouwer and the traditional intuitionists, have a much more modest conception of what mathematical intuition will accomplish. In this they follow a common paradigm of a philosophical conception of mathematical intuition derived from Kant, for whom mathematical intuition concerns space and time as forms of our sensibility. Gödel's remarks about intuition have also scandalized philosophers, even many who would count themselves platonists. I shall again try to give some explanation of what Gödel's conception of intuition is. It is not quite so intrinsically connected with his platonism as one might think and as some commentators have thought. I hope to convince you that even though it is far from satisfactory as it stands, there are at least genuine problems to which it responds, which no epistemology for a mathematics that includes higher set theory can altogether avoid. I will suggest, however, that Gödel aims at what other philosophers (in the tradition of Kant) would call a theory of reason rather than a theory of intuition. Gödel is, however, evidently influenced by a pre-Kantian tradition that does not see these two enterprises as sharply distinct and that admits "intuitive knowledge" in cases that for us are purely conceptual.[3]

In connection with these explanations I shall try to say something about the development of Gödel's views. Late in his career, Gödel indicated that some form of realism was a conviction he held already in his student days, even before he began to work in mathematical logic. Remarks from the 1930's, however, indicate that at that time his realism fell short of what he expressed later. But it appears in full-blown form in his first philosophical publication, "Russell's mathematical logic" *1944*. The strong conception of mathematical intuition, however, seems in Gödel's published writings to come out of the blue in the 1964 supplement to "What is Cantor's continuum problem?" Even in unpublished writings so far available it is at most hinted at in writings before the mid-1950's. In what follows I will trace this development in more detail.

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[2] From a draft reply to a 1971 letter from Kenneth Blackwell, quoted in Wang [25, page 112], The quotation is from Russell [18, page 169] Gödel was fond of this particular quotation from Russell. In commenting on it in *1944*, however, he stated erroneously (p. 127 n.) that it had been left out in later editions of *Introduction to Mathematical Philosophy*. See Blackwell [3]. Evidently Russell himself did not pay close attention to Gödel's footnote. The specific issue about "not" is not pursued elsewhere in Gödel's writings, and I shall not pursue it here. Godel also remarks that Russell's statement gave the impression that he had had many discussions with Russell, while he himself recalled only one.

[3] It is possible that Gödel was influenced by the remarks about intuitive knowledge in Leibniz's "Meditations on knowledge, truth, and ideas" [11]. Knowledge is intuitive if it is *clear*, i e., it gives the means for recognizing the object it concerns, *distinct*, i.e., one is in a position to enumerate the marks or features that distinguish an instance of one's concept, *adequate*, i.e., one's concept is completely analyzed down to primitives, and finally one has an immediate grasp of all these elements.

§1. Speaking quite generally, philosophers often talk as if we all know what it is to be a realist, or a realist about a particular domain of discourse: realism holds that the objects the discourse talks about exist, and are as they are, independently of our thought about them and knowledge of them, and similarly truths in the domain hold independently of our knowledge. One meaning of the term "platonism" which is applied to Gödel (even by himself) is simply realism about abstract objects and particularly the objects of mathematics.[4]

The inadequacy of this formulaic characterization of realism is widely attested, and the question what realism is is itself a subject of philosophical examination and debate. One does find Gödel using the standard formulae. For example in his Gibbs lecture of 1951, he characterizes as "Platonism or 'Realism' " the view that "mathematical objects and facts (or at least something in them) exist independently of our mental acts and decisions" (**1951*, p. 311) and that "the objects and theorems of mathematics are as objective and independent of our free choice and our creative acts as is the physical world" (p. 312 n. 17). In "Russell's mathematical logic"—as I have said the first avowal of his view in its mature form—he does not use this language to characterize Russell's (earlier) "pronouncedly realistic attitude" of which he approves, but he does in his well-known criticism of the vicious circle principle, where he says that the first form of the principle "applies only if the entities involved are constructed by ourselves. If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members which can be described ... only by reference to this totality" (136).[5]

Gödel is concerned in the Russell essay to argue for the inadequacy of Russell's attempts to show that classes and concepts can be replaced by "constructions of our own" (152), and the Gibbs lecture contains arguments against the view that mathematical objects are "our own creation", a view maybe more characteristic of nineteenth-century thought about mathematics than of that of Gödel's own time.

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[4] For a general discussion of mathematical platonism, see Maddy [12].

[5] Cf. also: "For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually . . " (*1964* p. 262).

Rather than exploring how Gödel himself understands these characterizations, I will note some points that are more distinctive of Gödel's own realism. Introducing the theme in "Russell's mathematical logic," he quotes the statement from Russell [18] quoted above and then turns to an "analogy between mathematics and natural science" he discerns in Russell:

> He compares the axioms of logic and mathematics with the laws of nature and logical evidence with sense perception, so that the axioms need not necessarily be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these "sense perceptions" to be deduced; which of course would not exclude that they also have a kind of intrinsic plausibility similar to that in physics. I think that ... this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future (127).

In other places, as is well known, Gödel claims an analogy between the assumption of mathematical objects and that of physical bodies:

> It seems to me that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions (ibid., 137).

In *1964* the question of the "objective existence of the objects of mathematical intuition" is said (parenthetically) to be "an exact replica of the question of the objective existence of the outer world" (272).

Thus a Gödelian answer to the question what the "independence" consists in is, for example, that mathematical objects are independent of our "constructions" in much the same sense in which the physical world is independent of our sense-experience. Gödel does not address in a general way what the latter sense is, although some evidence of his views can be gleaned from his writings on relativity. The main thesis of his paper *1949a* is that relativity theory supports the Kantian view that time and change are not to be attributed to things as they are in themselves. But this thesis is specific to time and change; it is perhaps for that reason that he is prepared in one place to gloss the view by saying that they are *illusions*, a formulation that Kant expressly repudiates.[6] Gödel is not led by the considerations he advances to reject a realist view of the physical world in general; for example he does not suggest that space-time is in any way ideal or illusory. In fact, he frequently reproaches Kant for being too subjectivist.[7] But he is quite cautious in what little he says about how far we can be realists about knowledge of the physical world. But in his discussion of Kant, he clearly thinks that modern physics allows a more realistic attitude than Kant held; for example he remarks that "it should be assumed that it is possible for scientific knowledge, at least partially and step by step, to go beyond the appearances and approach the things in themselves."[8]

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[6] *1949a*, pp. 557–8, Kant, *Critique of Pure Reason*, B69.

§2. I now want to approach the question of the meaning of Gödel's realism by inquiring into its development. One distinctive feature of Gödel's realism is that it extends to what he calls concepts (properties and relations), objects signified in some way by predicates. These would not necessarily be reducible to sets, if for no other reason because among the properties and relations of sets that set theory is concerned with are some that do not have sets as extensions.[9] It may be that this feature arose from convictions with which Gödel started. In an (unsent) response to a questionnaire put to him by Burke D. Grandjean in 1975, Gödel affirmed that "mathematical realism" had been his position since 1925.[10] In a draft letter responding to the same questions, Gödel wrote, "I was a conceptual and mathematical realist since about 1925."[11] The term "mathematical realism" occurs in Grandjean's question; the term "conceptual" is introduced by Gödel.

Gödel's response to Grandjean would suggest that he was prepared to affirm in 1975 that the realism associated with him was a position he had held since his student days. Moreover, in letters to Hao Wang quoted extensively in Wang [24], Gödel emphasized that realistic convictions, or opposition to what he considered anti-realistic prejudices, played an important role in his early logical achievements, in particular both the completeness and the incompleteness theorems.[12]

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[7] E g., *1964*, p. 272. However, he interprets Kant's conception of time as a form of intuition as meaning that "temporal properties are certain relations of the things to the perceiving subject" (**1946/9-B2*, p. 231), and he finds that there is at least a strong tendency of Kant to think that, interpreted in that way, temporal properties are perfectly objective.

[8] **1946/9-C1*, p. 257; cf. **1946/9-B2*, p. 240. Of course it is quantum mechanics that has been in our own time the main stumbling block for realism about our knowledge in physics. Gödel says little on the subject; what little he does say (e.g., **1946/9-B2*, notes 24 and 25) indicates a definitely realistic inclination without claiming to offer or discern in the literature an interpretation that would justify this.

[9] Thus "property of set" is counted as a primitive notion of set theory (*1947*, p. 520 n., or *1964*, p. 264 n.). This notion corresponds to Zermelo's notion of "definite property" (cf. Gödel *1940*, p. 2).

[10] Wang [25, pp. 17–18].

[11] Ibid., p. 20

[12] Köhler [9] contains interesting suggestions about the influences on Gödel as a student that might have encouraged realistic views. They are not specific enough as regards mathematics to bear on an answer to the questions of interpretation considered in the text. In

Before I turn to these statements, let me mention the remarks of Gödel from the 1930's, to which Martin Davis and Solomon Feferman have called attention, that do not square with the platonist views expressed in *1944* and later. We have the text of a very interesting general lecture on the foundations of mathematics that Gödel gave to the Mathematical Association of America in December 1933. Much of it is devoted to the axiomatization of set theory and to the point that the principles by which sets, or axioms about them, are generated naturally lead to further extensions of any system they give rise to. When he turns to the justification of the axioms, he finds difficulties: the non-constructive notion of existence, the application of quantifiers to classes and the resulting admission of impredicative definitions, and the axiom of choice. Summing up he remarks,

> The result of the preceding discussion is that our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent (**1933o*, p. 50).

It is clear that Gödel regards impredicativity as the most serious of the problems he cites and notes (following Ramsey) that impredicative specification of properties of integers is acceptable if we assume that "the totality of all properties [of integers] exists somehow independently of our knowledge and our definitions, and that our definitions merely serve to pick out certain of these previously existing properties" (ibid.). That is clearly a major consideration prompting him to say that acceptance of the axioms "presupposes a kind of Platonism."[13]

The other remarks are glosses on his work on constructible sets and the consistency of the continuum hypothesis. In the first announcement of his consistency results Gödel says,

> The proposition A [i.e., $V = L$] added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way (*1938*, p. 557).

Acceptance of $V = L$ as an axiom of set theory would not be incompatible with the *philosophical* realism Gödel expressed later, although it would be

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discussing Gödel's relations with the Vienna Circle, Köhler writes as if he already held at the beginning of the 1930's the position of *1944* and later writings. The evidence does not support that.

[13] The cautious and qualified defense of a kind of platonism in Bernays [2] was delivered as a lecture about six months later. We think of one of the influential tendencies in foundations of the time, logicism after Frege and Russell, as a platonist view. That was not the way its proponents saw it in the 1930's.

with the mathematical views he expressed in connection with the continuum problem. But regarding the concept of an arbitrary infinite set as a "vague notion" certainly does not square with Gödel's view in 1947 that the continuum problem *has* a definite answer.[14]

Another document from about this time indicates that, after proving the consistency of the continuum hypothesis and probably expecting to go on to prove its independence, Gödel did not yet have the view of the significance of this development that he later expressed. In a lecture text on undecidable diophantine sentences, probably prepared between 1938 and 1940, Gödel remarks that the undecidability of the sentences he considers is not absolute, since a proof of their undecidability (in a given formal system) is a proof of their truth. But then he ends the draft with the remarkable statement:

> However, I would not leave it unmentioned that apparently there do exist questions of a very similar structure which very likely are really undecidable in the sense which I explained first. The difference in the structure of these problems is only that variables for real numbers appear in this polynomial. Questions connected with Cantor's continuum hypothesis lead to problems of this type. So far I have not been able to prove their undecidability, but there are considerations which make it highly plausible that they really are undecidable (**193?*, p. 175).

It is hard to see what Gödel could have expected to "prove" concerning a statement of the form he describes other than that it is consistent with and independent of the axioms of set theory, say ZF or ZFC, and that this independence would generalize to extensions of ZFC by axioms for inaccessible cardinals in a way that Gödel asserts that his consistency result does. There seems to be a clear conflict with the position of *1947*; it's hard to believe that at the earlier time he thought that exploration of the concept of set would yield new axioms that would decide them. Moreover the statement is a rather bold statement. I don't think it can be explained away as a manifestation of Gödel's well-known caution in avowing his views.

Let me now turn to the most informative documents about Gödel's early realism, the letters to Wang. There he explains the failure of other logicians to obtain the results obtained by him as due to philosophical prejudices, in particular against the use of non-finitary methods in metamathematics, deriving from views associated with the Hilbert school, according to which non-finitary reasoning in mathematics is justified "only to the extent to which it can be 'interpreted' or 'justified' in terms of a finitary metamathematics"

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[14] Martin Davis notes that in *1940* Gödel refers to $V = L$ as an axiom, indicating that he still held the view expressed in the above quotation from *1938*. (See his introductory note to **193?* in CW III, at p. 163.) It would confirm, however, only the first of the two distinguishable aspects of the *1938* view.

(Wang [24, p. 8]). This is applied to the completeness theorem, of which the main mathematical idea was expressed by Skolem in 1922. Gödel also asserts that his "objectivistic conception of mathematics and metamathematics in general" was fundamental also to his other logical work; in particular "the highly transfinite conception of 'objective mathematical truth', as *opposed* to that of 'demonstrability' " is the heuristic principle of his construction of an undecidable number-theoretic proposition (ibid., p. 9).

It should be pointed out that only one of the examples Gödel gives essentially involves impredicativity and thus conflicts sharply with the view of **1933o*: his own work on constructible sets. Where the conflict lies is of course in accepting the conception of the constructible sets as an intuitively meaningful conception, but it's on this that Gödel lays stress rather than on the fact that at the end of the process one can arrive at a finitary relative consistency proof. Gödel is said to have had the idea of using the ramified hierarchy to construct a model quite early; whether by the time of the MAA lecture he had seen that it "has to be used in an *entirely nonconstructive way*" (Wang [24, p. 10]) is not clear. It seems to have been only in 1935 that he had a definite result even on the axiom of choice.[15]

It seems we cannot definitely know whether Gödel in December 1933 already thought the "kind of Platonism" he discerned more acceptable than he was prepared to say. But it seems extremely likely that, with whatever conviction he embraced impredicative concepts in first developing the model of constructible sets in the form we know it, his confidence in this point of view would have been increased by his obtaining definite and important results from it. The remarks from 1938 show that there was already a further step to be taken; one possible reason for his taking it may have been reflection on the consequences of $V = L$ for descriptive set theory, which could have convinced him that $V = L$ is false. But it should be pointed out that the idea that some mathematical propositions are absolutely undecidable is one that Gödel still entertained in his Gibbs lecture in 1951, and in itself it is not opposed to realism.[16]

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[15] Wang writes ([25, p. 97]:

> From about 1930 he had continued to think about the continuum problem .... The idea of using the ramified hierarchy occurred to him quite early. He then played with building up enough ordinals. Finally the leap of taking the classical ordinals as given made things easier. It must have been 1935, according to his recollection in 1976, when he realized that the constructible sets satisfy all the axioms of set theory (including the axiom of choice). He conjectured that the continuum hypothesis is also satisfied.

Seen in light of the remarks in **1933o*, the "leap of taking the classical ordinals as given" was a decisive step in the development of Gödel's realism about set theory. Wang's remarks (evidently based on Gödel's much later recollection) suggest, but do not explicitly say, that this leap was taken close enough to 1935 to be probably later than December 1933. On the other hand Feferman conjectures that the rather casual treatment in **1933o* itself of the problem of the axiom of choice may have been due to Gödel's having an approach to proving its consistency. (See his introductory note to **1933o* in CW III)

It can be documented that Gödel obtained the essentials of the proof of the consistency of CH in June 1937. See Feferman, [note s (CW I 36)]

There is another more global and intangible consideration that could lead one to doubt that Gödel's views of the 1930's were the same as those he avowed later. This is the evidence of engagement with the problems of proof theory, in the form in which the subject evolved after the incompleteness theorem. Gödel addresses questions concerning this program in the MAA lecture _*1933o_ and more thoroughly and deeply in the remarkable lecture _*1938a_ given in early 1938 to a circle organized by Edgar Zilsel. This lecture shows that he had already begun to think about a theory of primitive recursive functionals of finite type as something relative to which the consistency of arithmetic might be proved; it is now well known that he obtained this proof in 1941 after coming to the United States. The lecture at Zilsel's also contains a quite remarkable analysis of Gentzen's 1936 consistency proof, including the no-counter-example interpretation obtained later by Kreisel (see Kreisel [10]). What he says about the philosophical significance of consistency proofs such as Gentzen's is not far from what was being said about the same time by Bernays and Gentzen, in spite of somewhat polemical remarks about the Hilbert school in this text and in others.[17]

§3. I shall not try to trace the development of Gödel's realism further independently of the notion of mathematical intuition. As I said, it is firmly avowed in *1944* and further developed in *1946*, *1947*, and _*1951_. It is thus during the period from 1943 or 1944 through 1951 that it becomes Gödel's public position.[18]

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[16] Note that in *1946* Gödel explores the idea of absolute provability. In this connection it is reasonable to ask whether Gödel is a realist by one criterion suggested by the work of Dummett, according to which realism admits truths that are "recognition-transcendent", that is obtain whether or not it is even in principle possible for humans to know them. In the sphere of mathematics, an obstacle to this view for Gödel is his confidence in reason; he expresses in places the Hilbertian conviction of the solvability in principle of every mathematical problem. See Wang [24, pp. 324–325] (on which see footnote 49 below), cf _*1961/?_, pp. 378, 380.

However, the discussion in _*1951_ makes clear that Gödel regards the existence of recognition-transcendent truth as meaningful, since if the mathematical truths that the human mind can know can be generated by a Turing machine, then the proposition that this set is consistent would be a mathematical truth that we could not know. And this is presumably what is decisive for Dummettian realism rather than whether recognition-transcendent truths in fact exist, which Gödel was inclined to believe they did not, at least in mathematics

[17] I owe this observation to Wilfried Sieg. Cf. our introductory note to _*1938a_ in CW III, at p. 85.

[18] The conversation that was the basis of Russell's remark quoted on p. 44 above would have taken place near the beginning of this period.

I have discussed elsewhere the position of *1944*.[19] It is not easy to discern a definite line of argument for realism (which would in turn clarify the position itself); the form of a commentary on Russell works against this. A very familiar argument which is already present in **1933o* (as well as in Bernays [2]) is that particular principles of analysis and set theory are justified if one assumes a realistic view of the objects of the theory and not otherwise. Gödel applies this point of view particularly in his well-known analysis of Russell's vicious circle principle, where he argues from the fact that "classical mathematics" does not satisfy the vicious circle principle that this is to be considered "rather as a proof that the vicious circle principle is false than that classical mathematics is false" (135).

When Gödel says that assuming classes and concepts as "real objects" is "quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence" (137, quoted above), his claim is that classical mathematics is committed to such objects and moreover it must be interpreted so that the objects are independent of our constructions. Gödel reinforces this claim by his analysis of the ramified theory of types in the present paper and by discussions elsewhere in his writings such as the criticism of conventionalism in **1951* and **1953/9* (actually briefly adumbrated at the end of *1944*). In a way this is hardly controversial today; an impredicative theory with classical logic is the paradigm of a "platonist" theory. But Gödel's rhetoric has certainly led most readers to think that his reasoning is not just to be reconstructed as an application of a Quinean conception of ontological commitment. Why is this so?

One reason is certainly Gödel's remarks about intuition, of which we are postponing discussion. But that conception plays virtually no role in *1944*. Another reason more internal to that text is that Gödel makes clear that his realism extends to concepts as well as classes (which in this discussion he does not distinguish from sets). Standard set theories either quantify only over sets or, if they have quantifiers for (proper) classes, allow a predicative interpretation of class quantification. Thus at most realism about sets seems to be implied by what is common to Gödel and philosophers who have followed Quine. Gödel makes clear that he sees no objection to an impredicative theory of concepts (139–40), and the paper contains sketchy ideas for such a theory, which apparently Gödel never worked out in a way that satisfied him. But Gödel does not directly argue for a realism about concepts that would license such a theory; in particular he does not argue that classical mathematics requires such realism.

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[19] In my introductory note in CW II, on realism see particularly pp. 106–110.

In what sense does *1947* offer a further argument for realism?[20] The major philosophical claim of *1947*, that the independence of the continuum hypothesis should in no way imply that it does not have a determinate truth-value, is rather an inference *from* realism. Gödel makes such an inference in saying that if the axioms of set theory "describe some well-determined reality," then "in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality" (520). But Gödel then proceeds to give arguments for the conclusion that the continuum problem might be decided. The first is the point going back to **1933o* about the open-endedness of the process of extending the axioms. The second is that large cardinal axioms have consequences even in number theory. Here he concedes that such axioms as can be "set up on the basis of principles known today" (i.e., axioms providing for inaccessible and Mahlo cardinals) do not offer much hope of solving the problem.[21] The further statement, that axioms of infinity and other kinds of new axioms are possible, was more conjectural, and of course the stronger axioms of infinity that were investigated later (already taken account of to some degree in the corresponding place in *1964*) were shown not to decide CH. The third consideration is that a new axiom, even if it cannot be seen to have "intrinsic necessity," might be verified inductively by its fruitfulness in consequences, in particular independently verifiable consequences. It might be added that Gödel's plausibility arguments for the falsity of CH constitute an argument for the suggestion that axioms based on new principles exist, since any such axiom would have to be incompatible with $V = L$.

Another point, which hardly attracts notice today because it seems commonplace, is that the concept of set and the axioms of set theory can be defended against paradox by what we would call the iterative conception of set. In 1947, to say that this conception offers a "satisfactory foundation of Cantor's set theory in its whole original extent" (518) was a rather bold statement. Even the point (made in Gödel *1944*, p. 144) that axiomatic set theory describes a transfinite iteration of the set-forming operations of the simple theory of types was not a commonplace. Of course in what sense we do have a "satisfactory foundation" was and is debatable. But I think it would now be a non-controversial claim that, granted certain basic ideas (ordinal and power set) in a classical setting, the iterative conception offers an intuitive conception of a universe of sets,which, in Gödel's words, "has never led to any antinomy whatsoever" (*1947*, p. 518). I think Gödel wishes to claim more, namely that the axioms follow from the concept of set. That thought is hardly developed in *1947* and anyway belongs with the conception of mathematical intuition.[22] Overall, *1947* was probably meant to offer an indirect argument for realism by applying it to a definite problem and showing that the assumption of realism leads to a fruitful approach to the problem. It is worth noting that he offers arguments for the independence of the continuum hypothesis of which the main ones are plausibility arguments for its *falsity*. An "anti-realist" urging upon us the attempt to prove the independence would presumably dwell more on the obstacles to proving it.

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[20] I pass over *1946*, which might, like *1947*, be described as an application of Gödel's point of view to concrete problems. This is not uncharacteristic of Gödel; also in *1944* he often seems to treat realism as a working hypothesis.

[21] This had been partly shown by Gödel in extending his consistency proof to such axioms; it was subsequently shown that the independence proof also extended, and the consistency and independence of CH were proved even for stronger large cardinal axioms such as Gödel did not have in mind in 1947.

The Gibbs lecture "Some basic theorems of the foundations of mathematics and their implications" (**1951*) seems to complete for Gödel the process of avowing his platonistic position. In some ways, it is the most systematic defense of this position that Gödel gave. At the end it seems to see itself as part of an argument as a result of which "the Platonistic position is the only one tenable" (322–3).[23]

The main difficulty of the Gibbs lecture's defense, however, is not the omission he mentions at the end, of a case against Aristotelian realism and psychologism, but that its central arguments are meant to be independent of one's standpoint in the traditional controversies about foundations; the overall plan of the lecture is to draw implications from the incompleteness theorems. Gödel's main arguments aim to strengthen an important part of his position, which he expresses by saying that mathematics has a "real content".[24] But although this is opposed to the conventionalism that he discerns in the views of the Vienna Circle, and also to many forms of formalism, it is a point that constructivists of the various kinds extant in Gödel's and our own time can concede, as Gödel is well aware. But it is probably a root conviction that Gödel had from very early in his career; it very likely underlies the views that Gödel, in the letters to Wang, says contributed to his early logical work. It would then also constitute part of his reaction to attending sessions of the Vienna Circle before 1930.

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[22] In the revised version *1964*, the discussion of the iterative conception of set is somewhat expanded.

[23] This remark appears to be an expression of a hope that Gödel maintained for many years, that philosophical discussion might achieve "mathematical rigor" and conclusiveness. As he was well aware, his actual philosophical writings, even at their best, did not fulfill this hope, and these remarks are part of an admission that certain parts of the defense of mathematical realism had not been undertaken in the lecture.

[24] This conviction will come up in the discussion of intuition in sections §4 and §5, see also my introductory note to *1944* and Parsons [14].

§4. I now turn to the conception of mathematical intuition, beginning with some remarks about its development. I have outlined above the presentation of Gödel's realism in his early philosophical publications *1944* and *1947* and the lecture **1951*. For a reader who knows *1964*, it is a striking fact about these writings that the word "intuition" occurs in them very little, and no real attempt is made to connect his general views with a conception of mathematical intuition.

In *1944* the word "intuition" occurs in only three places, none of which gives any evidence that intuition is at the time a fundamental notion for Gödel himself. The first (128) is in quotation marks and refers to Hilbert's ideas. The second is in one of the most often quoted remarks in the paper, in which Russell is credited with "bringing to light the amazing fact that our logical intuitions (i.e., intuitions concerning such notions as: truth, concept, being, class, etc.) are self-contradictory" (131). Here "intuition" means something like a belief arising from a strong natural inclination, even apparent obviousness. In the following sentence these intuitions are described as "common-sense assumptions of logic." It's not at all clear to what extent "intuition" in this sense is a guide to the truth; it is clearly not an infallible one. In the third place (150), Gödel again speaks of "our logical intuitions," evidently referring to the earlier remarks, and it seems clear that he is using the term in the same sense.

One other remark in *1944* deserves comment. In his discussion of the question whether the axioms of *Principia* are analytic in the sense that they are true "owing to the meaning of the concepts" in them, he sees the difficulty that "we don't perceive the concepts of 'concept' and 'class' with sufficient distinctness, as is shown by the paradoxes" (151). Since "perception" of concepts is spoken of in unpublished writings of Gödel, this seems to be an allusion to mathematical intuition in a stronger sense. But the remark itself is negative; it's not clear what Gödel would say that is positive about perception of concepts.

The word "intuition" does not occur at all in *1946* and only once in *1947*. Concerning constructivist views, he remarks

> This negative attitude towards Cantor's set theory, however, is by no means a necessary outcome of a closer examination of its foundations, but only the result of certain philosophical conceptions of the nature of mathematics, which admit mathematical objects only to the extent in which they are (or are believed to be) interpretable as acts and constructions of our own mind, or at least completely penetrable by our intuition (518).

Since Gödel does not elaborate on his use of "intuition" at all, one can't on the basis of this text be at all sure what he has in mind. But it appears that intuition as here understood, instead of being a basis for possible knowledge of the strongest mathematical axioms, is restricted in its application, so that the demand that mathematical objects be "completely penetrable by our intuition" is a constraint that would strongly limit what objects can be admitted.[25]

The Gibbs lecture is again virtually silent about intuition. I have not found in it a single occurrence of the word "intuition" on its own.[26] But talk of perception where the object is abstract occurs again, this time more positively, but still without elaboration or explanation. Gödel defends the view that mathematical propositions are true by virtue of the meaning of the terms occurring in them.[27] But the terms denote concepts of which he says:

> The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe (320).

At the end, he says of the "Platonistic view":

> Thereby I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind (323).

There is nothing in these early writings to rule out the interpretation that the talk of "perception" of concepts is meant metaphorically. The last quoted statement could come down to the claim that the "non-sensual reality" that mathematics describes is *known* or *understood* very incompletely by the human mind. Thus although there are what might be indications as early as *1944* of a strong conception of mathematical intuition, in public documents before 1964 they are less than clear and decisive, and Gödel does not begin to offer a defense of it. Nonetheless the allusions to perception of concepts in *1944* and **1951* are very suggestive in the light of his later writings, and it is reasonable to conjecture that although he was not yet ready to defend his conception of intuition he already had some such conception in mind.

But of course there is one published writing before 1964 in which a concept of intuition figures more centrally, and that is the philosophical introduction to the *Dialectica* paper *1958*. The German word used is the Kantian term *Anschauung*. I shall not discuss this paper in any detail but only state rather dogmatically that what is at issue are conceptions of intuition and intuitive evidence derived from the Hilbert school. Gödel is concerned with the question of the limits of intuitive evidence, where these limits will clearly be rather narrow. It is contrasted with evidence essentially involving "abstract concepts." Thus the conception of intuition involved is not the strong one, a mark of which is that it yields knowledge of propositions involving abstract concepts in an essential way. There is no doubt that that was Gödel's view of the central concepts of set theory and the axioms involving them. The fact that in *1972 Anschauung* is translated as "concrete intuition" indicates both that in *1958* he was employing a more limited conception of intuition than that of *1964* and that it may be a special case of the latter.

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[25] The meaning of "intuition" here could agree with that of *Anschauung* in *1958*; see below The phrase is replaced in *1964* by "completely given in mathematical intuition" (262); it is hard to be sure whether Gödel saw this as more than a stylistic change.

[26] There are references to intuitionism (e.g., in n. 15, p. 310), and he does speak (p. 319) of the "intuitive meanings" of disjunction and negation.

[27] This is, of course, a sense in which mathematics could be said to be analytic; for further discussion see Parsons [14]

There is, however, a source earlier than *1964* for Gödel's thought about mathematical intuition, the drafts of the paper "Is mathematics syntax of language?" (**1953/9*), which Gödel worked on in response to an invitation from Paul Arthur Schilpp to contribute to *The Philosophy of Rudolf Carnap* but never submitted. Six versions survive in Gödel's *Nachlaß*.

The main purpose of the paper is to argue against the conception of mathematics as syntax that is found in logical positivist writings, especially Carnap's *Logical Syntax of Language*.[28] Gödel had already given a version of his argument in **1951*,[29] in a way that does not use the notion of mathematical intuition, and even sketched the ideas in the discussion of analyticity at the end of *1944*. The basic argument, related to arguments directed at Carnap by Quine, is that in order to establish that interesting mathematical statements are true by virtue of syntactical rules or conventions it is necessary to use the mathematics itself in its straightforward meaning.[30] In arguing, contrary to the view he is criticizing, that mathematics has a "real content," Gödel is, as I have said, affirming one aspect of his realism. It is, however, only one: The same argument would be open to an intuitionist, and Gödel himself argues that certain fallback positions of his opponent still leave him obliged to concede "real content" at least to finitist mathematics.

The presentation of his argument against Carnap in **1953/9* does not similarly eschew reference to mathematical intuition, although in the briefer, stripped down presentation of the argument in version V, it does not figure prominently. Before we go into it we should rehearse some elementary distinctions about intuition. In the philosophical tradition, intuition is spoken of both in relation to objects and in relation to propositions, one might say as a propositional attitude. I have used the terms intuition *of* and intuition *that* to mark this distinction. The philosophy of Kant, and the Kantian paradigm generally, gives the basic place to intuition of, but certainly allows for intuitive knowledge or evidence that would be a species of intuition that. But talk of intuition in relation to propositions has a further ambiguity, since in propositional attitude uses "intuition" is not always used for a mode of *knowledge*. When a philosopher talks of his or others' intuitions, that usually means what the person concerned takes to be true at the outset of an inquiry, or as a matter of common sense; intuitions in this sense are not knowledge, since they need not be true and can be very fallible guides to the truth. To take another example, the intuitions of a native speaker about when a sentence is grammatical are again not necessarily correct, although in this case they are, in contemporary grammatical theory, taken as very important guides to truth. In contrast, what Descartes called *intuitio* was not genuine unless it was knowledge. Use of "intuition" with this connotation is likely to cause misunderstanding in the circumstances of today; it may even lead a reader to think one has in mind something like intuitions in the senses just mentioned with the extra property of being infallible. It is probably best to use the term "intuitive knowledge" when one wants to make clear one is speaking of knowledge.[31]

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[28] Carnap [4] and [5].

[29] A large part of it (pp. 315–319), however, is in a section marked "wegzulassen"; it is possible that this was not included in the lecture as delivered. Cf. editorial note c, p. 315.

[30] For discussion see Parsons [14]. However, I barely touch there on the question whether the position Gödel criticizes is what Carnap actually holds This is questioned by Warren Goldfarb in his introductory note to **1953/9* in CW III. Cf. Goldfarb and Ricketts [8].

A difficulty in reading Gödel's writing on mathematical intuition is that he uses the term in both object-relational and propositional attitude senses, and in the latter it is not always clear what epistemic force the term is intended to have. Since, where a strong conception is involved, it is mainly concepts that are the objects of intuition, and Gödel does (as we have already seen) speak of perception of concepts, it might be well in discussing Gödel to use the word "perception" where intuition of is in question, and reserve the term "intuition" for intuition that. I will follow that policy in what follows.

In **1953/9* Gödel seems to take the propositional sense as primary. I think it is clear that he has first of all in mind what might be called rational evidence, or, more specifically, autonomous mathematical evidence. Thus in stating the view he is criticizing he writes, "Mathematical intuition, for all scientifically relevant purposes ... can be replaced by conventions about the use of symbols and their application" (version V, 356). Apart from the conventionalism his argument is directed against, the only alternative to admitting mathematical intuition that Gödel considers is some form of empiricism. Thus the deliverances of mathematical intuition are just those mathematical propositions and inferences that we take to be evident on reflection and do not derive from others, or justify on a posteriori grounds, or explain away by a conventionalist strategy.[32]

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[31] In the philosophy of mathematics, however, this has the disadvantage that "intuitive knowledge" has a more special sense, for example in Gödel *1958* and *1972*.

[32] One might ask, particularly in the light of later writing in the philosophy of mathematics, about the option of not taking the language of mathematics at face value. The only such option considered in Gödel's writings is if-thenism. Apart from other difficulties, in his view the translations have enough mathematical content to raise again the same questions.

It is clear Gödel has primarily in mind mathematical axioms and rules of inference that would be taken as primitive. He does not, however, distinguish mathematics from logic. An example given in a couple of places is modus ponens.[33] In application to logic, what we have presented up to now of Gödel's position does not differ from a quite widely accepted one, in declining to reduce the evidence of logic either to convention or to other forms of evidence. Such a view is even implied by Quine when he regards the obviousness of certain logical principles as a constraint on acceptable translation, although of course Quine would not agree that this implies an important distinction between logical and empirical principles.

With regard to the epistemic force of Gödel's notion of mathematical intuition, the remarks in the supplement to *1964* have given rise to some confusion. I think this can be largely cleared up by taking account of **1953/9*. I think it is clear that for Gödel mathematical intuition is not *ipso facto* knowledge. In a way the existence of mathematical intuition should be non-controversial:

> The existence, as a psychological fact, of an intuition covering the axioms of classical mathematics can hardly be doubted, not even by adherents of the Brouwerian school, except that the latter will explain this psychological fact by the circumstance that we are all subject to the same kind of errors if we are not sufficiently careful in our thinking (version III, 338 n. 12).[34]

In this context, "intuition" has something like the contemporary philosopher's sense, with perhaps more stability and intersubjectivity: Most of us who have studied mathematics find the axioms of classical mathematics intuitively convincing or at least highly plausible. According to Gödel, Brouwer (or for that matter a conventionalist) should grant this much.[35] Elsewhere, where it is clear that he regards mathematical intuition as a *source* of knowledge, it is still clear that possession of intuition isn't already possession of knowledge, for example when he talks of mathematical intuition producing conviction:

> However, mathematical intuition in addition produces the conviction that, if these sentences express observable facts and were obtained by applying mathematics to verified physical laws (or if they express ascertainable mathematical facts), then these facts will be brought out by observation (or computation) (version III, 340).

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[33] Version III, note 34 (p 347); version V, p 359.

[34] A parallel passage in version IV is clearer but more controversial in that it introduces the idea of intuition of concepts.

[35] I think Brouwer did grant a good part of what Gödel has in mind here But to sort this out would be a long story and belong to the discussion of Brouwer rather than Gödel.

> If the possibility of a disproval of mathematical axioms is frequently disregarded, this is due solely to the convincing power of mathematical intuition (version V, 361).

What he calls the "belief in the correctness of mathematical intuition" (version III, 341) is not a trivial consequence of acknowledging its existence.

Gödel does (as we shall see) regard mathematical intuition as significantly like perception, but that someone has the intuition that *p* does not imply *p* in the way that if he sees that *p* that implies *p*. (If someone claims to see that *p* and *p* turns out to be false, then he only seemed to see that *p*.) It is rather more like making a perceptual judgment, which may have a strong presumption of truth but which can in principle be false.

A conclusion I draw from this is that what is at issue between Gödel and his opponents about mathematical intuition is not any basic assertion of its existence, but some questions about its character and especially its ineliminability as an epistemic factor. Gödel attributes to Carnap the view that appeals to mathematical intuition need play no more than an heuristic role in the justification of mathematical claims. Something like this seems also to be true of Quine (although Gödel never comments on Quine's philosophical views).

Even if one grants to mathematical intuition in the sense explained so far a high degree of credence, the question will still arise why it should be called *intuition*. Other philosophers have held that there are non-empirical, non-conventional truths without calling the evidence that pertains to them intuition or using for them a term that could easily be understood as meaning something close to that. A very good example is Frege, who quite on the contrary insists that arithmetical knowledge, because it is a part of logic, does not depend on intuition. For him the term (that is, the German term *Anschauung*) has a roughly Kantian meaning. Gödel himself often speaks of reason in talking of the evident character of mathematical axioms and inferences. This is in agreement with the usage of Frege and others in the rationalist tradition. Yet in speaking of the source of knowledge in these cases as mathematical intuition, without the spatio-temporal connotation of the Kantian tradition, Gödel is not just differing with Frege about terminology.[36]

To analyze the differences between Gödel and Frege would require more exploration of Frege than I can undertake here. But we can see a major difference in the analogy Gödel stresses in places in **1953/9* between what he calls mathematical intuition and sense-perception. His claims about this analogy are strong but not very much developed. As I remarked earlier, he does not distinguish mathematics from logic. Thus even elementary logic seems to be an application of mathematical intuition:

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[36] Although Kantian intuition plays a role in Gödel's writings, it is not altogether clear whether he accepted some version of the notion or simply explored it as part of a philosophy (that of the Hilbert school) he wished to explore because of its connection with proof theory and constructivity.

> The similarity between mathematical intuition and a physical sense is very striking. It is arbitrary to consider "this is red" an immediate datum, but not so to consider the proposition expressing modus ponens or complete induction (or perhaps some simpler propositions from which the latter follows). For the difference, as far as it is relevant here, consists solely in the fact that in the first case a relationship between a concept and a particular object is perceived, while in the second case it is a relationship between concepts (version V, 359).

In this passage and in many others, we find a formulation that is very characteristic of Gödel: In certain cases of rational evidence (of which we can easily grant modus ponens to be one), it is claimed that "perception" of concepts is involved. Indeed, in this passage such perception is even said to be involved in a situation where one recognizes by sense-perception the truth of 'this is red' (with some demonstrative reference or other for 'this'). An inference seems to be made from 'a perceives that ... *F* ... ', where '*F*' is a predicate or general term occurring in '... *F* ... ', to 'a perceives (the concept) *F*'. Gödel does not formulate "the proposition expressing modus ponens," but presumably it would involve the concepts of proposition, being of the form 'if *p* then *q*', and implication, so that the claim is that in this case a relation of these concepts is involved. (I am assuming that perceiving a relation between concepts involves perceiving the concepts; I think that can be justified from the texts.[37])

If we ask what the analogy with perception is beyond that of providing an irreducible form of evidence, an appropriate answer is likely to be of the form that certain *objects* are before the mind in a way analogous to that in which physical objects are present in perception. Gödel's answer is "concepts", perhaps concepts of a particular kind. But that in the case of either 'this is red' or elementary logical truths and inferences concepts are present in this way seems to be an assumption, at best part of an explanation of how these things might be evident that is not carried further.[38] This is certainly a point on which Gödel can be criticized.

Gödel actually goes further and sees a close analogy between reason and an "additional sense." After discussing the idea of an additional sense that would show us a "second reality" separated from space-time reality but still describable by laws, Gödel says:

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[37] Cf. the formulation of the same point in version IV, ms. p. 19.

[38] Gödel does in one place (**1961/?*, pp. 382, 384) make brief remarks about language learning and suggests that when a child first understands a logical inference this is a step that brings him to a higher state of consciousness. Cf Parsons [14, section III].

> I even think this comes pretty close to the true state of affairs, except that this additional sense (i.e., reason) is not counted as a sense, because its objects are quite different from those of all other senses. For while through sense perception we know particular objects and their properties and relations, with mathematical reason we perceive the most general (namely the 'formal') concepts and their relations, which are separated from space-time reality insofar as the latter is completely determined by the totality of particularities without any reference to the formal concepts (version III, 354).[39]

In the corresponding passage in version IV, instead of the last sentence above Gödel has:

> For while with the latter [the senses] we perceive particular things, with reason we perceive concepts (above all the primitive concepts) and their relations (ms. p. 17b).

The difference suggests an important uncertainty or change of mind on Gödel's part, as to the exact sphere of reason (which would include mathematical intuition). In IV any perception of concepts seems to be an application of reason, but in III it seems that the concepts involved in 'this is red' belong rather to sense. The passage from V suggests the position of IV but may have been intended to be noncommittal.

It is disappointing that Gödel's logical example is a general principle that involves quantification over propositions or sentences and characteristically logical concepts like implication. He does not, here or so far as I know elsewhere, answer directly the question whether a particular logical truth such as 'it is raining or it is not raining' or a particular inference (say, by modus ponens) is an application of mathematical intuition. This could depend on the question just mentioned, whether any perception of concepts is an application of reason.

Such elementary logical examples would differ from the example that Gödel was most interested in, the axioms of set theory, in that the claim that they are rather directly and immediately evident has a great deal of plausibility. One couldn't argue for them from more theoretical considerations without using inferences or assumptions of much the same kind.

Gödel had another argument for the analogy between reason and perception, based on what in the Gibbs lecture he called the inexhaustibility of mathematics, which he argued for in two ways: from the incompleteness theorem, which implies that a sound formal system for a part of mathematics can always be properly extended, and from the iterative conception of set, where, on his understanding, the conception would always give rise to more sets than a given precise delineation of principles would provide and thus to new evident axioms. Thus there are an unlimited number of independent "perceptions":

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[39] What does Gödel mean by this last assertion? It seems to say, as Warren Goldfarb remarks in his introductory note to the paper, that "the empirical world is fixed independently of mathematics" (CW III 333). Gödel does not suggest such a view in his discussions of physics (**1946/9* and *1949a*), and it is difficult to reconcile with the talk in *1964* of the "abstract elements contained in our empirical ideas" (272).

> The "inexhaustibility" of mathematics makes the similarity between reason and the senses... still closer, because it shows that there exists a practically unlimited number of perceptions also of this "sense" (**1953/9*, version III, 353 n. 43).[40]

The concept of set was doubtless for Gödel the most favorable example of "perception" of concepts. (It is also a case where Gödel argued something I have not stressed here, although it is discussed in Parsons [14], that the propositions known in this way are in a way analytic.) Thus in *1964*, he emphasizes the fact that intuition gives rise to an "open series of extensions" of the axioms (p. 272), and of course the incompleteness theorem implies that any such series generated by a recursive rule would be incomplete and would, indeed, suggest a further reflection that would lead to a still stronger extension. Gödel interpreted these considerations by saying that the "mind, in its use, is not static, but constantly developing" (*1972a*, p. 306). This remark is directed against a mechanist view of mind such as Gödel attributed to Turing. He explicitly offers the generation of new axioms of infinity in set theory as an example.[41] It is interesting that the inexhaustibility of mathematics is used by Gödel both in drawing his analogy between perception and insight into mathematical axioms and in his critical discussion of mechanism.[42] The complex, iterated reflection involved in the uncovering of stronger mathematical axioms and the concepts entering into them strikes me intuitively as very different from perception, and I don't think Gödel has offered more than a rather undeveloped formal analogy. But it is a real problem for what I at the outset called a theory of reason to give a better account.

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[40] Gödel follows this remark by remarks about axioms of infinity in set theory. A parallel remark in IV (p. 19) is followed by an appeal to the second incompleteness theorem.

[41] One should, however, compare the version of what in CW II is Remark 3 of *1972a* with the version of the same remark published in Wang [24, p. 325].

[42] On this subject see **1951* (and Boolos's introductory note in CW III), Wang [24, pp. 324–326], and Wang [26]. It would be beyond the scope of this paper to pursue this subject further. But it should be pointed out that what is needed for Godel's case against mechanism is the inexhaustibility of our potential for acquiring mathematical *knowledge*. He himself makes clear in **1951* that that does not follow simply from the mathematical considerations such as the incompleteness theorems. On the other hand it is not intrinsically connected with platonism as opposed to, say, intuitionism.

§5. Let us now turn to the remarks about mathematical intuition in *1964*. Gödel presents a sketch of his epistemology of mathematics in four paragraphs (pp. 271–272). Some things that are obscure in the first and third paragraphs should be clearer in the light of our discussion so far.

Gödel begins with a remark that is among the most quoted in all his philosophical writing:

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves on us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future (271).

A first problem concerning this passage is how Gödel gets from the axioms "forcing themselves on us as being true", which we might accept as a form of intuition that, to the conclusion that we have something like a perception of the *objects* of set theory, an instance of intuition of. We can see that by "the objects of set theory" Gödel means not just sets but the primitive *concepts* of set theory, "set" itself, membership, what he calls "property of set" (264 n. 18). And it is clear from the above discussion that he understands rational evidence in general as involving perception of the concepts that are the constituents of the proposition in question. This, I think, is the unstated premiss of an inference that at first sight appears to be a *non sequitur*. Although Gödel never so far as I know denies that there is "something like a perception" of *sets*, it isn't on that idea that his conception of our knowledge of axioms of set theory rests.[43] The "new mathematical intuitions leading to a decision of such problems as Cantor's continuum hypothesis", in particular, would be simply insights into the truth of new axioms that would decide the continuum hypothesis.

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[43] Nonetheless there is still a problem, as was pointed out to me by Earl Conee. Perception of "the objects of set theory" does on the face of it include perception of sets, and it is not clear how perception of the concepts explicitly occurring in the axioms of set theory should lead to such perceptions.

Warren Goldfarb has pointed out that in some places in **1953/9* Gödel seems to take "concepts" also to include mathematical objects. (See pp. 332–3 of his introductory note and the texts cited there; of these the most persuasive to me is version III, note 45.) But even if that were Gödel's general usage, it would not solve this particular problem.

It can be said that some sets can be identified individually by concepts, and one, ω, is all but explicitly mentioned in the axiom of infinity Since Gödel would probably have regarded deduction as leading to further or clearer perceptions of concepts, he could very well have thought that individually identifiable sets are "perceived" by way of perception of the concepts that identify them uniquely. This view would have the consequence that natural numbers are also "perceived". So far as I know Gödel nowhere affirms or denies this.

In the third paragraph Gödel presents ideas that will be familiar to the reader of **1953/9*. He is considering a fallback position where intuition concerning the concepts of set theory is not the guide to knowledge that he himself takes it to be. It is still "sufficiently clear to produce the axioms of set theory and an open series of extensions of them," and this "suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis" (272). That reflection on the concepts of set theory gives rise to intuitions of this kind can hardly be doubted if one studies the work of set theorists, although how clear the intuitions are can be questioned, already concerning the axioms of replacement and power set. The meaning that is thus given to CH is that the progress of set theory could give rise to axioms that are supported by the intuitions of set theorists and decide CH. But of course the particular line of inquiry on which Gödel rested his original hopes, large cardinals, proved fruitful in other respects but has not resolved this particular problem. Whether something like what Gödel hoped for is at all likely to happen through the discovery of axioms of another kind deciding CH is so far as I know open, and I would defer to the judgment of experts in any case. But clearly the question can't be suppressed: Couldn't our intuitions concerning sets be conflicting, so that different axioms were discovered that have their own kind of intuitive support but have opposite implications regarding CH?

The spectre of the concept of arbitrary infinite set being a "vague notion" that needs to be "determined in a definite way" by new axioms isn't easily banished, and then one can't rule out the possibility that it might be determined in incompatible ways.

The opening sentence of the paragraph suggests that mathematical intuition might be developed altogether without any commitment as to the extent to which it is a guide to truth:

> However, the question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here (272).

Commenting on this remark, W. W. Tait compares mathematical intuition to the perceptions of a brain in a vat [23, p. 365]. That would, I think, not square with the view that intuition plays a role in elementary mathematics and logic without which one could not even answer the question whether "intuition" understood more noncommittally is able to decide CH. That Gödel is here making a concession for the sake of argument, and thinking primarily of the intuitions leading to strong axioms of set theory, is suggested by the fact that later in the same paragraph he talks of what "justifies the acceptance of this criterion of truth in set theory."[44] He also, in effect, repeats the point about there being a potentially unlimited number of independent intuitions needed to decide questions not only in set theory but also in number theory.

I now move back to the second paragraph, possibly the most difficult and obscure passage in Gödel's finished philosophical writing. Only a small part of it is much illuminated by the earlier writings that I have studied. The passage presents new ideas, possibly derived from the study of Husserl that Gödel began in 1959. But it is with Kant and perhaps Leibniz that he seems to make a more direct connection.

Since Gödel is making a comparison between mathematical concepts and those referring to physical objects, it may be helpful to recall the most basic elements of Kant's conception of the latter. Knowledge of objects has constituents of two kinds, intuitions and concepts. The former are contributed by the faculty of sensibility (at least at first approximation). But they can't be identified with sensations or sense-data: intuitions are of objects in space and time, and space and time are a priori contributions of the human mind. Intuition gives knowledge its particular reference, but knowledge is in the end propositional, and something must be predicated of objects. Concepts also have both empirical and a priori dimensions. Any objective knowledge at least subjects its objects to the categories, a priori contributions of the understanding (the faculty of thought). The categories are "concepts of objects in general"; referring our knowledge to objects means applying this abstract and a priori system of concepts.

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[44] Tait is concerned to argue that mathematical intuition is not "what confers objective validity on our theorems" according to Gödel. I am not sure that Gödel has said enough to make at all clear how he would understand the latter problem. Tait may be denying that according to Gödel mathematical intuition is necessary to us as a ground for asserting mathematical propositions; if so I disagree, and I think the argument of **1951* and **1953/9* would make little sense if the denial is right. The philosophical defense of the objective validity of mathematics is another matter. I agree with Tait that the previous paragraph contains something of what Gödel has to say about that.

Tait seems to reject the interpretation of Gödel as an "archetypical Platonist." In part he is rejecting, certainly rightly, the imputation to Gödel of the postulation of a faculty by which we "interact" with mathematical objects. It still seems to me that there is an important sense in which Gödel is a Platonist and Tait is not. Tait's view that "questions about the legitimacy of principles of construction or proof are not ... questions of fact" and the reasons he gives for this [23, p. 361] are alien to Gödel (see below). Tait himself sees some disagreement (ibid., p. 365, end of footnote 3).

Related remarks concerning Tait's discussion of Gödel are made in Yourgrau [27, pp. 394–5].

Note added in proof. In correspondence, Tait states that the principal concern of note 3 of [23], just discussed, was to criticize the interpretation, expressed by Paul Benacerraf, of Gödel as postulating a faculty with which we "interact" with mathematical objects. I have expressed agreement with Tait's rejection of this interpretation.

Tait raises the question what Gödel means by "the sense which I explained first" in the passage from **193?* quoted on p. 50 above. This is not completely clear, but it is very probable that Gödel refers to the discussion at the beginning of the lecture of Hilbert's conviction of the solvability of every well posed mathematical problem and intends that a proposition undecidable in this sense would contradict Hilbert's conviction.

Some additional confirmation of my reading of this passage and that from *1938* quoted on p. 49 is offered by the remark, in a 1939 Göttingen lecture, that "it is very plausible that with A [i.e., $V = L$] one is dealing with an absolutely undecidable proposition, on which set theory bifurcates into two different systems, similar to Euclidean and non-Euclidean geometry" (**1939b*, p. 155).

To return to Gödel, after saying that intuition doesn't have to be "conceived of as ... giving an *immediate* knowledge of the objects concerned" he says that "as in the case of physical experience we *form* our ideas of these objects on the basis of something else which *is* immediately given." It is clear that Gödel intends to say that in the case of physical experience something other than sensations is "immediately given." Here I think he doesn't mean what most analytic philosophers of today (and also, it should be noted, Edmund Husserl) would say, that in some sense real *objects* are immediately given (to the extent that it is appropriate at all to talk of the "given"). The picture resembles Kant's, for whom knowledge of objects has as "components" a priori intuition and concepts. It is, to be sure, un-Kantian to think of pure concepts as given, immediately or otherwise. But Gödel's picture seems clearly to be that our conceptions of physical objects have to be constructed from elements, call them primitives, that are given, and that some of them (whether or not they are much like Kant's categories) must be abstract and conceptual.

Gödel says, "Evidently the 'given' underlying mathematics is related to the abstract elements contained in our empirical ideas." But the only elaboration of this statement is the remark (footnote 40) that the concept of set, like Kant's categories, has as its function " 'synthesis', i.e., the generating of unities out of manifolds."

Anyway, the general idea is that at the foundation of our conceptions of the physical world *and* of mathematics are certain "abstract elements" which appear to be primitive concepts. So far Gödel is in very rough agreement with Kant. What he mysteriously calls "another kind of relationship between ourselves and reality" (than the causal, manifested in the action of bodies on our sense organs) either consists of, or would account for, the fact that these elements represent reality objectively. They are not "purely subjective, as Kant asserted." Gödel does not offer an interpretation of Kant's transcendental idealism, but it is pretty clear he means to reject it. But in talking of primitive concepts that are not subjective in Kant's sense, whatever that is, Gödel may be following the inspiration of Leibniz.

We should not forget that the concept of intuition is not the basis for Gödel's entire story about mathematical knowledge, since he holds that mathematical axioms can have an a posteriori justification through their consequences. He does not do very much to bring this and the more direct evidence of set-theoretic axioms together; it's as if there were two independent kinds of reasons for which one might accept them. A more holistic view seems to do more justice to the facts and seems even to underlie Gödel's actual argument about the continuum problem. Then one would contrast the more ground-level intuitions (logical inference and elementary arithmetic) with the more theoretical ones. There is a process of mutual adjustment of these. In mathematical practice there are also many "middle-level" intuitions, persuasive propositions about how things should turn out that no one would claim to be evident in themselves or would seriously propose as axioms for a fundamental theory such as set theory.

One can see where Gödel's conception is perhaps eccentric, or at least controversial, by comparing it with another account of rational justification, suggested by John Rawls's views concerning moral and political theories. On this account what would be most properly called "intuitions" are what he calls our "considered judgments" at lower levels, in the moral and political case concrete moral judgments, in particular concerning the justice of social arrangements.[45] Then one constructs theories. Theories may have intrinsic plausibility in their own right and may be defended on theoretical grounds against rivals. But an ineliminable part of their justification is that they yield our considered judgments, or, more likely, a corrected version of them. (If a theory tells us that these judgments are wrong, it explains why they are wrong.) But the process of mutual adjustment of theory and concrete judgment is a dialectical one, which might go through a number of back-and-forth steps. Ideally at least, this process ends in Rawlsian "reflective equilibrium." This view is in two ways more nuanced than Gödel's. First, it allows for a distinction between the kind of intrinsic plausibility possessed by ground-level judgments and that of high-level theoretical principles, and the intrinsic plausibility of the latter is not thought of as analogous to perception. Gödel, in talking about set theory, describes both as instances of intuition and closely analogous to perception. Second, with respect to more theoretical principles, it makes clear that integral to their justification is both their intrinsic plausibility and their ability to yield consequences that square with low-level intuition. It may be that that was Gödel's underlying view, but it hardly receives emphasis when he talks about these matters.

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[45] Rawls compares them to the intuitions of a native speaker concerning his language; he sees an analogy between their role and that of speakers' intuitions in a theory of grammar. See Rawls [15, p 47]. In later writings Rawls says that the considered judgments that are relevant are at all levels of generality ([16, p. 8]; [17, p. 8, 28]). Then the comparison with speakers' intuitions is less apt. The view sketched in the text (which still seems to me a reasonable interpretation of the position of [15]) does not take account of this aspect of Rawls's later view. In particular, I do not go into the question of the status on this view of the distinction between principles and considered judgments ([15, p. 20]).

If you have begun to think that ideas derived from Rawls offer the kind of theory of reason that the foundations of set theory require, a minimal further look at his writings, not to speak of the extensive controversial literature on them, should disabuse you. At least in later writings than *A Theory of Justice*, Rawls makes clear that the procedure of reflective equilibrium should not be expected to yield a unique theory.[46] In his later writings, Rawls seems to hold that different "comprehensive doctrines" about morality might be developed so as to achieve reflective equilibrium.[47] The nearest analogue in the philosophy of mathematics would be general philosophical and methodological views about mathematics such as constructivism or some kind or other of platonism.

§6. In conclusion, let me return to Gödel's platonism. I suggested at the beginning that the connection between it and Gödel's conception of mathematical intuition would prove not to be as intrinsic as might appear at first sight. One reason is clear: finitary mathematics, intuitionistic mathematics, and classical mathematics without the characteristic concept formations of set theory (say, what is predicative relative to the natural numbers) each have definite and coherent concepts, and Gödel does not deny intuition concerning these concepts. Indeed, part of his case for the indispensability of mathematical intuition is that attempts to reconstruct mathematics without it require taking some mathematics, at least finitary arithmetic, at face value and therefore appeal to intuition at that level. The position that finitism is the limit of what mathematical intuition underwrites may be blind concerning the obvious, and it closes the door to certain extremely natural forms of reflection (such as whatever convinces us that first-order arithmetic is consistent), but it is not logically incoherent.

A second kind of independence of the two views is that Gödel's epistemology of set theory involves not just recognizing the fact of intuition concerning the concepts and axioms, but giving credence to it. Of course he maintains that that is not just an arbitrary judgment. But he clearly admits that the concepts of higher set theory are not so clear that the claim of intuitions concerning the axioms to yield knowledge is as obvious and unquestionable as Descartes intended *intuitio* to be. Even though there is a difference here between Gödel and someone who rejects set theory or who thinks of the axioms either hypothetically or formalistically, the difference can be overestimated.

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[46] See Rawls [16, p. 9]. In [15, p. 50], he raises the question whether reflective equilibrium is unique and declines to offer a definite answer. It appears that whether a unique equilibrium is attainable depends on the particular context of application of the method.

[47] For an example see Rawls [17, pp 95–6]. Such a possibility is already mentioned in [15, p. 50] But Rawls also remarks that "the struggle for reflective equilibrium continues indefinitely, in this case as in all others" ([17, p. 97]), which counters the impression he sometimes gives that fully satisfying the demands of reason is a humanly attainable end.

There is a third point concerning Gödel's platonism that should be made. Even if we grant Gödel everything he could wish for concerning the clarity of our intuitions concerning the objects of set theory, it is far from clear that he has a case for the transcendental realism concerning these objects that he seems to adhere to, as when he says that the concepts in a mathematical proposition "form an objective reality of their own, which we cannot create or change, but only perceive and describe" (**1951*, p. 320) and that "the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false" (*1964*, pp. 263–4). The widespread impression that Gödel is not just affirming CH $\vee \neg$CH, i.e., allowing the application of the law of the excluded middle here, seems to me correct. The view he is expressing is that even if our grasp of the concept of set is not sufficiently clear to decide CH, the concepts themselves form an independent order that, as it were, guides us in developing set theory.[48]

Such a view clearly goes beyond saying that mathematical intuition is intuition concerning *truth*. In Gödel's conception, it is also the unfolding of certain concepts, and tied to a certain kind of development of the concepts. (Intuition concerning inaccessible cardinals requires a prior understanding of lower-level set theory, say ZF.) It is far from clear that it necessarily contains within itself the means of resolving certain disputes. Mathematical intuition itself doesn't tell us that there *must* be a truth of the matter on questions that intuition and other means of arriving at knowledge do not decide. It also does not tell us that given a question such as the continuum problem, it must be possible to develop our intuitions in such a way that we will arrive at principles sufficient for a solution, although Gödel's conviction appears to have been affirmative in both cases.[49]

Gödel would probably argue that unless they reflect an independent reality, we have no explanation of the convergence and the strength of the intuitions we have. It would require a lengthy exploration of foundational issues in set theory to decide whether this reply has any merit. I will only remark that it is prima facie an empirical question whether our intuitions in set theory do or do not have a high degree of convergence and strength, a question to be answered in part by investigating the actual development of set theory. To my not very expert eye, the claim I am here attributing to Gödel has not been at all decisively refuted, but the state of the subject leaves a lot of serious questions, in particular those surrounding the continuum problem which already occupied Gödel.

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[48] Gödel's position as expressed here is analogous to what Rawls calls rational intuitionism in moral theory ([17, pp. 95–6]), not surprisingly since Rawls has given Leibniz as an example of a rational intuitionist. (One cannot take for granted that Leibniz was a conceptual realist; see Mates [13, chapter 10]. Although I cannot justify this here, I believe that Leibniz still offered a model for Gödel's position.)

Still, in actual argument Gödel sometimes steps back from this position or treats it as a working hypothesis. Although I think it was a conviction of his, I doubt that it is a piece of his philosophy that he claimed to have defended at all adequately.

[49] On the second point, see Wang [24, pp. 324–325]. (Wang states ([26, p. 119]), that the passage cited (from p. 324, last line, through the end of the paragraph) was written by Gödel.) But Gödel nowhere claims that this belief itself is a deliverance of mathematical intuition.

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$\mathrm{HOD}^{L(\mathbb{R})}$ IS A CORE MODEL BELOW Θ

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In this paper we shall answer some questions in the set theory of $L(\mathbb{R})$, the universe of all sets constructible from the reals. In order to do so, we shall assume $\mathrm{AD}^{L(\mathbb{R})}$, the hypothesis that all 2-person games of perfect information on ω whose payoff set is in $L(\mathbb{R})$ are determined. This is by now standard practice. ZFC itself decides few questions in the set theory of $L(\mathbb{R})$, and for reasons we cannot discuss here, ZFC + $\mathrm{AD}^{L(\mathbb{R})}$ yields the most interesting "completion" of the ZFC-theory of $L(\mathbb{R})$.

$\mathrm{AD}^{L(\mathbb{R})}$ implies that $L(\mathbb{R})$ satisfies "every wellordered set of reals is countable", so that the axiom of choice fails in $L(\mathbb{R})$. Nevertheless, there is a natural inner model of $L(\mathbb{R})$, namely $\mathrm{HOD}^{L(\mathbb{R})}$, which satisfies ZFC. (HOD is the class of all hereditarily ordinal definable sets, that is, the class of all sets x such that every member of the transitive closure of x is definable over the universe from ordinal parameters (i.e., "OD"). The superscript "$L(\mathbb{R})$" indicates, here and below, that the notion in question is to be interpreted in $L(\mathbb{R})$.) $\mathrm{HOD}^{L(\mathbb{R})}$ is reasonably close to the full $L(\mathbb{R})$, in ways we shall make precise in §1. The most important of the questions we shall answer concern $\mathrm{HOD}^{L(\mathbb{R})}$: what is its first order theory, and in particular, does it satisfy GCH?

These questions first drew attention in the 70's and early 80's. (See [4, p. 223]; also [12, p. 573] for variants involving finer notions of definability.) During that period, descriptive set theorists proved a number of interesting results concerning $\mathrm{HOD}^{L(\mathbb{R})}$, some of which we shall enumerate in §1 below. Nevertheless, the GCH in $\mathrm{HOD}^{L(\mathbb{R})}$, as well as other basic questions of combinatorial set theory lurking beyond it, remained open. Our work explains why: the methods of "neoclassical" descriptive set theory (as represented, for example, by Moschovakis' book [12]) could not yield the key first-order property of $\mathrm{HOD}^{L(\mathbb{R})}$. This property is that, below $\Theta^{L(\mathbb{R})}$ anyway, $\mathrm{HOD}^{L(\mathbb{R})}$ is a *core model*. (Here $\Theta =_{\text{df.}} \sup\{\alpha \mid \exists f : \mathbb{R} \overset{\text{onto}}{\longrightarrow} \alpha\}$.) One needs the recent progress in relating large cardinals and determinacy ([9], [15], [14]) and the advances in core model theory which have accompanied it ([10], [11], [13]) in order to formulate precisely and prove this key fact. In the end, it is

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not $\mathrm{AD}^{L(\mathbb{R})}$, but an existence assertion about mice to which it is equivalent, which is most relevant.

It is part of the basic theory of core models that they satisfy GCH, and so indeed $\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{GCH}$. More than that, the powerful fine-structure theory of core models, based on condensation and comparison arguments, applies to the study of $\mathrm{HOD}^{L(\mathbb{R})}$.

Our work also solves some other problems from the neoclassical period. We shall show that $\mathrm{HOD}^{L(\mathbb{R})}$ is, in a loose sense, constructed by iteration from countable mice. From this construction, by well-known methods, we get that if $\kappa < \Theta^{L(\mathbb{R})}$ and $\mathrm{HOD}^{L(\mathbb{R})} \models \kappa$ is regular, then exactly one of the following holds: (1) $\mathrm{HOD}^{L(\mathbb{R})} \models \kappa$ is measurable, (2) $\mathrm{cof}^{L(\mathbb{R})}(\kappa) = \omega$. In particular, if $\kappa < \Theta^{L(\mathbb{R})}$ is a successor cardinal of $\mathrm{HOD}^{L(\mathbb{R})}$, then $\mathrm{cof}^{L(\mathbb{R})}(\kappa) = \omega$ (and so κ is not a cardinal of $L(\mathbb{R})$). Relativizing this to $\mathrm{HOD}^{L(\mathbb{R})}(x)$ for $x \in \mathbb{R}$, we get the "boldface GCH": in $L(\mathbb{R})$, every wellordered family of distinct subsets of κ, where $\kappa < \Theta^{L(\mathbb{R})}$, has size $\leq \kappa$. Finally, turning to $\neg(2) \Rightarrow (1)$ of the dichotomy above, and relativizing it to arbitrary $x \in \mathbb{R}$, we can show that in $L(\mathbb{R})$, every regular $\kappa < \Theta^{L(\mathbb{R})}$ is measurable. (Cf. [12, p. 221].) It is perhaps worth noting that these last two corollaries are statements about $L(\mathbb{R})$ which, at least superficially, do not involve $\mathrm{HOD}^{L(\mathbb{R})}$ at all.

Some of the results above (e.g., the "all regulars are measurable" result) we only proved from $\mathrm{AD}^{L(\mathbb{R})}$ for $\kappa < (\delta^2_1)^{L(\mathbb{R})}$. Our proof for $\kappa < \Theta^{L(\mathbb{R})}$ required a slightly stronger hypothesis. H. Woodin later found the proof of the full results quoted above in these cases.

Our work adapts to finer notions of definability. For example, H_{2n+1} is a core model below δ^1_{2n+1}, and so $H_{2n+1} \models \mathrm{GCH}$. (Here

$$\delta^1_{2n+1} =_{\mathrm{df}} \sup\{\alpha \mid \exists f (f : \mathbb{R} \xrightarrow{\text{onto}} \alpha\} \text{ and } \{(x,y) \mid f(x) \leq f(y)\} \text{ is } \Delta^1_{2n+1})\}.$$

The model H_{2n+1} is a fragment of $\mathrm{HOD}^{L(\mathbb{R})}$ corresponding, roughly, to Σ^1_{2n+2} definability from ordinals $< \delta^1_{2n+1}$. cf. [12, p. 573].) In the other direction, Woodin has extended our results to models of the form HOD^M, for certain $M \models \mathrm{AD}$ which are larger than $L(\mathbb{R})$. There are interesting open questions in both directions. Looking downward: is it true that for all $x \in \mathbb{R}$ of sufficiently large Turing degree, $\mathrm{HOD}^{L[x]} \models \mathrm{GCH}$? Looking upward: let M be the minimal model of $\mathrm{AD}_{\mathbb{R}}$ containing all ordinals and reals; does HOD^M satisfy GCH?

In §1 we outline some older work on $\mathrm{HOD}^{L(\mathbb{R})}$. In §2 and §3 we sketch a proof that $\mathrm{HOD}^{L(\mathbb{R})}$ is a core model below $\Theta^{L(\mathbb{R})}$. We have endeavored to make this sketch accessible, and even useful, to the reader who has only a loose grasp of the basics of descriptive set theory and core model theory. On the other hand, the reader for whom a mouse is simply a rodent should not bother with §2 and §3.

§1. Older results about HOD$^{L(\mathbb{R})}$. We shall be discussing facts about HOD$^{L(\mathbb{R})}$ which are expressible in the first order language of $L(\mathbb{R})$, and provable in ZFC + AD$^{L(\mathbb{R})}$. For notational convenience, we "live inside $L(\mathbb{R})$" throughout this section; that is, we assume ZF + AD + $V = L(\mathbb{R})$ throughout.

Since every wellordered set of reals is countable, $\mathbb{R} \cap$ HOD is countable; nevertheless, HOD is close to V. One manifestation of this is Vopenka's theorem ([5]), according to which every set of ordinals, and hence every real, is generic over HOD. In fact, many interesting reals are OD, and thus in HOD. Among the OD sets of reals only the countable ones are literally in HOD; however, Woodin has shown that every OD set of reals has an ∞-Borel code in HOD. (See [14]. An ∞-Borel set of reals is one which is constructed from basic open sets via repeated wellordered union and complementation; an ∞-Borel code is a set of ordinals naturally coding such a construction.) In related work, Woodin showed that HOD $= L[P]$ for some $P \subseteq \Theta$.

An ∞-Borel code of A gives an effective way of answering membership questions about A, but no way of producing members of A. For example, $\mathbb{R} \backslash$HOD has an ∞-Borel code in HOD, but no member in HOD. A *Suslin representation* of A gives more information. Let T be a tree on $\omega \times \kappa$ for some ordinal κ; we say T is a Suslin representation of

$$p[T] = \{x \in {}^{\omega}\omega \mid \exists f \in {}^{\omega}\kappa \forall n (x \restriction n, f \restriction n) \in T\}.$$

Notice that for such a tree T, if $p[T] \neq \emptyset$, then $p[T] \cap L[T] \neq \emptyset$. It follows that $\mathbb{R} \backslash$HOD is an OD (in fact $\Pi_1^{L(\mathbb{R})}$) set of reals having no Suslin representation in HOD. The paper [8] shows, however, that every $\Sigma_1^{L(\mathbb{R})}$ set of reals has a Suslin representation in HOD. This can be used to compute the correctness of HOD: any Σ_1 sentence true in $L(\mathbb{R})$ is true in $L(\mathbb{R})^{\text{HOD}}$. (But $L(\mathbb{R})^{\text{HOD}}$ satisfies "There is a wellorder of $\mathbb{R}$", a Σ_1 sentence which is not true in $L(\mathbb{R})$.)

Actually, [8] gives a Suslin representation T for any complete $\Sigma_1^{L(\mathbb{R})}$ set of reals which is the "tree of a $\Sigma_1^{L(\mathbb{R})}$ scale". Such a T has the following useful properties. First, T is on $\omega \times \delta_1^2$, where δ_1^2 is the least ordinal σ such that $L_\sigma(\mathbb{R})$ is a Σ_1 elementary substructure of $L(\mathbb{R})$. (No smaller T is posssible.) Second, for each $x \in p[T]$, let ℓ_x be the minimal witness that $x \in p[T]$; that is, let $\ell_x(n) =$ least α such that $(\exists f \supseteq \ell_x \restriction n^\frown\langle\alpha\rangle)\forall i((x \restriction i, f \restriction i) \in T)$. Then $\forall \alpha < \delta_1^2 \exists x \in p[T](\ell_x(0) = \alpha)$; moreover, the relation "$x, y \in p[T]$ and $\ell_x(0) < \ell_y(0)$" is $\Sigma_1^{L(\mathbb{R})}$.

According to the following theorem of Becker and Kechris, all of HOD below δ_1^2 is coded into any such tree.

THEOREM (Becker, Kechris; cf. [2]). *Let T be the tree of a $\Sigma_1^{(\mathbb{R})}$ scale on a complete $\Sigma_1^{L(\mathbb{R})}$ set; then $V_{\delta_1^2}^{HOD} = V_{\delta_1^2}^{L[T]}$.*

The proof we give in §3 that HOD is a core model uses this theorem. We shall construct a core model $M \subseteq \mathrm{HOD}$ such that for some tree T of a $\Sigma_1^{L(\mathbb{R})}$ scale on a complete $\Sigma_1^{L(\mathbb{R})}$ set, $T \in M$. The Becker-Kechris theorem then implies $V_{\delta_1^2}^{\mathrm{HOD}} = V_{\delta_1^2}^M$, so that HOD is a core model "below δ_1^2".

HOD is also large in that it satisfies certain large cardinal hypotheses. The roots of this are Solovay's method of constructing ultrafilters using AD, and Kunen's theorem that every ultrafilter on an ordinal $\kappa < \Theta$ is OD ([7]). Becker, Martin, Moschovakis, and the author used these ideas to show that HOD has respectably large cardinals. (For example, say that κ is β-strong if there is an elementary $j : V \to M$ with critical point κ such that $V_\beta \subseteq M$. The author showed δ_1^2 is β-strong in HOD, where β is the least measurable cardinal $> \delta_1^2$ in HOD.) Woodin found a far reaching and definitive extension of this work.

THEOREM (Woodin, [14]). *The following are true in HOD, for* $\delta = (\delta_1^2)^{L(\mathbb{R})}$ *and* $\theta = \Theta^{L(\mathbb{R})}$,

(1) $\forall \beta < \theta$ (δ *is* β*-strong*),
(2) θ *is Woodin.*

The definition of "Woodin cardinal" can be found, for example, in [9].

Like the Becker-Kechris theorem, this result helped motivate our work. Notice that by (1), if HOD is a core model below δ_1^2, then HOD is a core model below Θ.

Becker [3] showed that ω_1^V is the least measurable cardinal of HOD. Woodin (unpublished) showed that δ_1^2 is the least κ which is β-strong in HOD for all $\beta < \Theta$. These results also give some hints as to how to realize HOD as a core model.

Concerning the GCH in HOD, Becker proved the following.

THEOREM (Becker [1]). *Suppose that* $\alpha \leq \omega_1^V$ *or* α *is reliable; then* $\mathrm{HOD} \models 2^\alpha = \alpha^+$.

Adopting a variant of Becker's definition, we say α is *reliable* iff there is a tree T on $\omega \times \alpha$ such that $\forall \beta < \alpha \exists x \in p[T](\ell_x(0) = \beta)$, and the relation "$x, y \in p[T]$ and $\ell_x(0) < \ell_y(0)$" has a Suslin representation. Thus δ_1^2 is reliable. If α is reliable, then $\alpha \leq \delta_1^2$; moreover, there is a closed unbounded set of reliable cardinals $\alpha < \delta_1^2$, as well as an unbounded set of unreliable cardinals $\alpha < \delta_1^2$. Woodin (unpublished) has shown that if α is a reliable cardinal, then no $\kappa < \alpha$ is α-strong in HOD. Conversely, our work here shows that if $\alpha < \Theta^{L(\mathbb{R})}$ is a cardinal such that no $\kappa < \alpha$ is α-strong in HOD, then α is reliable. This then characterizes the reliable cardinals in terms of the large cardinal structure of HOD.

The most immediate inspiration for our work was the work of H. Woodin and the author described in §4 of [13]. It is shown there that if M_ω is the minimal iterable proper class model with ω Woodin cardinals, and N is the result of linearly iterating M_ω so as to send its least measurable cardinal out to ω_1^V, then N and HOD agree on sets of rank $\leq \omega_1^V$. This re-proves Becker's result that ω_1^V is the least measurable of HOD, and also re-proves Becker's result on the GCH below ω_1^V. (It is amusing to note that this also re-proves Solovay's theorem that every $A \subseteq \omega_1^V$ is constructible from a real. For $A \in \mathrm{HOD}(x)$ for some real x, and the results of §4 of [13], relativised to x, then show A is constructible from $M_\omega(x)$.) The results of §4 of [13] clearly suggest that the whole of HOD might be an iterate of M_ω, and it was the attempt to show this which led to the present work.

§2. Some core model theory. This section contains a sketchy summary of some of the core model theory we shall need. All lies are white. The reader can find a full and true account in [11] and [13]. He can find a brief history and overview of the subject in [6], or in the introduction to [10].

Let M be a transitive model; then a (κ, ν) *extender over* $\mathcal{M}$ is a system of $\mathcal{M}$-ultrafilters coding an embedding $i_E : \mathcal{M} \to \mathcal{N}$ such that $\kappa = \mathrm{crit}(i_E)$ and $\nu < i_E(\kappa)$. We write $\kappa = \mathrm{crit}(E)$, $\nu = lh(E)$, and $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, E)$. A sequence $\vec{E} = \langle E_\alpha \mid \alpha < \tau\rangle$ of extenders is *coherent* if for all $\alpha < \tau$,

(1) E_α is an extender over $L[\langle E_\beta \mid \beta < \alpha\rangle]$,
(2) $\beta < \alpha \Rightarrow lh(E_\beta) < lh(E_\alpha)$, and
(3) $i_{E_\alpha}(\vec{E}) \restriction \alpha + 1 = \vec{E} \restriction \alpha$.

One can think of a coherent sequence as one in which the extenders are listed in order of increasing strength, without leaving gaps. A *premouse* is a structure $\mathcal{M} = (J_\alpha^{\vec{E}}, \in, \vec{E})$, where $\vec{E}$ is coherent; we call $\vec{E}$ the $\mathcal{M}$-sequence, and for $\beta < \alpha$ we set $\mathcal{J}_\beta^{\mathcal{M}} = (J_\beta^{\vec{E}\restriction\gamma}, \in, \vec{E} \restriction \gamma)$, where $\gamma = \{\eta \mid lh(E_\eta) \leq \omega\beta\}$.

Let $\mathcal{M}$ be a premouse and τ an ordinal. The *full iteration game on* $\mathcal{M}$ *of length* τ, or $\mathcal{G}(\mathcal{M}, \tau)$, is played as follows. Players I and II cooperatively build an iteration tree $\mathcal{T}$ on $\mathcal{M} = \mathcal{M}_0^{\mathcal{T}}$. Player I moves at successor stages $\alpha + 1 < \tau$ by choosing an extender $E_\alpha^{\mathcal{T}}$ from the $\mathcal{M}_\alpha^{\mathcal{T}}$-sequence and a $\beta \leq \alpha$ such that

$$\forall \gamma (\beta \leq \gamma < \alpha \Rightarrow \mathrm{crit}(E_\alpha^{\mathcal{T}}) < lh(E_\gamma^{\mathcal{T}})).$$

We appoint β the $\mathcal{T}$ predecessor of $\alpha + 1$, and set

$$\begin{aligned}\mathcal{M}_{\alpha+1}^{\mathcal{T}} \;&=\; \mathrm{Ult}(\mathcal{P}, E_\alpha^{\mathcal{T}}), \text{ where } \mathcal{P} = \mathcal{J}_\gamma^{\mathcal{M}_\beta^{\mathcal{T}}} \text{ for the largest } \gamma \\ &\quad\;\; \text{such that } E_\alpha^{\mathcal{T}} \text{ is an extender over } \mathcal{J}_\gamma^{\mathcal{M}_\beta^{\mathcal{T}}}.\end{aligned}$$

Player I wins, and the game ends, if $\mathcal{M}_{\alpha+1}^{\mathcal{T}}$ is illfounded. We say $\mathcal{T}$ *drops at* $\alpha + 1$, and put $\alpha + 1 \in D^{\mathcal{T}}$, if $\mathcal{P} \neq \mathcal{M}_\beta^{\mathcal{T}}$ in the above. If $\mathcal{P} = \mathcal{M}_\beta^{\mathcal{T}}$, then we let $i_{\beta,\alpha+1}^{\mathcal{T}} : \mathcal{M}_\beta^{\mathcal{T}} \to \mathcal{M}_\alpha^{\mathcal{T}}$ be the canonical embedding. At limit stages $\lambda < \tau$, Player II must pick a cofinal wellfounded branch of the tree $\mathcal{T}$ built so far.

This is a cofinal-in-λ set $b \subseteq \lambda$ which is a branch of $\mathcal{T}$, such that $D^{\mathcal{T}} \cap b$ is finite, and such that $\mathcal{M}_b^{\mathcal{T}}$ is wellfounded, where $\mathcal{M}_b^{\mathcal{T}}$ is the direct limit of the $\mathcal{M}_\alpha^{\mathcal{T}}$ for $\alpha \in b - \sup(D^{\mathcal{T}} \cap b)$ under the $i_{\alpha\gamma}^{\mathcal{T}}$'s. If II fails to do this, I wins and the game ends. Otherwise, we let $\mathcal{M}_\lambda^{\mathcal{T}} = \mathcal{M}_b^{\mathcal{T}}$, let $i_{\alpha\lambda}^{\mathcal{T}}$ be the canonical embedding for $\alpha \in b - \sup(D^{\mathcal{T}} \cap b)$, and continue. If I does not win at some stage $< \tau$, then II wins.

An *iteration tree* on $\mathcal{M}$ is just a run $\mathcal{T}$ of some $\mathcal{G}(\mathcal{M}, \tau)$ in which II has not lost. A τ*-iteration strategy* for $\mathcal{M}$ is a winning strategy for II in $\mathcal{G}(\mathcal{M}, \tau)$. We say $\mathcal{M}$ is τ*-iterable* just in case there is a τ-iteration strategy for $\mathcal{M}$.

An iteration tree $\mathcal{T}$ is normal iff $lh(E_\beta^{\mathcal{T}}) < lh(E_\gamma^{\mathcal{T}})$ whenever $\beta < \gamma$, and the $\mathcal{T}$-predecessor of $\alpha + 1$ is the least β such that $\text{crit}(E_\alpha^{\mathcal{T}}) < lh(E_\beta^{\mathcal{T}})$, for all α. We say $\mathcal{T}$ is *almost normal* if it is a linear composition of normal trees. We require no more than almost normal iteration trees in this paper.

The importance of being iterable resides in the following lemma and its proof. Let us say that δ is a *cutpoint of* $\mathcal{M}$ iff for all E on the $\mathcal{M}$-sequence, $\text{crit}(E) \leq \delta \Rightarrow lh(E) < \delta$. We write $\mathcal{N} \trianglelefteq \mathcal{M}$ ("$\mathcal{N}$ is an initial segment of $\mathcal{M}$") iff $\mathcal{N} = \mathcal{J}_\delta^{\mathcal{M}}$ for some cutpoint δ of $\mathcal{M}$.

COMPARISON LEMMA. *Let Σ and Γ be $\tau + 1$-iteration strategies for $\mathcal{M}$ and $\mathcal{N}$ respectively, where $\tau = \max(|\mathcal{M}|, |\mathcal{N}|)^+$. Then there are normal iteration trees $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}$, played according to Σ and Γ and having last models $\mathcal{P}$ and $\mathcal{Q}$ respectively, such that $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{P}$. Moreover, if $\mathcal{P} \trianglelefteq \mathcal{Q}$ then the branch of $\mathcal{T}$ leading from $\mathcal{M}$ to $\mathcal{P}$ has no drops, and if $\mathcal{Q} \trianglelefteq \mathcal{P}$ then the branch of $\mathcal{U}$ from $\mathcal{N}$ to $\mathcal{Q}$ has no drops.*

§3. The proof of the main result. For simplicity, we shall outline a proof that $\text{HOD}^{L(\mathbb{R})}$ is a core model below $(\delta_1^2)^{L(\mathbb{R})}$ which uses a mouse-existence hypothesis slightly stronger than $\text{AD}^{L(\mathbb{R})}$. The proof which uses only $\text{AD}^{L(\mathbb{R})}$ is similar.

The idea of our proof is that $\text{HOD}^{L(\mathbb{R})}$ is the direct limit of a certain class $\mathcal{F}$ of countable, iterable mice, under the iteration maps given by the comparison process. The mice in $\mathcal{F}$ are "small" enough that they have $\omega_1 + 1$-iteration strategies in $L(\mathbb{R})$. They are as "full" as possible, granted this smallness condition. (One gets a typical member of $\mathcal{F}$ by iterating M_ω, the minimal iterable class-size premouse with ω Woodin cardinals, then cutting the iterate off at a successor cardinal below its bottom Woodin cardinal.)

DEFINITION 3.1. Let $\mathcal{M}$ be a premouse; then $\mathcal{M}$ is ω*-small* iff whenever $\kappa = \text{crit}(E)$ for some E on the $\mathcal{M}$-sequence, then $\mathcal{J}_\kappa^{\mathcal{M}} \not\models$ "There are ω Woodin cardinals". $\mathcal{M}$ is *properly small* iff $\mathcal{M}$ is ω-small, and $\mathcal{M}$ satisfies ZF^-+ "There are no Woodin cardinals" + "There is a largest cardinal".

If $\mathcal{M}$ is ω-small, so is any $\mathcal{J}_\beta^{\mathcal{M}}$; on the other hand, $\mathcal{M}$ may be properly small, and yet for some β, $\mathcal{J}_\beta^{\mathcal{M}} \models$ "There are ω Woodin cardinals".

It is easy to see that if $\mathcal{M}$ is countable and $L(\mathbb{R}) \models \Sigma$ is an ω_1-iteration strategy for $\mathcal{M}$, then Σ really is an ω_1-iteration strategy for $\mathcal{M}$. We have a converse for properly small $\mathcal{M}$:

THEOREM 3.2. *Assume ZFC+ AD$^{L(\mathbb{R})}$. Let $\mathcal{M}$ be countable, properly small, and ω_1-iterable (in V). Then*

(a) *$L(\mathbb{R}) \models \mathcal{M}$ is $\omega_1 + 1$-iterable,*

(b) *every countable iteration tree $\mathcal{T}$ on $\mathcal{M}$ has at most one cofinal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ is ω_1-iterable.*

We sketch a proof of 3.2 for the reader familiar with [13]. Since $\mathcal{M}$ is ω_1-iterable, II wins the weak iteration game $W\mathcal{G}(\mathcal{M},\omega)$. This is a $\mathrm{G}^{\mathbb{R}}\Pi^1_1$ assertion about $\mathcal{M}$, hence absolute to $L(\mathbb{R})$. (See [8].) But [13] shows that this weak iterability is enough to compare properly small mice. An argument from [10, §5] then shows that, in $L(\mathbb{R})$, $\mathcal{M}$ is ω_1-iterable via the strategy of choosing the unique cofinal branch b such that II wins $W\mathcal{G}(\mathcal{M}_b^{\mathcal{T}},\omega)$. This (unique) ω_1-iteration strategy for $\mathcal{M}$ extends to what is, in $L(\mathbb{R})$, an $\omega_1 + 1$ iteration strategy for $\mathcal{M}$. (At stage ω_1, choose the unique cofinal b such that in $L(\mathbb{R})^{\mathrm{Col}(\omega,\omega_1)}$, II wins $W\mathcal{G}(\mathcal{M}_b^{\mathcal{T}},\omega)$. AD$^{L(\mathbb{R})}$ implies there is a generic $j : L(\mathbb{R}) \to L(\mathbb{R})^{\mathrm{Col}(\omega,\omega_1)}$, by an unpublished result of Woodin, hence there is such a b in $L(\mathbb{R})^{\mathrm{Col}(\omega,\omega_1)}$. There is such a b in $L(\mathbb{R})$ by the homogeneity of $\mathrm{Col}(\omega,\omega_1)$. This is where we use AD$^{L(\mathbb{R})}$.)

By 3.2, a countable, properly small, ω_1-iterable $\mathcal{M}$ has exactly one ω_1-iteration strategy. We call countable tree $\mathcal{T}$ on $\mathcal{M}$ *correct* iff $\mathcal{T}$ is played according to this strategy. Equivalently, $\mathcal{T}$ is correct iff every model $\mathcal{M}_\alpha^{\mathcal{T}}$ on $\mathcal{T}$ is ω_1-iterable. By 3.2, correctness is a $\Sigma_1^{L(\mathbb{R})}$ property. We say that $\mathcal{M}$ iterates correctly to $\mathcal{N}$ iff there is an almost normal, correct $\mathcal{T}$ on $\mathcal{M}$ with last model $\mathcal{N}$, and such that the branch $\mathcal{M}$-to-$\mathcal{N}$ of $\mathcal{T}$ has no drops. There may in fact be more than one iteration tree witnessing that $\mathcal{M}$ iterates correctly to $\mathcal{N}$, but by the Dodd-Jensen lemma, any two such trees determine the same iteration map $\pi : \mathcal{M} \to \mathcal{N}$. Because properly small $\mathcal{M}$ satisfy ZF$^-$, π is fully elementary.

DEFINITION 3.3. A countable, properly small premouse $\mathcal{M}$ is *full* iff whenever $\mathcal{M}$ iterates correctly to $\mathcal{P}$, A is a bounded subset of $\mathrm{OR} \cap \mathcal{P}$, and A is $\mathrm{OD}^{L(\mathbb{R})}(\{\mathcal{P}\})$, then $A \in \mathcal{P}$.

Fullness is a $\Pi_1^{L(\mathbb{R})}$ property. One can show (cf. [13, §4]) that $\mathcal{M}$ is full iff whenever $\mathcal{M}$ iterates correctly to $\mathcal{P}$, and $\mathcal{P} \trianglelefteq \mathcal{Q}$ where $\mathcal{Q}$ is ω-small and ω_1-iterable, then $\rho_\omega(\mathcal{Q}) \geq \mathrm{OR} \cap \mathcal{P}$.

We can now define our direct limit system. Set

$$\mathcal{F} = \{\mathcal{M} \mid \mathcal{M} \text{ is countable, properly small, } \omega_1\text{-iterable, and full}\},$$

and for $\mathcal{M}, \mathcal{N} \in \mathcal{F}$, let $\mathcal{M} \prec^* \mathcal{N}$ iff $\exists \mathcal{P}\ (\mathcal{M}$ iterates correctly to $\mathcal{P} \wedge \mathcal{P} \trianglelefteq \mathcal{N})$. The Dodd-Jensen Lemma implies that if $\mathcal{M} \prec^* \mathcal{N}$, then there is exactly one

$\mathcal{P}$ such that $\mathcal{P} \trianglelefteq \mathcal{N}$ and $\mathcal{M}$ iterates correctly to $\mathcal{P}$, and exactly one iteration map from $\mathcal{M}$ to $\mathcal{P}$. We call this map $\pi_{\mathcal{M},\mathcal{N}}$. Clearly $\mathcal{F}$, $\prec^*$, and the function $(\mathcal{M},\mathcal{N}) \mapsto \pi_{\mathcal{M},\mathcal{N}}$ are $\mathrm{OD}^{L(\mathbb{R})}$.

If δ is a cutpoint of $\mathcal{N}$, then any normal iteration tree on $\mathcal{N}$ decomposes into an iteration tree $\mathcal{T}$ on $\mathcal{J}_\delta^{\mathcal{N}}$ with iteration map σ, followed by an iteration tree $\mathcal{U}$ on the last model of $\mathcal{T}$, with all extenders $E_\alpha^{\mathcal{U}}$ used in $\mathcal{U}$ satisfying $\sigma(\delta) < \mathrm{crit}(E_\alpha^{\mathcal{U}})$. This fact easily implies that $\prec^*$ is transitive, and that if $\mathcal{M} \prec^* \mathcal{N} \prec^* \mathcal{P}$, then $\pi_{\mathcal{M},\mathcal{P}} = \pi_{\mathcal{N},\mathcal{P}} \circ \pi_{\mathcal{M},\mathcal{N}}$.

The comparison lemma and fullness imply that $\prec^*$ is directed. For suppose $\mathcal{M},\mathcal{N} \in \mathcal{F}$, and let $\mathcal{T}$ and $\mathcal{U}$ be the correct trees given by the comparison lemma. Let $\mathcal{P}$ and $\mathcal{Q}$ be their respective last models, and suppose e.g., $\mathcal{P} \trianglelefteq \mathcal{Q}$. The comparison lemma implies $\mathcal{M}$-to-$\mathcal{P}$ has no drops, so $\mathcal{M}$ iterates correctly to $\mathcal{P}$. But $\mathcal{M}$ is full, so $\rho_\omega(\mathcal{Q}) \geq \mathrm{OR} \cap \mathcal{P}$. This implies that $\mathcal{N}$-to-$\mathcal{Q}$ has no drops, so $\mathcal{N}$ iterates correctly to $\mathcal{Q}$. Thus $\mathcal{M} \prec^* \mathcal{Q}$ and $\mathcal{N} \prec^* \mathcal{Q}$. A little more work shows $\prec^*$ is countably directed.

Now let $\mathcal{M}_\infty =$ direct limit of $(\mathcal{F}, \prec^*)$ under the $\pi_{\mathcal{M},\mathcal{N}}$'s or rather, let $\mathcal{M}_\infty$ be the transitive collapse of the direct limit, which exists since $\prec^*$ is countably directed. For $\mathcal{M} \in \mathcal{F}$, let $\pi_{\mathcal{M},\infty} : \mathcal{M} \to \mathcal{M}_\infty$ be the canonical embedding. Clearly, $\mathcal{M}_\infty$ is a premouse, and $\mathcal{M}_\infty \in \mathrm{HOD}^{L(\mathbb{R})}$.

A direct computation shows that $\mathrm{OR} \cap \mathcal{M}_\infty \leq (\delta_1^2)^{L(\mathbb{R})}$. For let $\alpha \in \mathrm{OR} \cap \mathcal{M}_\infty$, and fix $\mathcal{M} \in \mathcal{F}$ so that $\pi_{\mathcal{M},\infty}(\bar\alpha) = \alpha$ for some $\bar\alpha$. Let $\mathcal{G} = \{\mathcal{P} \mid \mathcal{M}$ iterates correctly to $\mathcal{P}\}$. Then $\mathcal{G} \subseteq \mathcal{F}$, and one can easily check that $\mathcal{G}$ is $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$. Also, the relation R is $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$, where $R(\langle \mathcal{P}, \bar\beta\rangle, \langle \mathcal{Q}, \bar\gamma\rangle)$ iff ($\mathcal{P}, \mathcal{Q} \in \mathcal{G}$ and $\bar\beta \in \mathrm{OR}^{\mathcal{P}}$ and $\bar\gamma \in \mathrm{OR}^{\mathcal{Q}}$ and $\pi_{\mathcal{P},\infty}(\bar\beta) \leq \pi_{\mathcal{Q},\infty}(\bar\gamma)$). This is because we can check whether $R(\langle \mathcal{P}, \bar\beta\rangle, \langle \mathcal{Q}, \bar\gamma\rangle)$ holds by comparing $\mathcal{P}$ with $\mathcal{Q}$. Since every $\beta < \alpha$ is of the form $\pi_{\mathcal{P},\infty}(\bar\beta)$ for some $\mathcal{P} \in \mathcal{G}$, there is a $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$ prewellorder of HC of order type at least α. Thus $\alpha < (\delta_1^2)^{L(\mathbb{R})}$.

Now suppose $\mathrm{OR} \cap \mathcal{M}_\infty < (\delta_1^2)^{L(\mathbb{R})}$. Since $\mathcal{M}_\infty$ can be coded simply by a subset of $\mathrm{OR} \cap \mathcal{M}_\infty$, we then have by the Coding Lemma ([12, Chap. 7]) that for some real x, $\mathcal{M}_\infty$ is coded by a $\Delta_1^{L(\mathbb{R})}(\{x\})$ set of reals. But, essentially via §4 of [13], one can show that the complete $\Sigma_1^{L(\mathbb{R})}$ set of reals is projective in any set of reals coding $\mathcal{M}_\infty$. (Thus the complete $\Sigma_1^{L(\mathbb{R})}$ set of reals is $\Delta_1^{L(\mathbb{R})}$ in a real, a contradiction.) The reason is that $\Sigma_1^{L(\mathbb{R})}$ truth is reducible to truth in symmetric extensions of mice. (This is where our mouse existence assumptions come in.) Call $\mathcal{M}$ a $\langle \varphi, x\rangle$ *witness*, for $\varphi(v)$ a Σ_1 formula and $x \in {}^\omega\omega$, if $\mathcal{M}$ has ω Woodin cardinals with supremum δ and for some symmetric submodel of $\mathcal{M}^{\mathrm{Col}(\omega,<\delta)}$ with set of reals R^*, we have $x \in \mathbf{R}^*$ and $L_\alpha(\mathbf{R}^*) \models \varphi[x]$ for $\alpha = \mathrm{OR} \cap \mathcal{M}$. One can show that if there is a sufficiently iterable $\langle \varphi, x\rangle$ witness, then $L(\mathbb{R}) \models \varphi[x]$. (Cf. [13, §4].) Conversely, if $L(\mathbf{R}) \models \varphi[x]$, then $\exists \mathcal{M} \in \mathcal{F}\, \exists_\beta\, (\mathcal{J}_\beta^{\mathcal{M}}$ is a $\langle \varphi, x\rangle$ witness). This last statement is the mouse existence hypothesis from which our proof proceeds. It follows from, say, the existence of ω Woodin cardinals plus a measurable above them all. (Cf. [13, §4].) It follows that for all φ, x: $L(\mathbb{R}) \models \varphi[x] \Leftrightarrow \exists\beta\exists\mathcal{M}$ ($\mathcal{M}$ is a $\langle\varphi, x\rangle$-witness and $\mathcal{M} \preceq \mathcal{J}_\beta^{\mathcal{M}_\infty}$). This shows the complete $\Sigma_1^{L(\mathbb{R})}$ set of reals is projective in any set of reals coding $\mathcal{M}_\infty$. Thus $\mathrm{OR} \cap \mathcal{M}_\infty = (\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$.

One proof that for $\alpha = (\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$, $V_\alpha \cap \mathrm{HOD}^{L(\mathbb{R})}$ is contained in $\mathcal{M}_\infty$ goes roughly as follows. Let T be the tree on $\omega \times \mathcal{M}_\infty$ of all attempts to build a pair $(x, \pi) \in^\omega \omega \times^\omega \mathcal{M}_\infty$ such that x codes a structure $\mathcal{P}_x$ for the language of premice whose universe is $\omega - \{0\}$, $\pi(0)$ is a successor-cardinal and cutpoint of $\mathcal{M}_\infty$, and $\pi \restriction (\omega - \{0\})$ is an elementary embedding from $\mathcal{P}_x$ into $\mathcal{J}_{\pi(0)}^{\mathcal{M}_\infty}$. One can show that T behaves enough like the tree of a $\Sigma_1^{L(\mathbb{R})}$ scale on a complete $\Sigma_1^{L(\mathbb{R})}$ set that the Becker-Kechris proof goes through, and shows $(V_{\boldsymbol{\delta}_1^2})^{\mathrm{HOD}^{L(\mathbb{R})}} \subseteq L[T]$. But $T \in L[\mathcal{M}_\infty]$, and one can show that $L[\mathcal{M}_\infty] \cap V_{(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}} \subseteq \mathcal{M}_\infty$.

The key properties of T for the Becker-Kechris proof are the following. First, one can compute $\Sigma_1^{L(\mathbb{R})}$ truth using $p[T]$. We have already sketched how to do thiws. Second, for any successor cardinal cutpoint δ of $\mathcal{M}_\infty$, $\exists x \in p[T](\ell_x(0) = \delta)$. To see this, pick any $\mathcal{R} \in \mathcal{F}$ and $\bar{\delta}$ such that $\pi_{\mathcal{R},\infty}(\bar{\delta}) = \delta$. Let $\mathcal{S} = \mathcal{J}_{\bar{\delta}}^{\mathcal{R}}$, and let x be such that $\mathcal{P}_x \cong \mathcal{S}$. Since $\pi_{\mathcal{S},\infty} : \mathcal{S} \to \mathcal{J}_\delta^{\mathcal{M}_\infty}$ is essentially an iteration map, the Dodd-Jensen lemma implies that $\mathcal{S}$ does not embed elementarily into any $\mathcal{J}_\gamma^{\mathcal{M}_\infty}$ with $\gamma < \delta$, so that $\ell_x(0) = \delta$. Third, fix y such that $\mathcal{P}_y \cong \mathcal{R}$ for some $\mathcal{R} \in \mathcal{F}$. Let $I(y) = \{x \in p[T] \mid \ell_x(0) \leq \ell_y(0)\}$; then $I(y)$ is $\Sigma_1^{L(\mathbb{R})}$ in y, uniformly in such y. This is because $x \in I(y)$ iff $\exists\mathcal{S}\exists\beta$ ($\mathcal{R}$ iterates correctly to $\mathcal{S}$ and $\mathcal{P}_x \preceq \mathcal{J}_\beta^{\mathcal{S}}$).

One can also show that $(V_{\boldsymbol{\delta}_1^2})^{\mathrm{HOD}^{L(\mathbb{R})}}$ is contained in $\mathcal{M}_\infty$ without using the Becker-Kechris ideas, but using instead generic elementary embeddings.

Quod erat sketchandum.

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EUROPEAN SUMMER MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

LOGIC COLLOQUIUM '93

Keele, England, July 20–29, 1993

The European Summer Meeting of the Association for Symbolic Logic was held at Keele University, Staffordshire, on 20–29 July, 1993. The program included twenty invited hour talks, two invited short courses and forty-eight contributed papers. There were also demonstrations of computerized logic teaching software, by Stephen Read, Steve Reeves and Richard Bornat. The program committee consisted of Dov Gabbay, Wilfrid Hodges (chair), Martin Hyland, Thomas Jech, David Makinson, Stephen Read, Charles Steinhorn, Anne Troelstra and John Truss. The local organizers were Jayne Beardmore, Peter Fletcher, Chris Johnson, John Stell, Alan Treherne (chair) and Roger Whitehouse from the Computer Science and Mathematics Departments at Keele.

The following invited addresses were presented at the meeting:

Avon Avron, *Calculi of hypersequents and their use.*

Andreas Baudisch, *A new uncountably categorical group*

Andreas Blass, *Ubiquitous cardinals.*

Kosta Došen, *An introduction to substructural logics.*

Randall Dougherty, *On left-distributive algebras and elementary embeddings.*

Lou van den Dries, *O-minimality and tame topology (three-hour course).*

Ronald Jensen, *An alternative to Bukovský-Namba forcing.*

D. H. J. de Jongh, *Propositional logics and theories.*

Ulrich Kohlenbach, *Analyzing proofs in analysis.*

Sabine Koppelberg, *Set theoretic questions on Boolean algebras.*

Daniel Lascar, *The reconstruction problem*

Daniel Lehmann, *Cut elimination in a nonmonotonic logic.*

Alain Louveau, *Countable Borel equivalence relations.*

Daniele Mundici, *A logical analysis of Ulam's game with lies.*

Hans Jürgen Ohlbach, *Aspects of logic engineering.*

Evgenia Rabinovich, *On structures interpretable in fields.*

Harold Simmons, *Logic and computation—taking the Curry-Howard correspondence seriously (four-hour course).*

Robert Staerk, $3 + 1 = 2 + 2$: *From logic programs to inductive definitions.*

Michiel van Lambalgen, *On a new notion of indiscernibles in set theory.*

Jaap van Oosten, *Axiomatizing realizability.*

Alex Wilkie, *On the decision problem for the real exponential field.*

Boris Zil'ber, *Model-theoretic dimensions in complex and ultrametric analysis.*

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Abstracts of contributed talks given (in person or by title) by members of the Association for Symbolic Logic follow.

For the Program Committee
WILFRID HODGES, PROGRAM CHAIR

Abstracts of contributed papers

MARAT M. ARSLANOV, *An interplay between d-r.e. and r.e. sets.*

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A set $A \subseteq \omega$ is called d-r.e. if there are r.e. sets A_1 and A_2 such that $A = A_1 \backslash A_2$. A Turing degree is called a d-r.e. degree if it contains a d-r.e. set; it is called properly d-r.e. if it is d-r.e. but not r.e.

Recently it was shown (S. B. Cooper and others) that the partial order of the d-r.e. degrees is not dense. Earlier the author proved the following statements.

THEOREM 1. *For each r.e. degree* $\mathbf{a} < \mathbf{0}'$ *there exists a properly d-r.e. degree* $\mathbf{b}$ *such that* $\mathbf{a} < \mathbf{b} < \mathbf{0}'$.

THEOREM 2. *For any high r.e. degree* $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ *there exists d-r.e. degrees* $\mathbf{b}$ *and* $\mathbf{c}$ *such that* $\mathbf{b} < \mathbf{a} < \mathbf{c}$ *and* $\mathbf{a}$ *is the only r.e. degree between* $\mathbf{b}$ *and* $\mathbf{c}$.

Studying related questions, the author proved the following two theorems.

THEOREM 3 *There is a low properly d-r.e. degree* $\mathbf{h}$ *such that for any r.e. degree* $\mathbf{a}$, *if* $\mathbf{a} < \mathbf{h}$ *then* $\mathbf{a} < \mathbf{b} < \mathbf{h}$ *for some r.e.* $\mathbf{b}$.

THEOREM 4. *There is a properly d-r.e. degree* $\mathbf{d}$ *such that for any r.e. degree* a, *if* $\mathbf{d} < \mathbf{a}$ *then* $\mathbf{d} < \mathbf{b} < \mathbf{a}$ *for some r.e. degree* $\mathbf{b}$.

JOHN T. BALDWIN AND MASANORI ITAI, *Properties of K-generic models.*

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Let $(K, \leq)$ be a pair where K is a class of finite relational structures and $\leq$, *strong submodel*, is a relation on the members of K such that $A \leq B$ implies $A \subseteq B$. If $(K, \leq)$ satisfies the amalgamation property and the joint embedding property, there exists a structure M called $(K, \leq)$-generic with the following properties:

i) every finite member of K can be strongly embedded in M,
ii) every finite substructure of M is in K,
iii) if A and B are finite strong submodels of M that are isomorphic, say by f, then there is an automorphism of M extending f.

Hrushovski constructed, as a $(K, \leq)$-generic model, a stable $\aleph_0$-categorical pseudoplane refuting a conjecture of Lachlan. He also defined, as yet another $(K, \leq)$-generic model, a strongly minimal set of a new type giving a counter example to a Zil'ber's conjecture. By generalizing the idea, Baldwin was able to define a family of almost strongly minimal, non-Desarguesian projective planes.

We study some model theoretic properties of various $(K, \leq)$-generic models, e.g., determining their Morley ranks, $I(T, \aleph_0)$ where T is the theory of a $(K, \leq)$-generic model.

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ALESSANDRO BERARDUCCI AND BENEDETTO INTRIGILA, *Some new results on easy terms*.

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An *easy* term, is a lambda term X such that for every closed lambda term Y, the $\lambda\beta$-calculus plus the equation $X = Y$ is a consistent lambda theory, written $\mathrm{Con}(X = Y)$. G. Jacopini proved that $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ is easy (see [2] for references). By some results of Visser and Statman we know that easy terms are suitable candidates for representing inside lambda calculus the undefined value of a partial recursive function (see [1]). However in some contrast with the intuition "easy = completely undefined", in [2] it is proved that if we identify all easy terms, then we obtain new equalities which equate an easy term with a non-easy term. We prove that if $N_1X = N_2X$ where N_1 and N_2 are distinct normal forms, then X is non-easy, actually there is a closed normal form N such that $\neg\mathrm{Con}(X = N)$. Motivated by this result we introduce the following notation: a term is *n.f.-easy* if for every closed normal form N, $\mathrm{Con}(X = N)$. It turns out that in the λI-calculus, n.f.-easy implies easy, however we show that this fails in the full $\lambda\beta$-calculus. We also prove that there is a closed lambda term X which can be consistently equated to every closed lambda term with the exclusion of the identity $\lambda x.x$ (or any given projector $\lambda x_1 \ldots x_n.x_i$). Both results are consequences of a general result which states that given two closed terms A and B with incompatible Böhm trees, there is a closed term X depending only on A, which can be consistently equated to B but not to A. This is proved by a general method to obtain Church-Rosser extensions of lambda calculus by which we obtain various other consistency results. Finally we prove that for X to be easy it suffices that X can be consistently equated with every closed lambda term whose Böhm tree has height at most one.

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JÖRG BRENDLE, *Extraspecial p-groups and Kurepa trees*.

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Given a group G, we let $Z(G)$ denote its center, G' its commutator subgroup, and $\Phi(G)$ its Frattini subgroup (the intersection of all maximal proper subgroups of G). Given $U \leq G$, we let $N_G(U)$ stand for the normalizer of U in G. A group G is *FC* iff every element $g \in G$ has finitely many conjugates. A p-group E is called *extraspecial* iff $\Phi(E) = E' = Z(E) \simeq \mathbf{Z}_p$, the cyclic group with p elements.

When generalizing a characterization of center-by-finite groups due to B. H. Neumann, M. J. Tomkinson asked the following question. Is there an it FC-group G with $|G/Z(G)| = \kappa$ but $[G : N_G(U)] < \kappa$ for all (abelian) subgroups U of G, where κ is an uncountable cardinal. We consider this question for $\kappa = \omega_1$ and $\kappa = \omega_2$. It turns out that the answer is largely independent of ZFC, and that it differs greatly in the two cases.

More explicitly, for $\kappa = \omega_1$, it is consistent with, and independent of, ZFC that there is a group G with $|G/Z(G)| = \omega_1$ and $[G : N_G(A)] \leq \omega$ for all abelian $A \leq G$. We do not know whether the same statement is still consistent if we drop *abelian*. On the other hand, for $\kappa = \omega_2$, the non-existence of groups G with $|G/Z(G)| = \omega_2$ and $[G : N_G(A)] \leq \omega_1$ for al (abelian) $A \leq G$ is equiconsistent with the existence of an inaccessible cardinal. In particular, there is an extraspecial p-group with this property if there is a Kurepa tree.

MARK A. BROWN, *Stit in Time*.

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Belnap and Perloff [2] continue development of their version of *stit* theory: the theory of the operator "See To It That" for a logic of action, in which for agent α and formula P one can always meaningfully form the formula $[\alpha\text{stit} : P]$. Their underlying semantics, based on models with forward-branching time, is extremely rich and subtle, but the corresponding system—with the *stit* operator (relativized to agents) as the only modal operator—is difficult, perhaps impossible, to characterize completely by syntactical means.

In this paper, I consider an approximation to *stit* theory, intended to preserve as much as possible of the richness and subtlety of Belnap and Perloff's account, but cast in a multimodal language in which the *stit* operator is a defined, not a primitive, modal operator. The multimodal language contains modal operators

$\langle \mathsf{F} \rangle P$, interpreted to mean that at some moment in the future P;

$\langle \mathsf{P} \rangle P$, interpreted to mean that at some moment in the past P;

$\langle \mathsf{S} \rangle_\alpha P$, interpreted to mean that at some alternative moment in the present, (i.e., at some simultaneous moment on another of time's branches); consistent with α's abilities, P

and $\boxed{\circ}_\alpha P$, interpreted to mean that α *does* P; in a rather minimal sense of *does*.

The operators $\langle \mathsf{F} \rangle$, $\langle \mathsf{P} \rangle$, and $\langle \mathsf{S} \rangle_\alpha$ are all normal modal operators, corresponding to binary relations (*forwards, backwards* and *sideways in time*) by means of which we can describe the relevant aspects of the tree of branching histories. They combine to give a rich temporal framework in our language. The action operator $\boxed{\circ}_\alpha$, also a normal modal operator, is conceived along lines indicated in Brown [4], but relativized to agents

The resulting multimodal system can help us understand, and explicitly represent, some of the interrelationships between agent, action, and time, in an approximation to *stit* theory, and can also help us explore the connections between *stit* theory and the simpler, but more simplistic, system of Brown [3]. The paper is concerned with the limitations, as well as the potential, of a multimodal approach.

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ANTONINO DRAGO, *Constructive mathematics and theoretical physics. From the wave equation to a general method.*

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In past two decades some authors found out by constructive mathematics (CoM) undecidability results in several theories of theoretical physics (TP) : statistical mechanics, thermodynamics, Hamiltonian mechanics, quantum mechanics, chaos theory [2]. In particular, Pour-El and Richards directed the attention on counter-examples of non-computable solutions of wave equation [4]. These counter-examples belong to the set of weak solutions, which Bishop's CoM is unable to represent, whereas by essentially adding Markoff's principle Aberth's CoM states them as undecidable—essentially by his Th. 6.1 [1]; i.e., some may be obtained by means of more hypotheses, some others cannot be obtained at all. However, all weak solutions are obtained when one adds a non-constructive hypothesis, i.e., extended Church's Thesis [5].

By generalizing to any differential equation of TP, we can say that almost all solutions which are relevant to TP—the regular ones—are represented by CoM. However, since singular solutions too may be relevant to TP, the correct mathematical conclusion to draw is that we cannot obtain all physical solutions by a general algorithm of a CoM. Hence, the correct philosophical conclusion with respect to a deductive physical theory of TP is that CoM does not allow to consider any differential equation as a universalistic principle In other words, TP cannot be reduced to a mere mathematical technique; or, even, mathematical-physics is essentially more restricted than TP.

All that qualify the general method for introducing Aberth's CoM in a deductive physical theory, as illustrated in [3]. There, the CoM is the Aberth's version. I can justify this choice by the following argument.

In the history of TP the authors of some theories rejected the deductive organization. The original texts stated by a double negated statement the crucial problem of the theory. L. Carnot's classical mechanics (1782), classical chemistry, Lobachevsky's non-Euclidean geometry, S. Carnot's thermodynamics were centered respectively on the following problems: "A body cannot change its state of motion if not by other bodies"; "It is not possible that matter is divisible at infinity"; "It is not true that heat is not work"; "It is not absurd the hypothesis of two parallel lines". In classical logic one is allowed to cancel the two negations; one obtains as affirmative statements just the principles of the corresponding deductive formulations of the same theories: "Any body perseveres in its state of motion"; "There are atoms"; "There exist two parallel lines"; "Heat is equivalent to work". Aberth's CoM, by including—through Markoff's principle—a typical formula for canceling double negations, results to be adequate to the above affirmative statements of the principles in the deductive formulations. On the contrary, the founders of the above theories rejected the affirmative statements for lack of experimental evidence. By accepting $\neg\neg A \neq A$, i.e., the characteristic law of non-classical logic, they organized their formulations in a not at all deductive way; thus the CoM resulting to be adequate to them is the Bishop's one, whose logic—being without Markoff's principle—is a non-classical one. In other terms, the radical variation of meaning suffered by principles w.r. to the option on kind of organization, requires that a suitable version of CoM be introduced in a physical theory.

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ROY DYCKHOFF, *Loop-free construction of counter-models for intuitionistic propositional logic.*

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The standard method of constructing counter-models to invalid formulae of intuitionistic propositional logic requires a loop checker, essentially because Contraction is a primitive rule of the standard Gentzen sequent calculus for this logic. We showed in [1] how to reformulate the calculus to avoid contraction—similar ideas were first presented in 1952 by Vorob'ev and more recently in 1992 by Hudelmaier and in 1991 by Lincoln *et al.* Here we make explicit the associated algorithm, without loop-checking, for the construction of Kripke counter-models.

Part of the work described here is joint work with Luis Pinto. A detailed exposition will appear in the Proceedings of the Second International Gauss Symposium (editor Rudolf Fritsch), held in Munich in August 1993.

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DANIEL DZIERZGOWSKI, *A saturation theorem for Kripke structures.*

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We propose an extension of the classical notion of recursively saturated structures (Barwise and Schlipf) to Kripke structures for intuitionistic logic.

Let $\mathcal{L}$ be a countable language. We define $\mathcal{D}$ to be a *countable $\mathcal{L}$-collection* iff $\mathcal{D}$ is a countable set of pairs $\langle \Phi, (x, x_1, \ldots, x_n) \rangle$, where Φ is a countable set of formulae whose free variables are among $x, x_1, \ldots, x_n$.

Let $\mathcal{M} = ((\mathcal{M}_k)_{k \in K}, (K, \leq, 0))$ be a Kripke structure (K is partially ordered by $\leq$, for all $k \in K$, $0 \leq k$ and $\mathcal{M}_k \subseteq \mathcal{M}_l$ if $k \leq l$). We suppose that $\mathcal{L} \subseteq \mathcal{L}_{\mathcal{M}_0}$, and say that $\mathcal{M}$ is *$\mathcal{D}$-saturated* iff for all $k \in K$, for all $\langle \Phi, (x, x_1, \ldots, x_n) \rangle \in \mathcal{D}$, and for all $c_1, \ldots, c_n \in \mathcal{M}_k$,

$$\text{there exists } m \in \mathcal{M}_k \text{ s.t. } \mathcal{M} \Vdash_k \Phi[m, c_1, \ldots, c_n]$$

$$\text{iff for all finite } \Psi \subseteq \Phi, \text{ there exists } m \in \mathcal{M}_k \text{ s.t. } \mathcal{M} \Vdash_k \Psi[m, c_1, \ldots, c_n].$$

Then we are able to prove that, *if $\mathcal{M}$ is a countable Kripke structure and $\mathcal{D}$ is a countable $\mathcal{L}$-collection, then $\mathcal{M}$ has a countable $\mathcal{D}$-saturated elementary extension*

On the other hand, let $\mathcal{M} = ((\mathcal{M}_k)_{k \in K}, (K, \leq, 0))$ and $\mathcal{N} = ((\mathcal{N}_k)_{k \in K}, (K, \leq, 0))$ be two countable Kripke structures (both are indexed by the same $(K, \leq, 0)$. Let $\mathcal{M} \oplus \mathcal{N}$ denote the "disjoint union" of $\mathcal{M}$ and $\mathcal{N}$: $\mathcal{M} \oplus \mathcal{N} = ((\mathcal{M}_k \oplus \mathcal{N}_k)_{k \in K}, (K, \leq, 0))$. For classical structures, it can be proved that if $\mathcal{M} \oplus \mathcal{N}$ is recursively saturated then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic as soon as $\mathcal{M} \equiv \mathcal{N}$. For Kripke structures for intuitionistic logic, this is likely to be false; in [2], we could only prove a weak form of this result, where, furthermore, $\equiv$ has been replaced with a stronger relation related to Fraïssé's characterization of elementary equivalence by means of partial isomorphism (classically this stronger relation is equivalent to $\equiv$; see also [1]).

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KENJI FUKUZAKI, *Definability results and a proof of a Nullstellensatz for regular rings.*

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We show some definability results for (commutative von Neumann) regular rings.

THEOREM. *For any $t, q, n \in N$ there exists $m_1(t, q, n)$ such that if R is any regular ring and $g, f_1, \ldots, f_t$ are polynomials in $R[x_1, \ldots, x_n]$ with $\deg f_i \leq q$ for each i and $g \in (f_1, \ldots, f_t)$, then there exist polynomials $h_1, \ldots, h_t \in R[x_1, \ldots, x_n]$ such that $\deg h_i \leq \deg g + m_1(t, q, n)$ for each i and $g = \sum_{i=1}^{t} h_i f_i$.*

THEOREM. *For any $t, q, n \in N$ there exists $m_2(t, q, n)$ such that if R is any regular ring and $g, f_1, \ldots, f_t$ are polynomials in $R[x_1, \ldots, x_n]$ with $\deg f_i \leq q$ for each i and $g \in \sqrt{(f_1, \ldots, f_t)}$, then $g^{m_2(t,q,n)} \in (f_1, \ldots, f_t)$.*

These are analogues of definability results for fields proved by G. Hermann. Using these results we give another proof of a Nullstellensatz for regular rings proved by Saracino and Weispfenning.

THEOREM. *Let R be a regular ring. R' a monically closed regular ring extending R, and $g, f_1, \ldots, f_t$ be in $R[x_1, \ldots, x_n]$. Let $I = (f_1, \ldots, f_t)$. We assume $I \cap R = \{0\}$. Then there exist $r \in N$ and $h_1, \ldots, h_t \in R[x_1, \ldots, x_n]$ such that $g^r = \sum_{i=1}^{t} h_i f_i$ (i.e., $g \in \sqrt{I}$) iff g vanishes at every common zero of $f_1, \ldots, f_t$ in R'*

S. S. GONCHAROV, *Families with positive computable enumerations.*

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Systematic studying of computable enumerations was initiated by A. N. Kolmogorov. As the result of intensive investigating by a large number of authors, the enumeration theory questions formed a new actual direction in classical recursion theory and reflected in the monograph [1].

In this paper we study the problems of existence and quantity of minimal enumeration of families of r.e. sets raised in [4]. These questions were also studied in [2]–[5]. In [3] we proved that there exists the family of r.e. sets that has exactly two non-equivalent one-one computable enumerations. Remind that enumeration ν is called positive provided the set $\{(n, m) : \nu(n) = \nu(m)\}$ is r.e. set.

THEOREM 1. *For every $n \in N$, there exists a family of r.e. sets with exactly n non-equivalent* *positive computable enumerations.*

THEOREM 2. *There exists a family of r.e sets with the unique but not the least positive computable enumeration.*

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VALENTIN GORANKO, *Refutation systems in modal logic.*

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A refutation system for a formal theory is a deductive system which infers the non-valid (refutable) statements of the theory. Refutation systems originate from Aristotle's attempt at systematic refutation of the non-valid syllogisms not by means of counterexamples, i.e., "refuting models", but rather applying certain rules of inference like *modus tollens*: if A implies B and B is refuted then A is refuted too. The idea of refutation systems was carried further by Łukasiewicz, Slupecki and his school, Scott, Skura and others.

In the first part of this investigation, *sentential* refutation systems are constructed and proved to be complete for a number of modal logics including **K**, **GL** (the logic of provability in Peano arithmetic), **T**, **S4**, **S4Grz** and others. Thus, complete syntactical characterizations in Łukasiewicz's sense are established for these logics and, in particular, purely syntactic decision procedures for them are obtained.

In the second part, *sequential* refutation systems in Gentzen style are introduced for the classical propositional calculus and for some normal modal logics including **K** and **GL**.

In conclusion, generalized deductive systems are discussed which involve both operators for provability and refutability, and employ combined inference rules for them.

JONANNA GRYGIEL, *Freely generated filters in Boolean algebras.*

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A set A of elements of a Boolean algebra is said to be independent provided that $a_1 \cdot \ldots \cdot a_n \leq b_1 + \;. + b_k$ does not hold for any system of different elements $a_1, \ldots, a_n, b_1, \ldots, b_k$ from A.

We say that a filer F is freely generated if an independent set of generators for F exists.

$m(F)$ denotes the minimal cardinality of the set of generators for F We can prove the following results.

(1) Every countably generated filter in any atomless Boolean algebra is freely generated.
(2) If $m(F)$ is a regular and uncountable cardinal and F is a filter of a free Boolean algebra, then F is freely generated.

The connection between the existence of the set of free generators for a filter F and $m(F_\tau)$, where F_τ is an image of F in τth Cantor-Bendixson derivative of a given Boolean algebra will be also considered

Application of the notion of independence to classical logic gives us the natural strengthening of the notion of logical independence. The results obtained in algebraical way can be easily applied to Lindenbaum algebras and give us an answer to questions about the algebraic axiomatizability of sets of formulae.

SACHIO HIROKAWA, *The proofs of $\alpha \to \alpha$ in $P - W$.*

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A syntactic structure is investigated for the system of pure implicational relevant logic $P - W$ [1]. The system is defined by the axioms $B = (b \to c) \to (a \to b) \to a \to c$, $B' = (a \to b) \to (b \to c) \to a \to c$ and $I = a \to a$. Martin and Meyer [3] showed a semantic proof for the following conjectures. But no syntactic proof is yet known.
Belnap's conjecture: If $\alpha \to \beta$ and $\beta \to \alpha$ are derivable in $P - W$ then $\alpha = \beta$.
Power's conjecture: No instance of $a \to a$ is derivable from B and B'.

Trig, Bunder and Hindley [4] defined the set of 'hereditary right-maximal linear' λ-terms which exactly corresponds to the set of $BB'I$-combinators. Thus the theorems in $P - W$ coincides the set of types of those λ-terms. By the analysis of type assignment to these λ-terms, we show that $\alpha \to \alpha$ is a type of such a λ-term iff the λ-term is $\beta\eta$-reducible to $\lambda x.x$. We prove this characterization using the underivability of $\alpha \to \alpha$ from B and B'.

Interesting is that we can prove this underivability syntactically from our characterization of the proofs of $\alpha \to \alpha$. We can prove Power's conjecture using our characterization and the characterization of invertible λ-terms (Dezani [2]).

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LEON HORSTEN, *Reflecting in epistemic arithmetic.*

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One of the principal aims of Shapiro's system of Epistemic Arithmetic (EA) is to give an axiomatic investigation of the notion of provability in principle in an arithmetical context (see [1]). By an old argument of Kaplan and Montague, it follows that for no consistent extension T of EA we have

$$\vdash_T Pr_T(\ulcorner A \urcorner) \to \Box A \tag{1}$$

or even

$$\vdash_T \Box(Pr_T(\ulcorner A \urcorner) \to \Box A \tag{2}$$

Nevertheless, for some consistent extensions of EA (for example EA itself), (1) and (2) are true reflection principles, expressing seemingly weak logical relations between the concept of absolute provability or provability in principle and provability in a formal system. In other words, the argument of Kaplan and Montague reveals a strong proof-theoretical limitation of consistent extensions of EA

For modal-structural formalizations of arithmetic (see [1]), the situation turns out to be different. We construct two (sound) epistemic formalizations T_1 and T_2 of arithmetic on a modal-structural basis such that T_1 contains (1) and (2) and T_2 contains (2). These systems contain enough arithmetic for the two incompleteness theorems to be proved for them, whereby they cannot prove their own Gödel sentence or their own consistency. Yet an easy fixed-point calculation shows that T_1 and T_2 can prove that their Gödel sentence and consistency are provable in principle.

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TANELI HUUSKONEN, *M-languages are not closed under negation.*

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The so-called *M-languages* $M_{\kappa\lambda}$, where $\kappa, \lambda \in \mathrm{Card} \cup \{\infty\}$, have conjunctions and disjunctions of size less than κ and well-ordered quantifier strings of length less than λ. Their definition does not allow for a syntactic negation, as there does not seem to be any satisfactory semantics. This leaves open the question as to whether every M-language is closed under negation, and if not, whether every M-formula has a semantic negation in some possibly larger M-language.

We prove that the following $M_{\omega_1\omega_1}$-sentence has no semantic negation in any $M_{\kappa\lambda}$:

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \cdots \exists u \left[\bigwedge_{n<\omega} x_n < u \wedge \forall v[v < u \rightarrow \bigvee_{n<\omega} v < y_n] \wedge A(u) \right],$$

where $<$ and A are a binary and a unary relation symbol, respectively

The proof is based on the fact that the falsity of the above sentence in well-ordered models is equivalent to the existence of certain stationary sets, and that stationarity reflects worse than the truth of a sentence in an M-language.

TAPANI HYTTINEN, *A short survey on strong nonstructure theorems.*

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By a strong nonstructure theorem we mean a theorem which says that if a theory belongs to some class of theories then it has very equivalent nonisomorphic models. Usually the equivalence is measured by the length of the Ehrenfeucht-Fraïssé games in which $\exists$ has a winning strategy (see Definition 1 below). These theorems are called nonstructure theorems because intuitively the models must be complicated if they are very equivalent but still nonisomorphic. Also structure theorems usually imply that certain degree of equivalence give isomorphism.

1 DEFINITION. *Let t be a tree and κ a cardinal The Ehrenfeucht-Fraïssé game of length t between models $\mathcal{A}$ and $\mathcal{B}$, $G_t^\kappa(\mathcal{A},\mathcal{B})$, is the following. At each move α:*

(i) *player $\forall$ chooses $x_\alpha \in t$, $\kappa_\alpha < \kappa$, and either $a_\alpha^\beta \in \mathcal{A}$, $\beta < \kappa_\alpha$ or $b_\alpha^\beta \in \mathcal{B}$, $\beta < \kappa_\alpha$, we will denote this sequence by X_α.*

(ii) *if $\forall$ chose from $\mathcal{A}$ then $\exists$ chooses $b_\alpha^\beta \in \mathcal{B}$, $\beta < \kappa_\alpha$, else $\exists$ chooses $a_\alpha^\beta \in \mathcal{A}$, $\beta < \kappa$, we will denote this sequence by Y_α.*

$\forall$ must move so that $(x_\beta)_{\beta\le\alpha}$ forms a strictly increasing sequence in t. $\exists$ must move so that $\{(a_\gamma^\beta, b_\gamma^\beta) \mid \gamma \le \alpha, \beta < \kappa_\gamma\}$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$. The player who first has to break the rule loses.

We write $\mathcal{A} \equiv_t^\kappa \mathcal{B}$ iff $\exists$ has a winning strategy for $G_t^\kappa(\mathcal{A},\mathcal{B})$. Among the strong nonstructure theorems we mention in this talk is the following.

2 THEOREM. ([1]) *Let $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) < \mu$, $\lambda^{<\kappa} = \lambda$ and $\mu^\kappa = \mu$. Assume T is an unsuperstable theory, $|T| \le \lambda$ and $\kappa(T) > \kappa$. Then there are models $\mathcal{A},\mathcal{B} \models T$ of cardinality λ such that $\mathcal{A} \equiv_{\mu\times\kappa}^\lambda \mathcal{B}$ and $\mathcal{A} \not\cong \mathcal{B}$.*

Under the assumptions of Theorem 2 the claim of the theorem is virtually the best possible, i.e., the theorem does not remain true if in the claim we replace $\mu \times \kappa$ by $\mu \times \kappa + 2$.

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TAKAO INOUÉ, *Partial interpretations of Leśniewski's epsilon in modal and intensional logics.*

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The propositional fragment $\boldsymbol{L}_1$ of Leśniewski's ontology $\boldsymbol{L}$ was introduced in [5] (for $\boldsymbol{L}_1$, see also [4]). In this paper, we propose the following sound translation from $\boldsymbol{L}_1$ to Lemmon-Scott's normal propositional modal logic $\boldsymbol{K}$.

(i) $I(A \vee B) = I(A) \vee I(B)$,

(ii) $I(\sim A) = \sim I(A)$,

(iii) $I(\epsilon ab) = p_a \wedge \Box(p_a \equiv p_b)$,

where p_a and p_b are propositional variables corresponding to name variables a and b, respectively. The translation is not an embedding. But we prove that it is faithful for a certain large decidable set of formulas (i.e., decent formulas) of $\boldsymbol{L}_1$. The (partial) faithfulness of the translation is generalized to an arbitrary extension of $\boldsymbol{K}$ under some conditions. With respect to similar translations, analogous theorems hold for Von Wright-type deontic logics, the system $\boldsymbol{PrL}$ of provability logic and its subsystem, for example. This paper is an endeavor to understand Leśniewski's epsilon on the basis of my recognition that Leśniewski's epsilon would be a variant of truth-functional equivalence. In the Appendix, an axiomatic rejection for $\boldsymbol{L}_1$ is proposed as a case study of [1]–[3].

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PETER KOEPKE, *A proof of the Martin-Steel theorem based on embedding normal forms.*

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In "A Proof of Projective Determinacy" by D. A. Martin and J. Steel, ***Journal of the AMS***, vol. 2, no. 1, 1989, the argument is based on the interplay between homogeneous trees and iteration trees in the presence of Woodin cardinals.

A set $A \subseteq ({}^{\omega}\omega)^n$ is the projection of a (sufficiently complete) homogeneous tree iff it possesses an *Embedding Normal Form with Witnesses* (ENFW), which means that there is a commutative system

$$\langle M_s \mid s \in ({}^{<\omega}\omega)^n \rangle, \langle \pi_{st} \mid s \subseteq t \in ({}^{<\omega}\omega)^n \rangle$$

and *witnesses* $f_s \in M_s$ such that:

(a) $M_0 = V$, M_s is transitive, $\pi_{st} : M_s \to M_t$ is elementary, and $f_s : ({}^{\omega}\omega)^n \to On$;
(b) $\vec{b} \in A$ implies that $\langle M_s \mid s \subset \vec{b} \rangle$ has a well-founded direct limit;
(c) $\vec{b} \notin A$ implies that $\pi_{st}(f_s)(\vec{b}) > f_t(\vec{b})$ for $s \subset t \subset b$, $s \neq t$.

The *critical point* of this system is the least ordinal moved by the maps π_{st}.

In analogy to results on projections of homogeneous trees we can give (rather short) proofs of:

(1) If A has an ENFW then A is determined.
(2) If $A \subseteq ({}^{\omega}\omega)^{n+1}$ has an ENFW with critical point $> \delta$, where δ is a Woodin cardinal, then $B = \{\vec{y} \in ({}^{\omega}\omega)^n \mid \neg\exists x \in ({}^{\omega}\omega)(x, \vec{y}) \in A\}$ has an ENFW with arbitrary high critical point below δ.

This immediately implies the Martin-Steel Theorem: If there are infinitely many Woodin cardinals then every projective set of reals is determined.

In my talk I am going to sketch some aspects of the proofs of (1) and (2) which are specific to the use of ENFWs.

OLIVER KULLMANN, *Methods for 3-SAT-decision in less than $2^{0.59\,n}$ steps.*

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In [2] and [3] the upper bound $2^{0.6943\,n}$ (worst case for the number of steps in abstraction from a (small) polynomial factor in the number of literal-occurrences) for 3-SAT-decision was demonstrated (n = number of variables). Starting out from Schiermeyer [4] (who achieved the upper bound $2^{0.659\,n}$) we improved the bound to $2^{0.5893\,n}$ (with a better understanding of what's going on).

For us the step forward in [4] depends on a refined measure of complexity for the clause sets, which causes that beyond the decrease of the number of variables in the branchings of the algorithm there comes another (structural) simplification of the clause sets into account (namely there arise new 2-clauses).

We now develop this (implicit) basis idea and introduce the following new methods (which yield our results):

1. We generalize the 'autarkness-lemma' ([3]) and get a natural and flexible way of reacting if there are not 'enough' 2-clauses.

2. We introduce the new 'blocking-lemma' which gives the opportunity of eliminating or adding new clauses. With this we can fill the gaps left from the autarkness-method (and probably we go beyond the realm of resolution.)
3. We can use certain global structures of the whole evaluation tree to 'accept' the new 2-clauses.
4. We refine the measurement of the clause-sets.

The general tool for the estimation of running time is the concept of a 'measure' for trees (which substitutes the ordinary calculation of recursion equations with the benefit of great simplification)—only with this 'global' concept (in contrast to the 'local' recursion equations) one can use global regularities of the behavior of the algorithm (see point 3 above).

This result goes on with the results from [1], where the upper bounds $2^{\frac{1}{9}l}$ and $2^{\frac{1}{3}k}$ for whole KNF (l = number of literals, k = number of clauses) with short algorithms (and proofs) were obtained.

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CLAUDE LAFLAMME, *Bounding and dominating number of families of functions on* $\mathbb{N}$.
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We survey a few general mathematical problems which have an equivalent form in terms of the structure of certain families of functions (or subsets) on the natural numbers.

The goal is to classify these families in ZFC, but forcing is useful to test what can't be done. The task appears to be difficult since not only any partial order of size at most ω_1 embeds in $\langle\omega^\omega, \leq^*\rangle$ but any partial order of size at most continuum consistently embeds. We feel that the bounding and dominating numbers is a good place to start, as it has been useful in the classification of gaps. A classification of unbounded families has essentially been completed a few years ago but the bounded case appears to be much more difficult and interesting.

LARISA MAKSIMOVA, *Explicit definition of implicitly definable in some modal logics*.
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In [1] the following analogue of the Beth Theorem was proved ($\vdash_L$ denotes derivability in L by modus ponens and necessitation rules).

Theorem 1. *Let L be any normal modal logic containing K4. Let* $\mathbf{p}$ *be a list of variables and* x *and* y *be outside of* $\mathbf{p}$. *For any formula* $A(\mathbf{p}, x)$, *if* $A(\mathbf{p}, x), A(\mathbf{p}, y) \vdash_L (x \leftrightarrow y)$, *then there exists a formula* $B(\mathbf{p})$, *such that* $A(\mathbf{p}, x) \vdash_L (x \leftrightarrow B(\mathbf{p}))$.

But the proof of this theorem given in [1] does not contain any way of constructing an explicit form of $B(\mathbf{p})$. In this paper we construct an explicit definition for some modal logics extending K4.

Denote by K4A a logic, which is characterized by the class FTA of all finite frames, both transitive and antisymmetric For $n < \omega$, let K4A$_n$ be characterized by the frames from FTA of height at most n. Denote $\Box A = (A \,\&\, \Box A)$.

THEOREM 2. *Let L be any normal modal logic containing K4A$_n$. Let* $\mathbf{p}$ *be a list of variables and x and y be outside of* $\mathbf{p}$ *For any formula* $A(\mathbf{p}, x)$, *if* $A(\mathbf{p}, x)$, $A(\mathbf{p}, y) \vdash_L (x \leftrightarrow y)$, *then* $A(\mathbf{p}, x) \vdash_L (x \leftrightarrow B_n(\mathbf{p}))$, *where*

$$B_n(\mathbf{p}) = \bigvee\{(C_{2i+1}(\mathbf{p}) \,\&\, \neg C_{2i}(\mathbf{p})) \mid 0 \leq 2i \leq n+2\},$$

$$C_0(\mathbf{p}) = 1,$$

$$C_1(\mathbf{p}) = \boxdot A(\mathbf{p}, T),$$

$$C_{i+1}(\mathbf{p}) = \boxdot A(\mathbf{p}, C_1(\mathbf{p}) \leftrightarrow \cdots \leftrightarrow C_i(\mathbf{p})).$$

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ALBERTO MARCONE, *Bqo theory and reverse mathematics*.
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The notion of better quasi-ordering (bqo) has been introduced by Nash-Williams in the 60's and has been studied by both combinatorists and logicians. We study bqos from the point of view of *reverse mathematics* ([3]) by trying to establish which subsystems of second order arithmetic are necessary to prove the theorems of bqo theory. We concentrate on one of the results proved by Nash-Williams ([1]) which we denote by **NWT** and which asserts that if a quasi-ordering Q is a bqo then $\tilde{Q}$ (the set of all countable sequences of elements of Q ordered by embeddability) is also a bqo.

Nash-Williams' original proof of **NWT** and all its refinements rely on the so-called *minimal bad array lemma*, a powerful technique that appears to require rather strong set-existence axioms (e.g., the system $\mathbf{\Pi}^1_2$-$\mathbf{CA}_0$). By using an entirely new proof we prove **NWT** in the weaker system $\mathbf{\Pi}^1_1$-$\mathbf{CA}_0$. Our proof uses hereditarily indecomposable sequences and a weak version of the minimal bad array lemma which suffices to prove a generalization of the classical Higman's theorem.

We also obtain some partial results toward the proof of **NWT** in the even weaker system $\mathbf{ATR}_0$ (the only step missing is a proof of the generalized Higman's theorem). If this proof exists it would settle the status of **NWT** because it follows from some recent results by Shore ([2]) that **NWT** implies $\mathbf{ATR}_0$ over the weak base system $\mathbf{RCA}_0$.

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JUAN CARLOS MARTÍNEZ, *Forcing construction of very thin-thick boolean algebras.*

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A *superatomic Boolean algebra* is a Boolean algebra in which every subalgebra is atomic. It is known that a Boolean algebra B is superatomic iff its Stone space $S(B)$ is scattered. The Cantor-Bendixson process for topological spaces can be transferred to the context of Boolean algebras, obtaining in this way a sequence of ideals. Suppose that B is a Boolean algebra. Then, for every ordinal α, the ideal I_α is defined by transfinite induction as follows: $I_0 = \{0\}$; if $\alpha = \beta + 1$, I_α is the ideal generated by $I_\beta \cup \{b \in B : b/I_\beta \text{ is an atom in } B/I_\beta\}$; and if α is limit, $I_\alpha = \bigcup\{I_\beta : \beta < \alpha\}$. Then, B is superatomic iff $B = I_\alpha$ for some α.

The *height* of a superatomic Boolean algebra B, $\mathrm{ht}(B)$, is the least ordinal α such that B/I_α is finite (which means $B = I_{\alpha+1}$). For every ordinal $\alpha < \mathrm{ht}(B)$ let $\mathrm{wd}_\alpha(B)$ be the cardinality of the set of atoms in B/I_α. Then, for an infinite cardinal κ, B is called *κ-very thin-thick* if $\mathrm{ht}(B) = \kappa^+ + 1$, $\mathrm{wd}_\alpha(B) = \kappa$ for $\alpha < \kappa^+$, and $\mathrm{wd}_{\kappa^+}(B) = \kappa^{++}$.

Roitman proved that the existence of an ω-very thin-thick superatomic Boolean algebra is consistent with the axioms of set theory. Using the notion of a simplified morass introduced by Velleman, we shall present a generalization of Roitman's theorem to larger cardinals.

CHRISTIAN MICHAUX AND ROGER VILLEMAIRE, *Expanding predicates in arithmetic and Cobham's theorem.*

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A subset L of natural numbers is said to be expanding if the distance between consecutive elements of L is not bounded.

We show that $L \subseteq \mathbf{N}$ is not first-order definable in Presburger Arithmetic iff there exists a subset of $\mathbf{N}$ first-order definable in $\langle \mathbf{N}, +, L\rangle$, which is expanding.

For example, if L is the Thue-Morse predicate (i.e., n belongs to L iff the number of 1's in the binary expansion of n is even), it is easy to show that the subset P_2 of the powers of two is first-order definable in $\langle \mathbf{N}, +, L\rangle$. Using a first-order version of Büchi's theorem (see [2]) we get a new proof of Cobham's theorem ([1]).

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CHRIS MILLER, *A growth dichotomy for o-minimal expansions of ordered fields.*
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Let $\mathcal{R}$ be an o-minimal expansion of an ordered field with underlying set R. There is a striking dichotomy in the orders of growth at $+\infty$ of $\mathcal{R}$-definable unary functions. In one case, $\mathcal{R}$ is *exponential*; i.e., there exists a unique nonconstant differentiable function $E : R \to R$, definable in $\mathcal{R}$ without parameters, with $E(x+y) = E(x)E(y)$ for all $x, y \in R$ and $E(0) = 1$ In the o-minimal context, such a function has most of the properties we associate with the real exponential function. If $\mathcal{R}$ is not exponential, then $\mathcal{R}$ is *power bounded*; i.e., every $\mathcal{R}$-definable function $f : R \to R$ is eventually dominated by some definable multiplicative homomorphism $g : (0, +\infty) \to (0, +\infty)$. Such a function g is called a *power function* for $\mathcal{R}$, and these functions behave essentially like real power functions $x \mapsto x^r : (0, +\infty) \to (0, +\infty)$.

In either case, the set of all power functions for $\mathcal{R}$ can be made into an ordered field, (with composition as multiplication, and pointwise multiplication in R as addition). If $\mathcal{R}$ is exponential, this field is isomorphic to R, and $\lim_{x\to+\infty} E(x)/g(x) = +\infty$ for all power functions g If $\mathcal{R}$ is power bounded, this field is isomorphic to a subfield of the prime model of $\mathcal{R}$, every power function is definable *without* parameters, and for every $\mathcal{R}$-definable function $f : R \to R$, not ultimately identically 0, there exists a power function g and $c \in R$, $c \neq 0$, such that $\lim_{x\to+\infty} f(x)/g(x) = c$.

MARGARITA OTERO, *The amalgamation property in normal open induction.*
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Let OI denote the fragment of Peano Arithmetic with the induction restricted to open formulae. It is known that OI fails to have the joint embedding property, a result due to Wilkie

On the other hand, we have proved that if we require our models to be normal, that is, integrally closed in their fraction fields, the corresponding theory NOI, extending OI, has the joint embedding property.

Here we prove that NOI does not have the amalgamation property.

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JOHN PAIS, *The topology of nonmonotonicity.*
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One current popular model-theoretic approach to the study of the mathematical properties or "logic" (e.g., [2], [3]) of nonmonotonic inference operators focuses its effort on the task of providing a semantics for various conditional logics. In this case, the primary allure of conditional logic is its well-known ability to reflect any extant *residual classical monotonicity*. However, since its model theory is too weak to provide a *faithful representation* ([5]) of ***intrinsic*** *nonmonotonicity* ([6]), one can ascribe too much significance to this fact at the expense of not focusing on the problem of a general mathematical characterization of nonmonotonicity.

This paper presents an abstract framework for representing nonmonotonic inference operators based on *exterior topology* ([4]), which is a topological approach employing nonmonotonic exterior operators in contrast to monotonic interior or closure operators. A metric concept of "closest" new information is developed in terms of exterior (inference) operators inducing *symmetric revision spaces*. These operators are then compared to inference operators constructed using the minimal elements of a preference relation and inducing *cumulative revision spaces*. It is shown that the former have properties preserving more residual monotonicity, while the later relinquish this residual monotonicity and instead possess more intrinsic nonmonotonicity. Furthermore, this intrinsic nonmonotonicity is now easily described in terms of standard topological concepts Finally, it is shown that exterior topology yields a framework subsuming both the AGM logic of theory change ([1]) and approaches employing preferential consequence relations.

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VALERIA DE PAIVA AND LUIZ CARLO PEREIRA, *A new proof system for intuitionistic logic*.

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Sequent Calculus formulations of Intuitionistic Logic (LJ) are usually derived from those of Classical Logic (LK) by restricting the form of the sequents, so that they are of the form

$$\Gamma \vdash \Delta$$

where Δ contains at most one formula (Gentzen 1933).

A less restrictive system, variously known as LJ′ or IL>, relaxes this condition. In LJ′ the rules are as with LK (i.e., they permit multiple conclusions) except for the rules for implication and universal quantification on the right, which are of the form

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma \vdash A(a)}{\Gamma \vdash \forall x A(x)}$$

(with the usual proviso in the universal quantification rule). Note that B (as well as $A(a)$) is a single formula. It can be shown that LJ′ satisfies cut-elimination and that it is equivalent to LJ.

This paper proposes a further relaxation of the form of the rules above. We define a new formulation, Full Intuitionistic Logic (FIL) which, under certain very natural conditions, allows for multiple conclusions in the above two rules. We show that FIL satisfies cut-elimination and that it is equivalent to LJ$'$.

We also consider a natural deduction-like formulation of FIL and, via the Curry-Howard Correspondence, a term calculus. This provides a new form of typed lambda-calculus, which suggests applications to functional programming.

To conclude we discuss briefly the application of this formulation to intuitionistic modal and linear logic.

HAYDEE WERNECK POUBEL AND LUIZ CARLOS PEREIRA, *A functorial approach to abstract introduction and elimination rules.*

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A natural extension of Natural Deduction was defined by Schröder-Heister where not only formulae but also rules could be used as hypothesis and hence discharged. It was shown that this extension induces the definition of abstract introduction and elimination rules and that the set $\{\wedge, \vee, \rightarrow, \bot\}$ of intuitionistic sentential operators form a *complete* set of operators modulo the abstract introduction and elimination rules, i.e., that any operator whose introduction and elimination rules are instances of the abstract rules can be defined in terms of $\{\wedge, \vee, \rightarrow, \bot\}$.

The aim of this work is to extend to Abstract Natural Deduction the well-known connections between Proof Theory and Category Theory. More precisely, we shall show how to systematically associate to any operator ϕ, whose introduction and elimination rules are instances of the abstract introduction and elimination rules, an adjunction between cartesian closed categories with finite coproducts whose objects are rules instead of formulae.

STEPHEN READ, *In praise of classical harmony.*

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It is sometimes claimed that classical logic is incoherent and unacceptable because it is not a conservative extension of its negative fragment; or at least, that the rules for classical negation are not in harmony, for proofs of certain theses, notably Peirce's Law, can be proved only via a maximal formula, its double negation, introduced by one negation rule, and eliminated by another. (See, e.g., M. Dummett, "The philosophical basis of intuitionistic logic", reprinted in his ***Truth and other enigmas***, Duckworth, 1978, esp. p. 221.)

The objection depends crucially on the presentation of classical logic which is considered. This is a familiar point. The standard natural deduction rules (in the system NK) for '&', '$\vee$' and '$\rightarrow$' are insufficient to yield proofs of Peirce's Law and its kin. They become provable only when negation is added. But this problem does not affect, for example, standard sequent calculus, LK. The reason is that LK allows multiple conclusions, and Peirce's Law is provable directly by two applications of → right, and one each of → left, Thinning and Contraction. Also truth-tables (or semantic trees), arguably the real basis of classical logic (i.e., the study of structures, proofs being only a means to that end), yield all the negation-free formulae of classical logic directly.

The purpose of the present paper is to propose a minor revision of NK whereby all the negation-free theses of classical logic will become provable without use of the rules for negation. The revised rules can be challenged, on grounds of impurity; but nonetheless, the rules are in harmony, in that all proofs in '&', '∨' and '→', complete for the theses of classical logic, are normalizable. Indeed, it is an easy step to obtain a full calculus for classical logic with negation in which all the rules are in harmony. Every proof can be put in normal form by eliminating maximal formulae, formulae introduced by introduction rules and acting as major premise of elimination rules.

WIM RUITENBURG, *Basic propositional calculus.*

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There exist two sources of motivation for Basic Logic; constructivism and constructivity. The approach through constructivism is expounded in [3]. In that paper we observe that the first-order logic of the well-known constructivisms is Intuitionistic Predicate Calculus IQC. Most explanations of the logical constants are variations upon the Brouwer-Heyting-Kolmogorov proof interpretation. Our modification of this interpretation yields the first-order logic Basic Predicate Calculus BQC, a proper subsystem of IQC.

The approach through constructivity precedes the one through constructivism, but was originally restricted to Basic Propositional Calculus BPC, see [5]. The traditional world of constructivity consisted of two related, but quite distinct, parts. One part included the forms of constructive mathematics alluded to above; the other involved Gödel's incompleteness theorem and the associated modal logics for provability, in particular PrL (identical to the system G of [2]), that satisfy Löb's rule. The syntax of Intuitionistic Propositional Calculus IPC differs in an essential way from that of PrL with its □, but IPC corresponds in a natural way to the modal logic S4. Albert Visser found a propositional logic, FPC, with a syntax identical to the one for IPC, to complete the informal equation

$$\frac{\mathrm{IPC}}{\mathrm{S4}} = \frac{\mathrm{FPC}}{\mathrm{PrL}}.$$

This enabled him to interpret implication as formal provability [5]. In [4] Craig Smoryński uses Basic Modal Logic BML, better known as K4, as the natural generalization of both S4 and PrL. Similarly, Visser went beyond the equation above by introducing the system BPC satisfying the informal equations

$$\frac{\mathrm{BPC}}{\mathrm{K4}} = \frac{\mathrm{IPC}}{\mathrm{S4}} = \frac{\mathrm{FPC}}{\mathrm{PrL}}.$$

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VLADMIR V. RYBAKOV, *Recognizing of admissibility via branching property and drop-point property.*

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Starting with the solution of Harvey Friedman problem of effective recognizing admissible for intuitionistic **H** inference rules in [1], a number of similar results for individual various logics were obtained. But, so far, no general results on decidability or admissibility for a sufficiently rich classes of logics have been found (except for extensions of **S4.3** and tabular logics). This paper presents such new results of algorithmic recognizing admissibility in classes of modal and intermediate logics. They were obtained using and developing the displacement method discovered in [1] and [2]

THEOREM 1. *If a modal logic $\lambda \supseteq$ **K4** has the m-branching property and the effective m-drop point property and there is an algorithm which recognizes finite λ-frames (in particular, if λ is finitely axiomatizable) then there is an algorithm which checks admissibility of inference rules in λ.*

THEOREM 2. *If an intuitionistic intermediate logic λ has the strong effective m-drop point property and the m-branching property and the finite λ-frames are effectively recognizable (if λ is finitely axiomatizable, for instance), then λ is decidable with respect to admissibility*

These theorems cover all known up to now results on recognizing admissibility and give a lot of new ones. For instance,

COROLLARY 3. *All logics below are decidable with respect to admissibility.*
(a) ***K4, K4.1, K4.2, K4.3, S4.1, S4.2, Grz.2, KC*** $:= \boldsymbol{H} + \neg p \vee \neg\neg p$.
(b) *any finitely axiomatizable logic λ of finite depth, where* **K4** $\subseteq \lambda$ *or λ is an intermediate logic.*
(c) *Logics* $\boldsymbol{K4}_{b,n}$, $\boldsymbol{S4}_{b,n}$, $\boldsymbol{Grz}_{b,n}$, $\boldsymbol{GL}_{b,n}$ *with branching limited by given n and the Gabbay-De-Jong intermediate logics* $\boldsymbol{D}_n$
(d) *modal logics* $\boldsymbol{K4I}_n$, $\boldsymbol{S4I}_n$, $\boldsymbol{GLI}_n$, *and intermediate logics* $\boldsymbol{HI}_n$, *where each of the logics is the logic of all finite rooted corresponding to the abbreviation frames with not more than n maximal clusters*

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V. YU. SAZONOV, *On polynomial-regular codings of hereditarily-finite sets.*

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Let $\theta : \{0,1\}^* \to HF$ be any coding of the universe HF of *hereditarily finite sets* by binary strings. Define $\mathcal{P}_\theta$ to be the class of operations $F : HF \to HF$ such that $F\theta = \theta f$ for some polynomial-time computable (shortly, P-) function $f : \{0,1\}^* \to \{0,1\}^*$. Let $\theta \leq_P \eta := (\theta = \eta f$ for some $f \in P)$.

Set-theoretic Δ-*language* (cf. [1]) consists of Δ-terms $a, b, \ldots$ and Δ-*formulas* $\phi, \psi, \ldots$ and is defined by the following clauses:

$$\begin{aligned} \Delta\text{-}\mathit{formulas} &:= a \in b \mid \neg\phi \mid \phi \& \psi \mid \phi \vee \psi \mid \phi \Rightarrow \psi \mid \\ &\quad\; \forall x \in a.\phi \mid \exists x \in a.\phi \\ \Delta\text{-}\mathit{terms} &:= \langle\text{set-variables}\rangle \mid \{a, b\} \mid \cup\{b(x) : x \in a \& \phi(x)\} \\ &\quad\; \mid [p = p \cup \{x \in a : \phi(x, p)\}] \end{aligned}$$

Here x and $p \notin FV(a)$. These constructs have the evident semantics in HF (and even in any universe V for ZF), except probably *inductive* Δ*-separation* $[p = p \cup \{x \in a : \phi(x, p)\}]$. This is considered as a term (not formula!) to denote the result of stabilization (in $\leq \operatorname{card}(a)$ steps) of the monotonic sequence $\emptyset = p_0 \subseteq p_1 \subseteq \ldots \subseteq a$, where $p_{n+1} := p_n \cup \{x \in a : \phi(x, p_n)\}$.

THEOREM 1. *There exists $i : HF \to \{0,1\}^*$ such that arbitrary P-function $f : \{0,1\}^* \to \{0,1\}^*$ satisfies $fi = iF$ for some Δ-definable $F : HF \to HF$.*

P-regular coding system of HF is any four-tuple of mappings $\theta, i^R : \{0,1\}^* \to HF$ and $\theta^R, i : HF \to \{0,1\}^*$ satisfying the following five conditions:

1. $\theta\theta^R = 1_{HF}$ and $ii^R = 1_{\{0,1\}^*}$,
2. theorem 1 holds for i,
3. the predicate $\theta(a) \in \theta(b)$ is P-computable,
4. two P-computable mappings $a \mapsto a_1, \ldots, a_k$ and $a_1, \ldots, a_k \mapsto a$ between codes and lists of codes exist such that both satisfy the equality $\{\theta(a_1), \ldots, \theta(a_k)\} = \theta(a)$, and
5. $\theta^R i^R, i\theta$ and $\theta^R\theta : \{0,1\}^* \to \{0,1\}^*$ are in P.

(Example: *well-founded graphs collapsing* [1].) Let Δ_θ be Δ-language extended by operations $\tilde{\theta} := \theta i$ and $\tilde{\theta}^R := i^R\theta^R : HF \to HF$.

THEOREM 2. *For all P-regular codings θ, η.*

(1) *$\Delta_\theta \equiv \mathcal{P}_\theta$, i.e., Δ_θ-language represents exactly $\mathcal{P}_\theta$-operations (and $\mathcal{P}_\theta$ predicates) in HF;*
(2) *$\eta \not\leq_P \theta$ implies $\mathcal{P}_\theta \not\subseteq \mathcal{P}_\eta$ and $\mathcal{P}_\eta \not\subseteq \mathcal{P}_\theta$ (with counter-examples $\tilde{\theta}^R$ and $\tilde{\eta}$, respectively; in fact, $\eta \leq_P \theta \Leftrightarrow \tilde{\eta} \in \mathcal{P}_\theta \Leftrightarrow \tilde{\theta}^R \in \mathcal{P}_\eta$).*

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V. L. SELIVANOV, *A universal hierarchy of formulas.*

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In our contributed papers for the two previous Logic Colloquiums as well as in [1] we considered a universal hierarchy of subsets of ω which is in a sense a “finite analog” of the Wadge hierarchy. It was defined abstractly, namely with any sequence $\{L_n\}_{n<\omega}$ of distributive lattices satisfying $L_n \cup \check{L}_n \subseteq L_{n+1}$ (where $\check{L}_n = \{\overline{x} \mid x \in L_n\}$ we related a sequence $\{S_\alpha\}_{\alpha<\varepsilon_0}$ (where $\varepsilon_0 = \sup sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$) satisfying

$$S_\alpha \cup \check{S}_\alpha \subseteq S_\beta \text{ for } \alpha < \beta < \varepsilon_0 \text{ and } \bigcup_{\alpha<\varepsilon_0} S_\alpha = \bigcup_{n<\omega} L_n.$$

By universal hierarchy of formulas (modulo a given theory T) we mean the sequence $\{S_\alpha\}$ associated with the distributive lattices L_n of Σ^0_{n+1}-formulas (modulo equivalence in the theory T). This hierarchy refines the classical quantifier-alternation hierarchy the aim of our talk is to discuss some interesting properties of the universal hierarchy, namely:

a) $S_\alpha \cap \check{S}_\alpha = \bigcup_{\beta<\alpha}(S_\beta \cup \check{S}_\beta)$ for all $\alpha < \varepsilon_0$, so the universal hierarchy cannot be further refined;
b) there is a clear model-theoretic characterization of S_α-formulas extending the classical characterization of Σ^0_{n+1}-formulas;

c) it is closely related to the universal hierarchy $\{\Sigma_\alpha\}_{\alpha<\varepsilon_0}$ of subsets of ω, which can be used to give some explanation of the fact that natural definable index sets are usually m-complete in one of Σ_α, Π_α.

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A. STERN AND S SWIERCZKOWSKI, *Connected theories of order*

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A theory T is called *connected* if T interprets in the union $T_1 \cup T_2$ of two theories T_1, T_2 with *disjoint languages* only if T interprets in T_1 or T_2. Interpretations are meant here in the very general sense (multidimensional and with parameters) introduced by J. Mycielski in 1977 (general reference: a survey in the ***American Mathematical Society Memoirs 86*** (1991) by J. Mycielski, P. Pudlak and A. Stern).

Theorem. *Let T be a theory which proves the axioms of linear ordering and which provides for each model $\mathcal{M}$ of T and every non-empty interval $(a, b) \subseteq |\mathcal{M}|$ a definable injection $|\mathcal{M}| \to (a, b)$ Then T is connected*

Here T can be any extension of the theory of ordered fields. In particular, it follows that RCF is connected (as it was conjectured in the ***Memoirs 86***). J. Mycielski has shown that a finitely axiomatizable theory T is connected iff:

(*) For any structures $\mathcal{A}, \mathcal{B}$, T interprets 1-dimensionally in the theory $\mathrm{Th}(\mathcal{A} \dot{\times} \mathcal{B})$ of the disjoint product $\mathcal{A} \dot{\times} \mathcal{B}$ iff T interprets in $\mathrm{Th}(\mathcal{A})$ or in $\mathrm{Th}(\mathcal{B})$.

Using (*), one can reduce the above theorem to the following:

Lemma. *For any structures $\mathcal{A}, \mathcal{B}$ and any $(\mathcal{A} \dot{\times} \mathcal{B})$-definable linear ordering of $\mathcal{A} \dot{\times} \mathcal{B}$, there is a non-trivial open interval in $\mathcal{A} \dot{\times} \mathcal{B}$ which projects finitely on $\mathcal{A}$ or on $\mathcal{B}$.*

(Since in (*) the equality in T need not interpret as equality in $\mathcal{A} \dot{\times} \mathcal{B}$, one actually has to prove a generalization of this lemma concerning a linear ordering of $(\mathcal{A} \dot{\times} \mathcal{B})/\equiv$, where $\equiv$ is any $(\mathcal{A} \dot{\times} \mathcal{B})$-definable equivalence relation.)

FRANK O. WAGNER, *Hyperstable Theories*.

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Generalizing the work in [1] on theories without dense forking chains, we define an analogue of Krull dimension (which generalizes Shelah degree). We call a theory *hyperstable*, if every type is nonorthogonal to a formula which has this dimension. In such a theory, every type is domination equivalent to a product of regular types, which may however be infinite. Examples for hyperstable theories are those of [1], but there the weight must be finite. The regular types in a hyperstable theory are arranged in a hierarchy; every type has ordinal hereditary weight with respect to any type in the highest level it is not hereditarily orthogonal to. Every regular type has a domination equivalent representative based on a finite set. Lachlan's conjecture holds for these theories, and in between any two models there is a realization of a regular type. In the nonmultidimensional case, hereditary weight with regard to any level of the hierarchy is definable and continuous on some $\emptyset$-definable set of nonzero weight. Similarly if p is regular nontrivial, hereditary p-weight is definable and continuous on some set of nontrivial weight.

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HEINRICH WANSING, *Sequent systems for simple tense logic.*
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This talk deals with sequent calculi for the most important systems of simple tense logic within a modified version of Nuel Belnap's display logic. Various axiomatic extensions of the minimal normal tense logic K_t are captured by adding purely structural rules to a display calculus for K_t, so that the proof-theoretic meaning of the connectives, as laid down in their introduction rules, remains unchanged. The display calculi under consideration enjoy cut-elimination and the subformula property.

MICHAEL ZAKHARYASCHEV, *Decidability via canonical formulas.*
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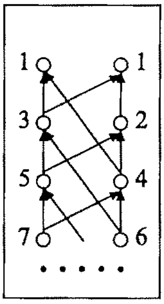
This paper presents two new results on the finite model property and decidability of logics in **NExtK4**, **ExtK4**, and **ExtInt**, the classes of normal and quasi-normal extensions of **K4** and extensions of intuitionistic logic **Int**, respectively. Both of them were obtained using the apparatus of the canonical formulas introduced in [2].



Let $\mathfrak{F}$ be a finite frame. By induction on n define a notion of *n-stable point* in $\mathfrak{F}$. Each point of depth 1 is *1-stable* in $\mathfrak{F}$, and a point is *$n+1$-stable* if it is not m-stable, for any $m \leq n$, and does not see at least one n-stable point in $\mathfrak{F}$. Say that an antichain $\mathfrak{d}$ in $\mathfrak{F}$ is *n-stable* if it contains at least one n-stable point *in the subframe of $\mathfrak{F}$ generated by $\mathfrak{d}$* and no $n+1$-stable point in the subframe. For example, each point in every generated subframe of the frame in the picture is n-stable, where n is the number located near the point.

Theorem 1. *If, for some $d > 0$, a logic $L \in$ NExtK4 or $L \in$ ExtInt can be axiomatized by a (finite or infinite) set of modal canonical formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ or, respectively, by a set of intuitionistic canonical formulas $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ in which every closed domain $\mathfrak{d} \in \mathfrak{D}$ is n-stable in $\mathfrak{F}$ for some $n \leq d$, then L has the finite model property*

Example. As is known, every intermediate logic with additional axioms of one variable can be axiomatized by one or two intuitionistic canonical formulas of the form $\beta(\mathfrak{F}, \mathfrak{D}^{\#}, \bot)$ where $\mathfrak{F}$ is a generated subframe of the frame depicted above and $\mathfrak{D}^{\#}$ is the set of all antichains in $\mathfrak{F}$. Therefore, by Theorem 1, every such logic has FMP, and so is decidable. This fact was first proved in 1977 by Sobolev, who used a completely different technique. Theorem 1 gives also a positive solution to the conjecture of Shehtman [1] on FMP of the logics

$$\begin{aligned}\mathbf{D4G}_n &= \mathbf{D4} \oplus \Box \bigvee_{j=0}^{n} \Box^{+}(p_j \wedge \bigwedge_{i \neq j} \neg p_i) \rightarrow \bigvee_{j=0}^{n} \Box\neg(p_j \wedge \bigwedge_{i \neq j} \neg p_j) \\ &= \mathbf{K4} \oplus \alpha(\bullet, \emptyset, \bot) \oplus \alpha(\overset{\overbrace{\circ \cdots \circ}^{n+1}}{\bullet}, \mathfrak{D}^{\#}, \bot).\end{aligned}$$

THEOREM 2. *Every quasi-normal logic $L \in$ ExtK4, which is axiomatizable by a finite number of normal and quasi-normal modal canonical formulas, without closed domains (i.e., formulas of the form $\alpha(\mathfrak{F}, \emptyset, \bot)$ and $\alpha^-(\mathfrak{F}, \emptyset, \bot)$, is decidable, though may have no Kripke frames at all, like Solovay's system $S(\mathbf{K4} \oplus \alpha(\circ, \emptyset)) + \alpha(\bullet, \emptyset)$.*

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Abstracts of papers submitted by title

ALEXANDER CHAGROV AND MICHAEL ZAKHARYASCHEV, *You can't independently axiomatize all modal logics.*
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An old open problem of the modal logic folklore is to prove or disprove that every normal logic has an independent axiomatization (with modus ponens, necessitation and substitution as the only inference rules). The following simple observation opens a way for solving this problem.

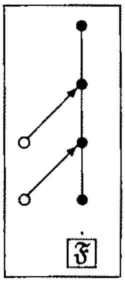
LEMMA 1. *If a logic L_1 has an independent axiomatization then, for every finite axiomatizable logic $L_2 \subset L_1$, $L_2 \neq L_1$, there is an immediate predecessor of L_1 containing L_2.*

Let L_2 be the minimal logic containing **K4** and the following seven formulas:

$$\Diamond^+\Box\bot, \quad \gamma \to \Diamond\gamma, \quad \neg\beta_0, \quad \Box^+(q \to \psi)\vee\Box^+(\neg q \to \psi), \quad \Diamond\alpha \to \Diamond^+\beta\vee\Diamond^+\alpha',$$

where

$$\Box^+\phi = \phi \vee \Box\phi, \Diamond^+\phi = \phi \vee \Diamond\phi, \psi \in \{\alpha, \beta, \gamma\},$$
$$\alpha = p \wedge \Box\neg p, \alpha' = \alpha(\Diamond p/p), \alpha'' = \alpha'(\Diamond p/p),$$
$$\beta = \Diamond\Diamond\alpha \wedge \neg\Diamond\alpha', \beta_0 = \beta(\top/p), \beta' = \beta(\Diamond p/p),$$
$$\gamma = \Diamond\beta' \wedge \Diamond\alpha'' \wedge \neg\Diamond\beta, \gamma' = \gamma(\Diamond p/p).$$



By L_1 we denote the logic which is characterized by the class of all general frames for L_2 that are based on the Kripke frames of the form shown in the picture (points in $\mathfrak{F}$, if any, see all the depicted irreflexive points $\bullet$ and no depicted reflexive $\circ$).

LEMMA 2. *L_1 is a proper extension of L_2 having no immediate predecessor containing L_2*

THEOREM 3. *L_1 has no independent axiomatization*

We can also construct logics without independent axiomatization which contain **Grz**, **GL** and intuitionistic propositional logic; however their description is somewhat bulky. As a consequence of these results we obtain an extended answer to a question of Blok [1]:

THEOREM 4. *The lattices of varieties of interior (Grzegorczyk, diagonalizable, Heyting) algebras are not strongly atomic.*

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S. T. FEDORYAEV, *Decidable algorithmic problems on relatively complemented distributive lattices which cannot be simultaneously decidable.*

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Here we study the algorithmic problems with mentioned properties along with the spectrum property and the correlation problem of algorithmic dimensions [2, 1] for the relatively complemented distributive lattices. By the algorithmic problem on a constructive model $\mathfrak{A}$ we mean the relation on $\mathfrak{A}$ that is invariant with respect to the automorphism group of this model [2] (equivalently, that is definable in $L_{\omega_1\omega}$).

Let $\mathcal{A}$ be the class of all relatively complemented distributive lattices with zero element admitting a constructivization with the decidable set of atoms.

Theorem. *For every lattice $D \in \mathcal{A}$ with infinitely many atoms, there exists a countable family W of algorithmic problems on D such that every algorithmic problem in W is decidable with respect to some constructivization of D, but no problems $W_1, W_2 \in W$ can be simultaneously decidable with respect to any constructivization of D*

Corollary 1. *The algebraic reducibility structure L (see* [1]*) of a lattice $D \in \mathcal{A}$ (or a lattice $\langle D, F\rangle$ in the language extended by Fréchet ideal) with infinitely many atoms has a countable set of elements such that no pair of these ones has a lower bound. In particular, L has no least element.*

Let dim-D be one of the algorithmic dimensions [2] (i.e., algebraic, program, uniform, or auto-dimension) of a lattice D.

Corollary 2. *For any lattice $D \in \mathcal{A}$, the following statements are equivalent*:

(i) *dim-$D \neq 1$;*
(ii) *dim-$D = \omega$;*
(iii) *D contains infinitely many atoms*

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L. GORDEEV, *Strong normalization and realizability.*

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Let T be an intuitionistic formal system in the language of set theory (with equality) enriched by the canonical abstraction-term operator $\{x \mid A\}$ for certain formulas A. Nonlogic axioms of T are of the form $(z \in \{x \mid A\} \leftrightarrow A[x/z])$ as corresponding to $\exists y \forall x (x \in y \leftrightarrow A)$ where y does not occur free in A. Let SEQT extend the canonical intuitionistic predicate sequent calculus (with cut) in the language of T by the corresponding rules of inference (τ

— arbitrary abstraction term):

$$\frac{A[x/\tau],\Gamma\Rightarrow B}{\tau\in\{x\mid A\},\Gamma\Rightarrow B}\qquad\qquad\frac{\Gamma\Rightarrow A[x/\tau]}{\Gamma\Rightarrow\tau\in\{x\mid A\}},$$

Strong normalizability of T expresses that every primitive recursive (say) chain of successive (canonical) cut-reductions in SEQT terminates. [This expression reduces by standard techniques to an equivalent purely combinatorial sentence.]

With any such T we correlate an appropriate (classical, say) formal system $T^\times$ whose language has objects of two sorts: (a) natural numbers and (b) sets. Moreover, the language of $T^\times$ includes standard number theoretic operations $0, 1, +, \cdot$, and it does not include abstraction terms. As compared to T, atomic formulas of $T^\times$ include $m = n$ as well as $(n, x) \in y$ instead of $x \in y$, where m, n and x, y are numbers and sets, respectively. Nonlogic axioms of $T^\times$ are the ones of Peano arithmetic (PA) and certain set formation axioms $\exists y\forall n\forall x((n, x) \in y \leftrightarrow A)$ where y does not occur free in A.

THEOREM. *The consistency of $T^\times$ implies strong normalizability of T, provably in* PA.

In particular, let T be the intuitionistic variant of Quine's New Foundations minus Extensionality (NF^-). The corresponding theory $T^\times$ preserves stratification, and its consistency reduces to the existence of κ-iterated power-set for some cardinal κ being a solution to the familiar proposition $(\forall m < \omega)(\kappa \to (\omega)^m_\omega)$ of infinite combinatorics.

On the other hand, strong normalizability of classical (sequent calculus corresponding to) theory NF^- easily follows from strong normalizability of the intuitionistic one.

COROLLARY. *Strong normalizability of (either intuitionistic or classical variant of) Quine's New Foundation minus Extensionality follows from the consistency of Zermelo's set theory extended by the existence of power-set hierarchy along* $\aleph_\omega$, *provably in* PA.

TAKAO INOUÉ AND ARATA ISHIMOTO, *A uniform formulation of some syllogistic systems with or without Leśniewski's epsilon.*

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We shall be here concerned with a uniform treatment of some syllogistic systems including classical propositional logic **CP**, in particular the Brentano-type syllogistic with Leśniewski's epsilon ε, denoted by **B-**ε, which has already been presented by the second named author in Japan in 1991. The system **B-**ε contains both the Brentano-type syllogistic and the propositional fragment $\mathbf{L}_1$ of Leśniewski's ontology **L** introduced in [5] (cf. [4]). Starting with **CP**, the systems to be presented are developed, step by step, by way of additional reduction rules and postulates. All the systems considered are rendered both in tableau method (in the style of [4]) and Hilbert-style formulation provided with Hilbert-style axiomatic rejection with Hintikka formulas as axioms as a case study of [1]. Some meta-theorems on cut elimination theorem and Ł-decidability for those systems are proved as a case study of [2] (cf. [3]).

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K. ZH. KUDAIBERGENOV, *On resplendency and homogeneity.*

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We indicate some conditions for "resplendency implies homogeneity" that are more or less similar to the standard notions of stability theory and compute the spectra of resplendency for some classes of superstable theories. The problem of finding such conditions and computing such spectra was posed by J. Baldwin.

GUN-WON LEE, *A construed number.*

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The use of number words has been already a part of our natural language as we can see in "lucky seven." It was said as that the number words are used as both *referential* and *predicative*. In order to have a stable theory the restriction of the uses of number words as predicative use only has been thought as an adequate at least temporarily in the formal languages. For this purpose, the number construed in set theory has been utilized widely already in the predicative use. And the use of the numbers unconstrued can be as usual to allow both usages of the number words in the above.

As in Quine, the construed numbers are the numbers construed in terms of Zermelo-Fraenkel-type set theory or von Neumann-type set theory as used usually in pure mathematics. Here these numbers aer different from the usual numbers in our ordinary life which have been used in various ways without the original meaning of them but as a 'name' also, which has been said to be "referential" in Cheng [4]. As in Belair [1], in order to construct a handy model it is in practice useful to use the Zermelo-Fraenkel-type set theory or von Neumann-type set theory without giving any useful meanings or interpretations at all but the artificially defined formal definitions only.

In principle, this method is useful in a constructive method to define a system. By doing so, it can be verified finitely. However, it cannot be the whole system which can handle number theory in general. What we have here is that (1) this system cannot say anything in general about the logistic thesis at all, (2) this system does not give any positive answers to the decision problems in general, but it tried not to get involved in those problems, (3) this system did not provide any way to skirt the logical or semantic paradoxes at all, but it constructs step by step avoiding the elements which can possibly generate those contradictions in the axiomatic systems, and (4) finally as an artificial language, this system cannot give any algorithms at all in the practical applications or measurements, however this system itself keeps its own analyticity, well-definedness, or well-order.

It is undeniable also that this construed number has been regarded as the duration which can be the paradigm of all the various disciplines also. However I can hardly find enough evidence to assert that it can be the absolute duration conceived up to now especially as in Newton, I think.

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S. I. MARDAEV, *Smallest-fixed-point theorem for intuitionistic propositional logic and Grzegorczyk logic.*

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Modal propositional formulas are formed in the usual way from variables, the constant $\perp$ and the connectives $\&, \vee, \neg, \Box, \Diamond$. Connectives $\supset, \equiv$ are usual abbreviations.

Theorem 1. *Let $\phi(p, q_1, \dots, q_n)$ be a modal formula such that all occurrences of p are positive. There is a formula $w(q_1, \dots, q_n)$ such that*

$$\vdash_{Grz} \omega(q_1, \dots, q_n) \equiv \phi(\omega(q_1, \dots, q_n), q_1, \dots, q_n),$$

$$\vdash_{Grz} \Box(p \equiv \phi(p, q_1, \dots, q_n)) \supset (\omega(q_1, \dots, q_n) \supset p).$$

Propositional formulas are formed in the usual way from variables, the constant $\perp$, and connectives $\&, \vee, \neg, \supset$.

Theorem 2. *Let $\phi(p, q_1, \dots, q_n)$ be a propositional formula such that all occurrences of p are positive. There is a formula*

$$\vdash_{Int} \nu(q_1, \dots, q_n) = \phi(\nu(q_1, \dots, q_n), q_1, \dots, q_n),$$

$$\vdash_{Int} (p \equiv \phi(p, q_1, \dots, q_n)) \supset (\nu(q_1, \dots, q_n) \supset p).$$

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MAARTEN MARX, *Relational semantics for (conjugated) modal logics.*

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Modal logics are usually presented using Kripke semantics. One well known exception is **S5** which one can also present without using an accessibility relation. Recent research in Lambek calculus and Arrow logic has shown us that a complete relational semantics can be given for them. Both Lambek calculus and Arrow Logic might also be viewed as modal logics. In [1] we present a generalization of these results to modal logics with n-adic operators. We end with a discussion and some applications of this generalization.

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LARRY MATHEWS, *Model theory of p-adically closed rings.*

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We develop some basic model theory for p-adically closed rings, extending some fundamental work on these structures by Belair in [1] and [2]. We consider the theory Th(pCR) of

p-adically closed rings in a suitable first-order language in which it has elimination of quantifiers. We prove that Th(pCR) has prime models over arbitrary subsets, and also identify the substructure of models of this theory.

We derive some elementary consequences of quantifier elimination, namely that Th(pCR) is *t-minimal* (see [3]), model-theoretically bounded, and satisfies the Exchange Principle. We also prove that Th(pCR) has definable Skolem functions in a suitable extended language.

We also prove some basic results concerning the structure of definable sets in models of Th(pCR). We prove a cell decomposition theorem for definable sets, similar to that for p-adically closed fields in [4]. From this we deduce a theory of dimension for definable sets.

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A. S. MOROZOV, *Properties of automorphism groups of some recursive models.*
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A model M is called *recursive* (*decidable*) provided its universe is a recursive set of natural numbers and its (elementary) diagram is decidable.

Here we study properties of automorphism groups of models with specific properties.

THEOREM. *Let M be a recursive model and let G be a subgroup in $\mathrm{Aut}M$. Denote $I_n(M) \rightleftharpoons \{\langle \overline{a}, \overline{b}\rangle \in |M|^n \times |M|^n \mid \exists f \in G. f(\overline{a}) = \overline{b}\}$. Let the model $(M, I_n)_{n<\omega}$ be decidable. Then the following statements are valid:*

1. *The $\forall$-positive theory of G is a Π^0_1-set.*
2. *If all G-orbits are finite then the $\forall$-negative theory of G is a Σ^0_1-set.*

The conditions of the Theorem is often the case in recursive linear orders, fields, Boolean algebras.

D. E. PAL'CHUNOV, *Elementary types semigroup of boolean algebras with distinguished ideals.*
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Boolean algebras with distinguished ideals (we call them I-algebras) are considered in the signature $\langle \cup, \cap, -, 0, 1, I_1, \dots, I_\lambda\rangle$, where I_j are unary predicates which distinguish ideals; number λ is fixed.

Let $S \rightleftharpoons \{\mathbf{A} \mid \mathbf{A} \text{ is an } I\text{-algebra}\}/\equiv$ be the set of elementary types of I-algebras. Let $\mathfrak{S} \rightleftharpoons (S, \times)$, where $\times$ denotes the direct product: $[\mathbf{A}]_\equiv \times [\mathbf{B}]_\equiv = [\mathbf{A} \times \mathbf{B}]_\equiv$.

Notice that $\mathbf{S}$ is a commutative semigroup with unit; the unit of S is the class consisting of the degenerated Boolean algebra with $I_j = \{0\}$.

For $a, b \in \mathbf{S}$ we write $a \leq b$ if $(\exists c)(b = a \times c)$.

Designate $N(a) \rightleftharpoons (\forall c, d)(a = c \times d \to (a = c \vee a = d))$.

Remark. The relation $\leq$ is a partial order on $\mathbf{S}$.

Consider the following predicates $L(a)$, $F(a)$, and $C(a)$ on the set

$$S : \begin{cases} L([\mathfrak{A}]_{\equiv})\ \&\ \text{iff the } I\text{-algebra } \mathfrak{A} \text{ is local (see [1])}, \\ F([\mathfrak{A}]_{\equiv})\ \&\ \text{iff } \mathfrak{A} \text{ is finitely axiomatizable}, \\ C([\mathfrak{A}]_{\equiv})\ \&\ \text{iff } \mathfrak{A} \text{ is } \omega\text{-categorical}. \end{cases}$$

THEOREM. *Locally, finite axiomatizability and ω-categoricity of I-algebras can be expressed in the language of the semigroup* $\mathbb{S}$ *by the following first-order formulae:*

$$L(a) \sim (\forall b \leq a)(\exists c < b)(\forall d < b)((c \leq d \,\&\, N(d)) \rightarrow c = d),$$

$$\begin{aligned} &F(a) \sim L(a) \,\&\, (\forall b, d, e) \\ &\quad ((e = d \times e \,\&\, a \neq e \,\&\, d = d \times d \,\&\, b \leq d \,\&\, b \neq b \times b \\ &\quad \&\, N(b) \,\&\, N(d)) \rightarrow (\exists c)(N(c) \,\&\, b < c < d)), \end{aligned}$$

and

$$C(a) \sim (\forall b \leq a)(F(b)).$$

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D. SINGH, *An extension of the program KRIPKE.*

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In this paper I propose an extension of the Program KRIPKE of [2] by interpreting a multiset of formulae, viz., $[A, B^m, C^n, \ldots]$ as an ordinal tree, viz., $[A \cdot B^m \cdot C^n \cdots]$, where A, B and C can be treated as events and A^k denotes A occurring k times. This extension can enable KRIPKE to capture temporal environments by way of interpreting an ordinal tree as a temporal data structure.

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ALDO URSINI, *An easy way to phase semantics for linear logic.*

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The aim is to give a simple interpretation of the basic connectives of (classical or intuitionistic, commutative or non-commutative) linear propositional logics. If R is a binary relation on a set M, $a \in M$ and $X \subseteq M$, let $R(a) = \{x \mid aRx\}$, $R(M) = \{x \mid \text{ for some } a \in M, aRx\}$. A *phasoid* is a structure $\langle M, R, \cdot, 1\rangle$ where $\cdot$ is a binary operation in M with neutral element 1, R is a binary relation on M and: (i) for all $a \in M, X \subseteq M$, $R(a) \subseteq R(M)$ iff $\check{R}(a)] \subseteq \check{R}(M)$; (ii) for all $a, b, c \in M$, $(ab)Rc$ iff $aR(bc)$ (Here $\check{R}$ is the converse of R; $a\check{R}b$ iff bRa.) The R-facts are those $X \subseteq M$ such that if $R(a) \subseteq R(X)$, then $a \in X$. The basic linear operations on R-facts are defined very simply. For instance:

$$X^{\perp} = \{x \mid R(x) \cap X = \emptyset\};$$

$$X \text{ PAR } Y = \{x \mid \text{ for any } b, c \in M, \text{ if } (bc)Rx \text{ then } b \in \check{R}(Y) \text{ or } c \in \check{R}(X)\}.$$

Various completeness results are obtained.

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YU. G. VENTSOV, *Effectively infinite algorithmic dimension of recursive models.*

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We study recursive models of infinite algorithmic dimension [4]. A class of recursive models of Δ^0_2-effectively infinite algorithmic dimension is characterized in terms of effective density of some subset of the set of finite isomorphisms which can not be extended up to an automorphism. This class is sufficiently wide for it to contain such well-known classes, defined by Goncharov [1, 2], as of branching and unbounded recursive models. According to [3], this is the best result on effectively infinite dimension we can expect.

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XIAOKANG YU, *Measurability of analytic sets, formalized in second-order arithmetic.*

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Measurability of Borel sets of compact complete separable metric spaces, formalized in the language of second-order arithmetic, is established in the subsystem ATR_0 with the aid of Borel codes. Measurability of analytical sets, coded in the language of second-order arithmetic, can be established in the subsystems Π^1_1-CA_0.

SHU-GUO ZHANG, *A characterization of strongly normal ideals on $P_\kappa\lambda$.*

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Proposition 1. *If $I_{\kappa\lambda} \subseteq I$ and I is a normal ideal over $P_\kappa\lambda$ (in fact a little weaker assumption is enough) then I is κ-complete*

Lemma 2 [Johnson]. *If $\mu < \kappa$, $\mu < \nu \leq \lambda$ and I is normal, then I is (μ, ν)-distributive iff whenever $A \in I^+$ and $f : A \rightarrow P(\nu)$ is such that for each $x \in A$, $f(x) \in [x]^{\leq \mu}$, there is a $B \in P(A) \cap I^+$ such that $f \restriction B$ is constant.*

We give a characterization of strongly normal ideals over $P_\kappa\lambda$ as follows, by using Lemma 2.

Theorem 3. *If κ is weakly inaccessible, a normal ideal I over $P_\kappa\lambda$ is strongly normal iff it is (μ, λ)-distributive for each $\mu < \kappa$.*

Corollary 4. *If $\lambda^{<\kappa} = \lambda$ and κ is λ-ineffable or almost λ-ineffable or has the λ-Shelah property, then the associated ideals $NIn_{\kappa\lambda}$, $NAIn_{\kappa\lambda}$ and $NSh_{\kappa\lambda}$ are (μ, λ)-distributive for every $\mu < \kappa$*

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1994 ANNUAL MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

Gainesville, Florida, March 5–8, 1994

The Annual Meeting of the Association was held at the University of Florida in Gainesville, Florida, March 5–8, 1994. The program included eight invited hour talks; Special Sessions in Set Theory, and Proof Theory and Computer Science each consisting of five invited half-hour talks; forty-eight contributed talks; and five talks contributed by title. In addition, Donald A. Martin delivered the Fifth Annual Gödel Lecture entitled, $L(\mathbb{R})$: *A Survey*, and Charles Parsons delivered his Retiring Presidential Address entitled, *Platonism and Mathematical Intuition in Kurt Gödel's Thought*. The Ulam Lecture, sponsored by the Southeastern Logic Symposium and by the Department of Mathematics of the University of Florida, was entitled *Transparent Proofs and Limits to Approximation*, and was given by László Babai. The meeting took place shortly after the death of one of the most important mathematical logicians of the twentieth century, Stephen Cole Kleene, on January 26, 1994. Kleene's major contributions to both recursive function theory and intuitionistic mathematics were described in remarks made by Yiannis Moschovakis at the meeting. The Program Committee consisted of Steven Buechler, Manual Lerman (Chair), Dale Miller, William Mitchell, and Gila Sher. The Local Organizing Committee consisted of Douglas Cenzer, Jean A. Larson (Chair), William J. Mitchell, Greg Ray, and Rick Smith.

The following invited addresses were presented at the meeting:

Lou van den Dries (University of Illinois), *New O-Minimal Expansions of the Field of Reals*.

Jean-Yves Girard (CNRS, Marseilles), *Geometry of Interaction: A Monist Approach to Logic*.

Stewart D. Shapiro (Ohio State University), *Logical Consequence: Models and Modality*.

Robert I. Soare (University of Chicago), *Models of Arithmetic and Upper Bounds for Arithmetical Sets*.

Simon Thomas (Rutgers University), *The Cofinality of the Infinite Symmetry Group*.

James Baumgartner (Dartmouth College), *New Ultrafilter Problems on ω*.

Julia Knight (University of Notre Dame), *Nest Priority Constructions*

Harold T. Hodes (Cornell University), *Introduction and Elimination, in General and in Particular*.

The following addresses were presented at the Special Session on Proof Theory and Computer Science:

Natarajan Shanker (SRI International), *Proof Search in Intuitionistic and Linear Logic*.

Gopalan Nadathur (Duke University), *Some Applications of Proof Theory in Logic Programming*.

Douglas Howe (Bell Laboratories), *Functional Programming Logics Based on Set Theory*.

Maria Luisa Bonet (University of Pennsylvania), *On Finding Hard Tautologies for Frege Systems*.

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Radha Jagadeesan (MIT), *Games and Linear Logic*

The following addresses were presented at the Special Session on Set Theory:

Menachem Kojman (Carnegie-Mellon University), *Negations of GCH and Universality Spectrums.*

Stevo Todorčević (University of Toronto), *Classifying Finite Partitions of* X^n.

Matthew Foreman (University of California-Irvine), *How Different are* $\aleph_2$ *and* $\aleph_1$*?*

Andreas Blass (University of Michigan), *Some Cardinality Characteristics of the Continuum.*

Ernest Schimmerling (MIT), *Inner Models with Woodin Cardinals.*

Abstracts of contributed talks given (in person or by title) by members of the Association for Symbolic Logic follow.

For the Program Committee
MANUEL LERMAN

Abstracts of contributed papers

STÅL AANDERAA, *A Trachtenbrot inseparability theorem for groups.*
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Given a finitely presented group G. Then we may define two sets of words in the generators of G. Let A be the set of words w in the generators of G such that w becomes the identity in G. Let B be the set of words w in the generators of G such that there exists a finite homomorphic image of G in which w becomes different from identity. Then A and B are both r.e. sets and they are disjoint. I can prove that there exists a finitely presented group G such that the sets A and B are recursively inseparable, i.e., there exists no recursive set C of words such that A is a subset of C, and such that C and B are disjoint.

TUNA ALTINEL, *Simple groups of finite Morley rank which contain a strongly embedded subgroup.*
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We prove results related to the Cherlin-Zilber conjecture which states that an infinite simple group of finite Morley rank should be an algebraic group over an algebraically closed field.

A proper definable subgroup M of a group G of finite Morley rank is called *strongly embedded* if it contains involutions (elements of order 2) and for any $g \in G \backslash M$, $M \cap M^g$ does not contain any involutions.

A connected nonsolvable group of finite Morley rank whose proper definable connected subgroups are nilpotent is called a bad group. A structure $\langle K, +, \cdot, A\rangle$ of finite Morley rank where K is an algebraically closed field and A is a proper infinite subgroup of the multiplicative group of K is called a bad field.

We prove that a simple group of finite Morley rank whose proper definable simple sections are algebraic, which contains a strongly embedded subgroup, does not interpret a bad field and in which the Sylow 2-subgroups contain infinitely many involutions, is isomorphic to $\mathrm{PSL}_2(\mathrm{K})$ where K is an algebraically closed field of characteristic 2.

The connected component of a Sylow 2-group of a group of finite Morley rank is the central product of a definable, connected nilpotent group of bounded exponent and a divisible abelian group. A group G of finite Morley rank is said to be of mixed type if both of these subgroups of a Sylow 2-subgroup of G are nontrivial. We discuss potential applications of the theorem to the groups of mixed type.

PAUL BANKSTON, *The model-theoretic topology of compact Hausdorff spaces.*

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We describe a mechanism, based on the Keisler-Shelah ultrapower theorem, which translates a substantial amount of classical first-order model theory into the setting of compact Hausdorff spaces. To achieve such a translation formally, all one needs is a category equipped with products and enough directed limits to allow an autonomous notion of ultraproduct, as well as a cardinal-valued function defined on the class of objects and reasonably like the usual cardinality in the presence of certain morphisms (e.g., monos and epis). In our situation the category is the opposite of the category of compact Hausdorff spaces and continuous maps, and the cardinal-valued function is topological weight.

Given a model-theoretic notion (or assertion) N, let $[N]^*$ be its translation (when possible) into the compact Hausdorff setting. For example, if $f : X \rightarrow Y$ is an [elementary embedding]*, then f is a continuous surjection that preserves and reflects such properties as connectedness, total disconnectedness, covering dimension, and finite cardinality. (On the other hand, f need not preserve or reflect strong inductive dimension, and need not reflect either metrizability or local connectedness.)

If N is a theorem of general model theory, then the assertion $[N]^*$ is not guaranteed to be a topological theorem; however, we know of no examples where this fails. Examples of success include the compactness theorem, as well as all known versions of the Löwenheim-Skolem theorem (with topological refinements). There are also some more specialized results. For example, if $f : X \rightarrow Y$ is a function between compact Hausdorff spaces and X is an arc (i.e., a homeomorphic copy of the real closed unit interval), then f is an [elementary embedding]* if and only if Y is an arc and f is a continuous monotone surjection.

STEPHEN BLOCH, *On parallel hierarchies and* R_k^i.

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In this paper I define hierarchies of function and relation classes, constructed from parallel complexity classes in a manner analogous to the polynomial hierarchy. There is a surprising correspondence between them and the levels of the polynomial hierarchy, and a Buss-style witnessing theorem relating the levels of these hierarchies to definability in the bounded arithmetic theories R_k^i; these results provide a more elegant proof of a logical characterization due to Buss, Krajíček, and Takeuti (1993). Finally, I show results analogous to those of Krajíček, Pudlák, and Takeuti (1991): if R_k^{i+1} is conservative over S_k^i with respect to Boolean combinations of $\Sigma_{i+1,k}^b$ formulae, then every $\Pi_{i,k}^b$ optimization problem can be approximated to within a constant factor by a finite student-teacher game at level $i+1$ of the parallel hierarchy; this in turn would imply that the parallel hierarchy, and hence the (quasi)polynomial hierarchy, collapses to level $i+2$.

MOGAN BROWN, *A characterization of* Q_3.

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Q_3 is an analog of the set of hyperarithmetical reals, $\Delta_1^1 \cap \mathbb{R}$, at level three of the analytical hierarchy. A. Kechris proved that $\Delta_1^1 \cap \mathbb{R}$ is the largest Π_1^1 set of reals (not equal to $\mathbf{R}$) which is closed downward under hyperarithmetical degrees. Using the recent result of J. Steel, that under certain large cardinal assumptions, the core model is Σ^1_3 correct, and the result of H. Woodin that assuming Δ^1_2 determinacy, all uncountable Δ^1_3 sets of reals contain a member of every Δ^1_3 degree above that of y^0_3, I prove the following:

THEOREM. (Assume Δ^1_2 determinacy) *Q_3 is the largest Π^1_3 set of reals (not equal to $\mathbb{R}$) closed downward under Δ^1_3 degrees.*

In particular, assuming determinacy there is no uncountable Π^1_3 set of reals (not equal to $\mathbb{R}$) which is closed downwards under Δ^1_3 degrees.

This solves a conjecture posed in A. Kechris, D. Martin, and R. Solovay, *Introduction to Q-theory*, ***Cabal Seminar 79–81.***

PAUL J. CORAZZA, *Justifying large cardinals with the wisdom of the ancients.*
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Large cardinals have become pervasive in set theory, but how do we justify them? Motivating principles like generalization, reflection and resemblance are sometimes helpful but still do not account for all large cardinals, especially the really big ones. Motivating principles must draw upon a fundamental intuition about the structure of the mathematical universe. One of the most explicit examples of such intuition is Cantor's notion of the Absolute Infinite. We present interesting details of the ancient Vedic conception of the Absolute Infinite that seem to suggest not only that there ought to be an elementary embedding of the universe to itself but also how to get around Kunen's proof that such embeddings don't exist! This Vedic model suggests that the "right" set theory would admit such a (modified) elementary embedding; this embedding is still strong enough to prove the existence of all large cardinals through super n-huge for every n. This model also suggests a number of unexpected research problems concerning large cardinals and the structure of the universe.

DENNIS F. CUDIA, *The thermodynamic limit.*
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The thermodynamic limit essential to the theory of information is given in the following corollary.

THEOREM 1. *If $\{Y(mx)/m : m = 1, 2, \dots\}$ is convergent to $Y_\infty(x)$ then, Y_∞ is an extensive variable, i.e., $Y_\infty(nx) = nY_\infty(x)$ for all integers $n > 0$.*

COROLLARY 2. *If $\{Z(mx)^{1/m} : m = 1, 2, \dots\}$ is convergent to $Z_\infty(x)$, then* $\log Z_\infty$ *is an extensive variable.*

The thermodynamic limit allows three things to be done:

1) exercise the dichotomy between information I and entropy S, that $S = -I$,
2) determine properties without using Stirling's approximation,
3) produce an extensive entropy S_∞ from a non-extensive thermodynamic probability W_M. For notation and general background see [1].

All probability distributions discussed here are multinomial distributions. A macrostate is any n-tuple of positive integers $(N_1, \dots, N_n)$. The distribution corresponding to the macrostate is $p_i = N_i/N$, $i = 1, \dots, n$ where $N = N_1 + \dots + N_n$. It is intuitively obvious that some information is lost in going from the macrostate to the distribution. That is, $I(N_1, \dots, N_n) > I(p_1, \dots, p_n)$. Since $S = -I$ it follows that $S(N_1, \dots, N_n) < S(p_1, \dots, p_n)$. In order to formulate a precise inequality recall that $W_M(N_1, \dots, N_n) = N!/(N_1! \cdot \dots \cdot N_n!)$ is the thermodynamic probability of mixing. Denote $W_m(N_1, \dots, N_n) = W_M(mN_1, \dots, mN_n)^{1/m}$ and denote

$$(*) \qquad W_\infty(N_1, \dots, N_n) = \lim W_m(N_1, \dots, N_n) = (1/p_1)^{N_1} \cdot \dots \cdot (1/p_n)^{N_n}$$

Then we have the following fundamental inequality.

THEOREM 3. $W_M(N_1, \ldots, N_n) < W_\infty(N_1, \ldots, N_n)$.

SKETCH OF PROOF It must be shown that

$$(**) \qquad N!/N^N < (N_1! \cdot \ldots \cdot N_n!)/(N_1^{N_1} \cdot \ldots N_n^{N_n})$$

Proceeding by mathematical induction on N, if (**) is true for $(N_1, \ldots, N_n)$ then it is true for $(N_1 + 1, N_2, \ldots, N_n)$ provided $(N_1/N_1 + 1)^{N_1} > (N/N + 1)^N$. But this last inequality is true because

$$1 < (1 + x^{-1})^x < (1 + y^{-1})^y < e, 0 < x < y.$$

Proceeding to 2) without Stirling's approximation and omitting the macrostate arguments we have the following.

THEOREM 4. $\lim(W_{m+1}/W_m) = 1$ *Therefore the sequence* $\{W_m : m = 1, 2, \ldots\}$ *has exactly one point of accumulation.*

Using Stirling's approximation and the fundamental inequality we have the following.

THEOREM 5. *The sequence* $\{W_m : m = 1, 2, \ldots\}$ *converges to* W_∞ *from below.*

Turning now to 3) the connection with the laws of thermodynamics is obtained.

DEFINITION 6. For any macrostate $(N_1, \ldots, N_n)$ the *limit thermodynamic probability of mixing* is denoted by $W_\infty(N_1, \ldots, N_n)$ and defined by (*).

Employing the Boltzmann Principle the following definition is obtained.

DEFINITION 7. For any multinomial distribution $p_1, \ldots, p_n$ and any integer N such that $Np_i, i = 1, \ldots, n$ are integers, the *limit entropy of mixing* is denoted by $S_\infty(p_1, \ldots, p_n, N)$ and defined by $S_\infty(p_1, \ldots, p_n, N) = k \log W_\infty(Np_1, \ldots, Np_n)$

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RUTH EBERLE, *Syntax and semantics of diagrams in natural deduction.*

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Hyperproof is a logical system, implemented in a computer program by the same name [2], [1], that extends first-order logic to include diagrammatically represented information With the formal addition of diagrammatically represented information, proofs and other analytical reasoning problems can be solved far more naturally and efficiently than is possible using purely sentential first-order logic.

Both the diagrams and sentences of *Hyperproof* concern a domain of discourse consisting of blocks. The blocks may have certain properties, and certain relations may exist between them. The diagrams, which can be used in proofs just like sentences, are visual representations of situations consisting of such blocks. In this paper I formalize the syntax and semantics of the diagrams in *Hyperproof* as well as the rules of inference for reasoning between sentences and diagrams.

Hyperproof-like systems can also be used to reason about other domains of discourse, using diagrammatic representation that differ from those of *Hyperproof*. Certain of these differences are due to the fact that unlike sententially represented information, diagrammatically represented information is strongly context-dependent. Thus, I also discuss which aspects of the syntax and semantics are generalizable to include different domains of discourse and which are not.

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TODD EISWORTH, *Selective ultrafilters and* $\omega \rightarrow (\omega)^{\omega}$.

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It is a theorem of Mathias that given a selective ultrafilter $\mathcal{U}$ and a partition of $[\omega]^{\omega}$ into $\mathcal{X}$ and $[\omega]^{\omega}\backslash\mathcal{X}$, there is an infinite set $X \in \mathcal{U}$ all of whose infinite subsets lie in one piece of the partition. Assuming the existence of a Mahlo cardinal, he also proved the consistency of the stronger partition property in which $\mathcal{X}$ is only assumed to be a member of $L(\mathbb{R})$. In this talk, I shall outline a proof that if CH holds (so in particular selective ultrafilters exist), this strengthened partition theorem implies that $\aleph_1$ is inaccessible in L.

SY D. FRIEDMAN AND DORSHKA WYLIE, *Embedding hierarchies and the fine structure of K.*

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An approach to the theory of mice and the core model K is described, using the concept of embedding hierarchy. The aim is to obtain a hierarchy for the core model which permits a more satisfying fine structure theory than the existing hierarchies. This should facilitate simpler proofs of combinatorial principles as well as allow for a good theory of coding over core models.

WILLIAM GASARCH AND JEFFREY HIRST, *Reverse mathematics and recursive graph theory.*

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We examine several results of recursive graph theory within the proof theoretic framework of reverse mathematics. The interplay of the two approaches yields some new combinatorial arguments and recursion theoretic insights.

A typical result is the following theorem concerning graph colorings, inspired by the work of Bean [1]. We say that a graph G is *k-chromatic* if there is a function mapping the vertices of G into k which always assigns different colors to neighboring vertices. If every finite subgraph of G is k-chromatic, we say that G is *locally k-chromatic*. The following theorem strengthens a result in [3].

THEOREM ($\mathbf{RCA}_0$). *For every $k \geq 2$, the following are equivalent*

(1) $\mathbf{WKL}_0$.

(2) *If G is locally k-chromatic, then G is k-chromatic*

(3) *If G is locally k-chromatic, then G is $2k-1$-chromatic*

We also consider results on bounded graphs, Euler paths, and Hamiltonian paths. For an overview of recursive combinatorics, see [2]. A nice introduction to reverse mathematics is given in [4].

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A. HEATON, *Axiomatic subrecursion theory.*
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We attempt an axiomatization of the kind of subrecursive classes naturally arising in a proof-theoretic context:

$$\mathrm{REC}(<|T|) = \text{PROVABLY RECURSIVE IN } T.$$

The axioms are a modification of the Moschovakis-Fenstad theory for "full" recursion theory, obtained by regulating the enumeration scheme according to a given well-ordering $\prec$ on codes, thus

$$f_i(e,x) = \begin{cases} f_e(x) & \text{if } e \prec i \\ 0 & \text{otherwise.} \end{cases}$$

We study various natural models and embeddings between them, our methods making use of subrecursive hierarchies.

JAMES E. HELMREICH, *Ultrahomogeneous expansions of ultrahomogeneous graphs.*
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We say a countable structure **M** is *ultrahomogeneous* provided every isomorphism of finite substructures of **M** extends to an automorphism of **M**. Lachlan and Woodrow have completely classified the countable ultrahomogeneous graphs. We expand the language of graphs to include a new unary predicate. In this expanded language, ultrahomogeneous 2-colorings of ultrahomogeneous graphs are found to be uniquely determined by the isomorphism type of each color.

GREG HJORTH, *A new Borel hierarchy theorem.*
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It is possible to strengthen the usual Borel hierarchy theorem by showing that ZFC proves that there is no Borel definable injection from the Borel sets at one level into the Borel sets of some lower level of the Borel hierarchy.

CHRYSAFIS HARTONAS, *Algebraic and Kripke-style semantics for substructural logics.*
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The central topic of this paper is to present a systematic approach to the algebraic and

Kripke-style semantics for logics with restricted structural rules, notably for logics on an underlying non-distributive lattice. We provide a new topological representation theorem for general lattices, using the filter space X. Our representation involves a Galois connection on subsets of X, hence a closure operator Γ, and the image of the representation map is characterized as the collection of Γ-stable, compact-open subsets of the filter space (which is characterized as a certain kind of lattice-ordered Stone space). The original lattice L is thus imbedded in the complete lattice $(\Gamma X, \subseteq, \bigcap, \bigvee)$ of stable sets. The representation is shown to be functorial and is extended to a Stone-type duality

We then consider lattices with additional, normal, operators An operator f on a lattice L is normal if $f : L^{(0)} \times \cdots \times L^{(n-1)} \to L^{(n)}$ is a monotone map, where each $L^{(i)}, i \in n+1$, is either L or L^{op} and f sends finite joins (or meets) of $L^{(i)}$ to joins (respectively, meets) in $L^{(n)}$. Every familiar operator, such as $\circ$ (times), $+$ (plus), $\leftarrow$, $\rightarrow$, $\neg$, $\Box$, $\Diamond$ is normal. We extend the lattice representation and duality theorems to the case of lattice-ordered algebras with normal operators.

We also raise and resolve the question of representation of operators by relations and conclude with a duality of algebraic and Kripke-style semantics. We give an extensive number of examples of representations of specific operators and provide explicit constructions of canonical Kripke frames, based on our representation arguments, for a number of logical systems, notably for Intuitionistic and Classical Linear Logics, or Noncommutative Linear Logic (without exponentials) and of fragments thereof.

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JOSÉ IOVINO, *The geometry of stable Banach spaces.*

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In [1] we introduced the concept of stability for theories of Banach spaces and Banach space structures. Significant classes of Banach spaces (e.g., Hilbert spaces, $L_p(\mu)$ spaces for $1 \leq p < \infty$) are stable.

We proved in [1] that a complete Banach space theory T is stable if and only if the following condition holds in every model of T. If $\mathcal{R}(\bar{x}, \bar{y})$ is a definable real-valued relation and $(\bar{a}_m)$, $(\bar{b}_n)$ are bounded sequences,

$$\lim_{n\to\infty} \lim_{m\to\infty} \mathcal{R}(\bar{a}_m, \bar{b}_n) = \lim_{m\to\infty} \lim_{n\to\infty} \mathcal{R}(\bar{a}_m, \bar{b}_n),$$

where the equality means that if the limits exist, they are equal.

On any Banach space, the function $x \mapsto \|x\|$ is a definable real-valued relation. Thus, if T is a stable Banach space theory, every separable model of T is "stable" in the sense of Krivine and Maurey [2]. In fact, when T is model complete, the converse is also true. The examples of stable theories mentioned above admit quantifier elimination.

I shall discuss some of the conditions stability imposes on the geometry of a Banach space.

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HERMAN RUGE JERVELL, *A Ramsey theorem for trees.*

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We generalize the results of Paris-Harrington and Ketonen-Solovay to get a Ramsey theorem for certain wellfounded homogeneous trees. The techniques used come from Girard's Π^1_2-logic.

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CARL G. JOCKUSCH AND ALEXANDRA SHLAPENTOKH, *Non-standard presentations of $\mathbb{Q}$ and $\mathbb{Z}$.*

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Given two r.e. degrees $a \leq b$, there exists a one-to-one recursive function $f : \mathbb{Q} \to \mathbb{N}$ such that $f(\mathbb{Z}) \in a$, $f(\mathbb{Q}) \in b$, and there exist recursive functions $P+, P-, P\cdot, P/ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $P + (f(x), f(y)) = f(x+y), P - (f(x), f(y)) = f(x-y), P \cdot (f(x), f(y)) = f(xy), P/(f(x), f(y)) = f(x/y)$, assuming $y \neq 0$.

STEVEN M. KAUTZ AND PETER BRO MILTERSEN, *Relative to a random Oracle, NP is not small: An application of the independence properties of algorithmically random sequences.*

Let A_0 and A_1 be infinite binary sequences which are 1-*random*, that is, algorithmically random in the sense of Martin-Lõf, Levin-Schnorr, or Chaitin. We say that A_0 and A_1 are *independent* if A_0 is 1-random relative to A_1 (the condition is symmetric, by a result of van Lambalgen). Independence properties for various kinds of subsequences of 1-random sequences have been obtained by van Lambalgen and Kautz; for example, if A is 1-random and A_0, A_1 respectively denote the even and odd bits of A, then A_0 and A_1 are independent. In this paper we prove a particularly strong independence property. if A is 1-random and A_1 is a subsequence chosen from A by means of a *bounded Kolmogorov-Loveland place selection*, and if infinitely many bits are left unselected, then the sequence A_0 of unselected bits is 1-random relative to A_1, i.e., A_0 and A_1 are independent. A Kolmogorov-Loveland place selection is a very general type of recursive selection rule which may be interpreted, for example, as the sequence of oracle queries in a relativized computation.

The application we have in mind concerns *resource-bounded measure theory* as formulated by Lutz, which provides a means of probabilistically describing the relative sizes of complexity classes and of defining the "random" sequences (languages) within a complexity class. Lutz has proposed the hypothesis that NP does not have measure zero within the class of exponential time languages, meaning loosely that a randomly chosen language in exponential time has non-negligible probability of being in NP. Lutz' hypothesis implies a strong separation of P from NP and is supported by the existence of a growing body of plausible consequences which are not known to follow from the weaker assertion P $\neq$ NP. What we prove is that relative to a random oracle (in fact, a 1-random oracle) NP does not have measure zero within exponential time, improving the earlier result of Bennett and Gill that P $\neq$ NP relative to a random oracle.

FARID KHAFIZOV, *AD and ultrapowers by some measures in* $L(\mathbb{R})$.

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Assume AD. We use techniques developed by Steve Jackson in his computation of $\delta^1_5 = (\sup_\eta j_\eta(\delta^1_3))^+ = \aleph_{\omega^{\omega^\omega}+1}$ (η ranges over the measures occurring in the homogeneous tree construction for a complete Π^1_3 set) to investigate the ultrapowers by measures, ν, occurring in the homogeneous tree construction for a complete Π^1_1 set. In particular, we compute $j_\nu(\omega_n)$ for all such measures ν which seem indispensable for precise computation of $j_\eta(\delta^1_3)$.

BORIS A. KUSHNER, *On Specker series.*

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Let α be a non-negative recursive sequence of constructive real numbers (c.r.n.; see, [2]). The series $\Sigma\alpha(i)$ is said to be a Specker series if its partial sums are bounded above by a number B and the series does not converge recursively. It is evident that the general term of a Specker Series converges to 0 classically; at the same time, as is well known, it may or may not converge to 0 recursively. The following theorem shows that general terms of all Specker series are equivalent from the point of view of recursive convergence.

THEOREM. *Let* $\Sigma\alpha(i)$ *be a Specker series, and* β *a recursive sequence of c.r.n.s that recursively converges to 0. Then the series* $\Sigma\beta(i)\alpha(i)$ *converges recursively.*

COROLLARY 1. *Let* $\Sigma\alpha(i)$ *be a Specker series, and* s *a c.r.n.,* $s > 1$. *The series* $\Sigma(\alpha(i))^s$ *is a Specker series iff* α *converges to 0 recursively.*

COROLLARY 2. *There is a Specker series* $\Sigma\alpha(i)$ *such that all the series* $\Sigma(\alpha(i))^s$ *where* s *is a c.r.n,* $s > 1$ *are Specker series.*

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CLAUDE LAFLAMME, *Strong meager properties for filters.*

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We discuss several "strong meager" properties for filters on the natural numbers between the classical Baire property and a filter being F_σ. Two such properties have been studied by Talagrand and a few more combinatorial ones are investigated. In particular, we define the notion of a P$^+$-filter, a generalization of the traditional concept of P-filter, and prove the existence of a non-meager P$^+$-filter. Our motivation lies in understanding the structure of filters generated by complements of members of a maximal almost disjoint family.

THOMAS E. LEATHRUM, *Completions of almost disjoint families*.

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Many interesting questions arise from considering what must be added to a nonmaximal almost disjoint family in order to make it maximal. In order to provide a general context for many such questions, define

$$adc_N(\mathcal{A}) \;=\; \min\{|\mathcal{A}'| : \mathcal{A}\cup\mathcal{A}' \text{ is an infinite maximal almost disjoint family of infinite subsets of the countable set } N\}.$$

So, for example, $\mathcal{A}$ is a maximal almost disjoint family if and only if $adc_\omega(\mathcal{A}) = 0$. The standard small cardinal invariant $\mathfrak{a}$ can be defined easily using adc : $\mathfrak{a} = adc_\omega(\emptyset)$. It is natural to consider $adc_N(\mathcal{A})$ for various particular families of $\mathcal{A}$. For example, if $\mathcal{A}$ is at most countable, then $adc_\omega(\mathcal{A}) = \mathfrak{a}$. However, not all of the questions are so trivial, as can be seen with any of several examples. among the results presented here will be a condition called *nowhere maximality* such that if $\mathcal{A}$ is nowhere maximal then $adc(\mathcal{A}) \geq \mathfrak{b}$. This result has certain extensions and strengthenings, but the nowhere maximality of $\mathcal{A}$ is *not* equivalent to $adc(\mathcal{A}) \geq \mathfrak{b}$. Is there a similar condition which *is* equivalent? This suggests several other questions: Under what conditions (on $\mathcal{A}$) is $adc_N(\mathcal{A}) > \omega? = 2^\omega?$ or $\geq \mathfrak{a}$?

STEVEN D. LEONHARDI, *A non-low$_2$ R.E. degree which does not bound any Slaman triple*.

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The high-low hierarchy is based on the Turing jump operator and provides a non-trivial partitioning of **R**, the recursively enumerable (r.e.) Turing degrees. Results of Cooper and Shore have shown that any class of the form high$_n$ or low$_n$ within this hierarchy is definable in the broader class of (not necessarily r.e.) Turing degrees. However, it is not yet known whether any such class (with $n > 0$) is definable within **R**.

Some partial progress has been made. Shore and Slaman have used both the property of bounding a nonsplitting pair, and the property of bounding what we call a Slaman triple, to separate the high from the low$_2$ r.e. degrees.

We prove the following:

THEOREM. *There exists a non-low$_2$ r.e. degree which does not bound any Slaman triple, that is, a triple* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ *of r.e. degrees such that*

(i) $\boldsymbol{u}$ *is not recursive,*
(ii) $\boldsymbol{w}$ *is not below or equal to* $\boldsymbol{v}$, *and*
(iii) *any nonrecursive* $\boldsymbol{b} \leq \boldsymbol{u}$ *has* $\boldsymbol{w}$ *below or equal to the join of* $\boldsymbol{b}$ *and* $\boldsymbol{u}$.

This result shows that the property of not bounding a Slaman triple does not define the low$_2$ degrees within **R**, and may be viewed as (albeit weak) evidence that the low$_2$ degrees may not be definable in **R**.

ADAM H. LEWENBERG, *Elementary pairs of o-minimal structures without poles*.
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An o-minimal structure without poles is an o-minimal structure in which every total 0-definable unary function is locally bounded. Similarly, an o-minimal theory T has no poles if every model of T is o-minimal without poles. Fix such a theory T. Let T_{tame} be the theory of structures of the form $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ where $\mathcal{N}$ is a proper elementary substructure of a T-model $\mathcal{R}$ such that $\mathcal{N}$ is Dedekind complete in $\mathcal{R}$, and st is the induced standard map on $\mathcal{R}$ induced by $\mathcal{N}$. If T is universally axiomatizable and admits quantifier elimination then so does T_{tame}. Furthermore, T_{tame} is complete; if T is model complete then so is T_{tame}.

MARK MCKINZIE AND CURTIS TUCKEY, *Elementary properties of infinite polynomials*.
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We have explored applications of hyperfinite polynomial theory to the elementary transcendental funcitons. Our techniques yield results traditionally thought to require concepts from the calculus; we formulate our concepts in extensions of nonstandard weak arithmetic. This work has potential applications in logic, history of mathematics, and calculus reform.

CHRIS MILLER, *A conjectured dichotomy for o-minimal expansions of ordered groups*.
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CONJECTURE.. *Let* $\mathfrak{R}$ *be an o-minimal expansion of an ordered group* $(R, <, +)$ *Then either* $\mathfrak{R}$ *defines a multiplication* $\times : R^2 \to R$ *such that* $(R, <, +, \times)$ *is a real closed field; or, for every* $\mathfrak{R}$*-definable function* $f : R \to R$*, there exists a definable additive function* $T : R \to R$*, a constant* $c \in R$*, and a definable function* $\varepsilon : R \to R$ *with* $\lim_{x \to +\infty} \varepsilon(x) = 0$ *such that* $f(x) = c + T(x) + \varepsilon(x)$.

It follows from a recent result of R. Poston that the conjecture is true if the elementary theory of $\mathfrak{R}$ has an archimedean model. Other evidence for the conjecture is given, and partial results are established. Some general results are proved for the case that no multiplication compatible with $(R, <, +)$ is definable in $\mathfrak{R}$; in particular, the ordered division ring of all $\mathfrak{R}$-definable additive functions is invariant under elementary equivalence.

MISAO NAGAYAMA, *Recent results in substructural implicational logics*.
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We would like to talk about recent results in substructural implication logics, such as BCK-logic and implicational relevant logics. The main result is a syntactic solution to the $P - W$ problem. The problem asks whether a is identical to b provided that $a \to b$ and $b \to a$ are both provable in $P - W$. E. P. Martin and R. K. Meyer originally gave an affirmative solution to this problem in [2] (for its concise version, refer to [1], pp. 375–384), however their method was semantical and not easy to grasp. Here we obtain a syntactical solution to this problem, based on a Gentzen-type system for $P - W$: The system $P - W$, without exchange rule, consists of the identity axiom, the left and right introduction rules of the implication and cuts. The left introduction rule of the implication and one of the cuts are defined by so-called *guarded merges*. The cut-elimination theorem for this system has been known among relevant logicians. Along with its proof, we obtain a normalization theorem for proofs in $P - W$, where we show that if $a \Rightarrow b$ is provable in $P - W$, we can in fact find its cut-free proof of a special form, giving us some syntactic information on a and b. Finally we briefly discuss the normalization theorems for proofs in other logical systems, and their applications.

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ANIL NERODE AND BAKHADYR KHOUSSAINOV, *Monadic second order theories of trees and semi-regular algebras.*
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Let $r_0, \ldots, r_n$ be symbols for partial unary operations. Every tree $\mathcal{A}$ over alphabet $R = (r_0, \ldots, r_n)$ is a unary algebra, where the unary partial operation r_i is $r_i(a) = ar_i$ if ar_i belongs to $\mathcal{A}$, undefined otherwise. We investigate the monadic second order theory it MSO($\mathcal{A}$) of such an $\mathcal{A}$ and the corresponding Rabin Automata (RA) [1, 2]. (Rabin's SnS deals only with the free algebra.) $\mathcal{A}$ is *regular* if empty or $\mathcal{A}$ has only finitely many non-isomorphic 1-generated subalgebras and each element has a successor. Let L be a nonempty subsignature of R. Path $d_0d_1d_2\ldots$ of $\mathcal{A}$ is an *L-path* if $(\exists n)(\forall i > n)(d_i \in L)$. A (Rabin) run is *$L$-successful* if its restriction to the L-paths is successful in Rabin's sense. This is a definition in $MSO(\mathcal{A})$ having a corresponding L-fragment in $MSO(\mathcal{A})$.

THEOREM 1. *The following are equivalent:*

(1) *L-emptiness over $\mathcal{A}$ is decidable. (Does there exist an L-successful run of RA on $\mathcal{A}$?)*
(2) *GH game determinacy is decidable (Who wins the Gurevich-Harrington game with corresponding winning L-acceptance condition?)*
(3) *The L-fragment of $MSO(\mathcal{A})$ is decidable.*

For ordinal η, we define $\mathbf{LQR}_\eta$. 1) $\mathbf{LQR}_0$ is all regular L-algebras. 2) Suppose given $\eta = \theta + 1$ and $\mathcal{A} \in \mathbf{LQR}_\theta$. If $a \in \mathcal{A}$ put $\mathcal{A}_\eta = \mathcal{A}_\theta \bigcup_{a \in \mathcal{A}} \{arx \mid a \in \mathcal{A} \,\&\, x \in \mathcal{A}_{I_a, r} \,\&\, r \in I_a\}$ into the next class. For limit ordinal η, put $\mathcal{A} \in \mathbf{LQR}_\eta$ if there exists a seqeunce $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_\theta \subset \ldots$ such that $\forall \theta < \eta(\mathcal{A}_\theta \in \mathbf{LQR}_\theta)$, and $\mathcal{A}_\eta = \bigcup_{\theta < \eta} \mathcal{A}_\theta$. Elements $\mathcal{A}$ of $\bigcup_\eta \mathbf{LQR}_\eta$ are the *L-quasi-regular algebras*. The *regularity degree* of $\mathcal{A}$ is the least η with $\mathcal{A} \in \mathbf{LQR}_\eta$, and is proved finite or ω. Call $\mathcal{A}$ *L-semi-regular* if the L-subalgebra generated by any $a \in \mathcal{A}$ is regular We prove that $\mathcal{A}$ is L-semi-regular iff $\mathcal{A}$ is L-quasi-regular. Algebra $\mathcal{A}$ is *recursively algebraic* if there exists a recursive sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots$ such that

(1) for each i, $\mathcal{A}_i \in \mathbf{LQR}_i$;
(2) for each i, if x and y generate isomorphic L-subalgebras in $\mathcal{A}_i$, then elements $r(x)$ and $r'(x)$ for which $r(x), r'(x) \in \mathcal{A}_{i+1} \backslash \mathcal{A}_i$ also generate isomorphic L-subalgebras;
(3) There exists a procedure which for all $i, x \in \mathcal{A}_i$, and $r \notin L$ such that $r(x) \in \mathcal{A}_{i+1} \backslash \mathcal{A}_i$ finds the number of isomorphism types of the 1-generated L-subalgebras of he L-algebra generated by $r(x)$.

THEOREM 2. (1) *For $\mathcal{A}$ a recursive algebraic L-semi-regular algebra with finite regularity* *degree, the L-fragment of the MSO($\mathcal{A}$) is decidable.*

(2) *For $\mathcal{A}$ a recursively algebraic L-semi-regular algebra with infinite regularity degree, the L-degree of it MSO($\mathcal{A}$) is co r.e.*

(The proof uses the GH determinacy theorem [1] at each stage of the construction.)

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ANDRÉ NIES, *Undecidable fragments of elementary theories.*

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If an elementary theory T is known to be undecidable, an interesting further question is for which k (if for any) the framgent of the Π_k-sentences in T or the set of Σ_k-sentences in T becomes undecidable: for small k, these sentences can be viewed as the "feasible" sentences and, if T is the theory of a particular structure (e.g., a degree structure in recursion theory), they normally coincide with the sentences about the structure actually considered in mathematical research. A first result in this direction has been given in [3], where the Π_3-theory of the structure of all Turing degrees is shown to be undecidable.

Undecidability of an elementary theory Th($\mathbf{C}$), $\mathbf{C}$ a class of structures of a given type, is usually proved indirectly, by coding in a uniform way the members of a complicated class of $\mathbf{D}$ of structures. It is required that each subtheory of Th($\mathbf{D}$) containing the valid sentences is undecidable (such a theory is called hereditarily undecidable). Using some ideas from [3], we introduce a general framework where the notions involved are adapted and the coding method is refined to obtain undecidability results for fragments. The method is applied first to classes of finite graphs and finite lattices, and, in a second step, to degree structures. For instance, we show that the Π_3-theory of the class of finite distributive lattices is hereditarily undecidable, by coding the class of finite bipartite graphs by Σ_1-formulas with parameters. Next, using the fact that each finite distributive lattice is isomorphic to an initial segment of the r.e. many-one degrees ([1]), we conclue that this degree structure has undecidable Π_3-theory. The method also applies to other r.e. degree structures. Here results can be obtained by coding the class of finite bipartite graphs directly ([2]).

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GREG RAY, *Truth in an intended model.*

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Tarski gave a precise characterization of truth for formal, interpreted non-modal languages. However, in our formal characterization of the logical properties—logical truth, logical consequence—we do not use this notion of truth. Rather, we use the model-theoretic notion of truth-in-a-model. Since, presumably, logical truth is a subspecies of truth, and logical consequence involves at least truth preservation, there had better be some connection between our analysis of truth and the technical notion of truth-in-a-model. In the case of non-modal language, we bridge the gap between the two, by claiming that a sentence is true just in case it is true in an intended model. Where an intended model is a model which has as domain the actual objects of discourse, and which relates these objects in an appropriate manner. However, the same strategy applied to the case of modal languages seems to require an intended modal model whose domain includes mere possibilia — things which might have existed but do not. If such things don't exist, then there can be no intended modal models. If we insist that there are such models, then we seem committed to the actual existence of the possibilia. Building on recent work by C. Menzel, I will sketch a strategy for giving a purely extensional, model-theoretic semantics for modal lnaguages requiring nothing but pure sets and the actual objects of discourse. To do this, we must define a notion of *intended model* which does not require possibilia.

MITCH RUDOMINER, *A connection between inner model theory and descriptive set theory*

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We discuss an extension of some work done by Donald A. Martin, John Steel, and Hugh Woodin. This work establishes connections between Inner Model Theory and Descriptive Set Theory by proving theorems of the following form: We start with a particular countable set of real numbers, A. This set is defined using concepts that are purely Descriptive Set Theoretic. A is the se tof reals which are Δ-definable from a countable ordinal parameter, for some particular pointclass Δ. The set A has much independent interest to Descriptive Set Theorists, who study it for their own reasons. The theorems of Martin, Steel, and Woodin then give an entirely different characterization of the set A using concepts from Inner Model Theory. It is shown that $A = \mathbb{R} \cap M$, where M is the canonical inner model for some large cardinal axiom. For example, let A be the famous set Q_3. Martin, Steel, and Woodin show that $A = \mathbb{R} \cap M$, where M is the canonical inner model with a Woodin cardinal.

We discuss an extension of this work to other pointclasses Δ, and other inner models M.

We then go on to discuss one particular application. A conjecture of Steel's concerning Inner Model Operators is proved. Also a generalization of Steel's conjecture is stated and proved.

ANDRE SCEDROV, *Stochastic interaction and linear logic.*

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In this joint work with P. D. Lincoln and J. C. Mitchell we present stochastic interactive semantics for propositional linear logic without modalities. The framework is based on prover/verifier interactions considered in computational complexity theory, in which the prover has unlimited power while the verifier can only toss fair coins or perform simple tasks when presented with the given formula or with subsequent messages from the prover. The additive conjunction & is described as random choice, which reflects the intuitive idea that the verifier can perform only "random spot checks." This stochastic interactive semantic framework is shown to be sound and complete. Furthermore, the prover's winning strategies are basically proofs of the given formula. In this framework the multiplicative and additive connectives of linear logic are described by means of probabilitistic operators, giving a new basis for intuitive reasoning about linear logic and a potential new tool in automated deduction. Extended abstract is available by anonymous ftp from ftp.cis.upenn.edu as pub/papers/scedrov/ipcup.{dvi,ps}.Z or by WWW on URL http://www.cis.upenn.edu/~andre.

G.Y. SHER, *A new solution to the problem of truth (Or: how general and how substantive is Tarski's theory of truth?).*

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The idea of a theory of truth that is both general and substantive is problematic (even contradictory, according to Kant). Roughly, we can identify the problem as follows: Different things make different sentences true; a *substantive* definition of truth has to capture these differences, but a *general* theory of truth must abstract from them. In the literature, the problem is usually articulated in vague terms, but one way to give it precise content is to view it as involving the kind of consideration that Frege brought to the question "What is number?" According to Frege, an informative definition of "number" will not just distinguish between numbers and non-numbers, but will also give us the identity conditions of numbers, i.e., distinguish between one number and another. More generally, an informative definition of X has to show not only how to distinguish X's from non-X's, but also how to distinguish one X from another. For a definition of truth to be informative it has to distinguish not only between truths and falsities (or truths and non-truths) but also between one truth and another. Clearly, to say that all true sentences denote The True is not informative. But Aristotle's characterization of truth is also not informative; even Tarski's definition, many claim, is uninformative. But how can we define truth informatively? How can we define truth in a way that captures both what is common to all truths and what distinguishes between them?

What I call "the problem of truth" consists of two questions (the second possibly empty): (A) Is a general and substantive theory of truth possible? (B) What is accomplished by such a theory? In this paper I propose an outline of a positive answer to the problem of truth. I analyze Tarski's theory in light of this answer, and I explore the possibility of extending and generalizing Tarski's theory.

MASARU SHIRAHATA, *A linear conservative extension of Zermelo Fraenkel set theory.*

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In this paper, we present the system of set theory LZF with the unrestricted comprehension, formulated in linear logic. The resource awareness provided by linear logic makes it possible to have more sets than in classical set theory by the unrestricted comprehension, while avoiding the set-theoretical paradoxes.

LZF is built on top of the Zermelo-Fraenkel set theory without the axiom of regularity, or ZF^-. We formulate LZF as a sequent calculus with abstraction terms. The terms which correspond to the sets in ZF^- are called Z-terms and the formulas which contain only Z-terms are Z-formulas. We establish the partial cut-elimination result for LZF, which implies that if a Z-formula is provable in LZF, it is provable by using only Z-terms. It follows from this that LZF is a conservative extension of ZF^-.

The partial cut-elimination theorem is proved in two steps. First, we eliminate all cuts over complex formulas by the standard method. Then we proceed by induction on the size of a proof to eliminate the rest of the cuts except those over Z-formulas. LZF is designed so that no contraction is involved in the second step, and this guarantees that the size of a proof gradually descreases in cut-elimination process. This nice property of linear logic has been observed by many authors and used for the cut-elimination of various systems.

WILFRIED SIEG AND JOHN BYRNES, *A mathematical explication of Turing's argument.*

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Turing addressed in [5] the fundamental question: *What are the possible processes a human computer can carry out in calculating values of a number-theoretic function?* He claimed that all such processes can be carried out by his "automatic machines" or, in modern terms, by deterministic Turing machines. Turing's consideration for this claim can be reconstructed as involving: first, a formulation of finiteness conditions (for symbolic configurations and mechanical operations), second, a proof that computational processes (satisfying the finiteness conditions) can be carried out by a Turing machine; third, the thesis that all (mechanical) processes carried out by human computors must satisfy the finiteness conditions In [3] this is called *Turing's central thesis.* Turing made the latter highly plausible by appealing to the limitations of the human sensory apparatus; its ultimate justification, for Turing, is given by "the fact that the human memory is necessarily limited." In Turing's presentation (section 9 of [5]) the three aspects of his considerations are intermixed, and important steps in the proof are only hinted at; e.g., the reduction of computations on two-diemsnional configurations to computation on sequences is suggested by " ... I think it will be agreed that the two-dimensional character of apper is no essential of computation." Though a systematic and detailed analysis is given in [4], the proof mentioned above is still lacking mathematical rigor.

To formulate Turing's Theorem properly we have to specify the finite symbolic configurations and the mechanical operations that can be carried out on them. For this purpose we introduce *Kolmogorov machines.* These machines are inspired by the analysis of algorithms in [2], cf. also [6]; but they directly generalize Turing machines, when the latter are prsented via production systems as in [1, pp. 6–7]. These machines do not just operate on strings of symbols with one scanned element, but rather on finite labeled graphs with one *central vertex* that is distinguished by a special label, the graphs have the further property that distinct paths (containing the central vertex) have uniquely associated sequences of labels. The opeations of such machines replace distinguished subgraphs (with the central vertex) by finite graphs (also containing the central vertex); they are gien by a finite list of generalized production rules. We can prove:

THEOREM. *Computations of Kolmogorov machines can be simulated by Turing machines.*

Given this mathematical analysis, Turing's central thesis is turned into the thesis that Kolmogorov machines (that clearly satisfy the required finiteness conditions) do directly the work of human computors. The Theorem above constitutes then the general reduction of mechanical calculations to Turing machine computations.

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SŁAWOMIR SOLECKI, *Equivalence relations and the theory of continua.*
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The following is a particular case of the theorem proved in the paper.

Let X be a Polish space. Let $E \subset X \times X$ be a K_σ equivalence relation with at least two distinct equivalence classes. Assume each equivalence class of E is dense. Then there is no Borel subset of X which has exactly one point in common with each equivalence class of E.

A corollary to this theorem answers an old question in the theory of indecomposable continua. Namely, there is no Borel subset of an indecomposable continuum which has exactly one point in common with each composant.

ZORAN SPASOJEVIĆ, *(ω_1, ω_1)-gaps under $MA(\omega_1)$.*

I prove that $\mathrm{MA}(\omega_1)$ implies that every $\subset^*$-increasing ω_1 sequence of subsets of ω is a lower half of some (ω_1, ω_1)-gap This result can be seen as a refinement of the famous theorem by Hausdorff on (ω_1, ω_1)-gaps, when $\mathrm{MA}(\omega_1)$ is assumed.

KATHERINE ST. JOHN, *Substructural logics and the formal language of recursion.*
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In [1], Moschovakis interprets the meaning, or intension, of a formula as an algorithm that computes its truth value (following Frege). We outline a semantics for substructural logics based on the intensions of the Formal Language of Recursion presented in [2]. That is, we will assign to every formula a recursor that represents its meaning (ignoring timing). A natural question to ask is if A and B are provably equivalent, must their meanings—or some aspects of their meanings—be equal?

More precisely, say two formulas, A and B, are *provably equivalent* in a substructural logic S, if $A \Rightarrow B$ and $B \Rightarrow A$ are provable in a Gentzen-style proof system for S. Also say $A \leftrightarrow B$ is a *linear tautology* in a substructural logic S, if A and B are propositional formulas using logical symbols only from S, every variable occurs exactly once in A and exactly once in B, and $A \leftrightarrow B$ is a truth-table tautology.

We have shown for the multiplicative fragment of linear logic (following the notation of [3]):

THEOREM 1. *For arbitrary formulas of $\{+, *\}$, A and B are provably equivalent if and only if $int(A) \equiv int(B)$, where $int(A)$ is the intension of the formula of A, ignoring timing*

This result extends to the system $\{+, *, \sim, \rightarrow\}$ for linear tautologies.

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KATRIN TENT, *Modules and 1-based groups.*

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We show that certain 1-based groups (e.g., totally categorical ones) can be embedded into a $\bigwedge$-definable larger group on which all structure is given by its ring of definable endomorphisms modulo the algebraic closure of $\emptyset$. In this way the larger group becomes a module which is almost quantifier eliminable and is generated by the smaller group. The ring is finite and as a consequence there is some bound $k < \omega$ such that every type is implied by its restriction to formulas with at most k parameters.

JOHN THURBER, *Turing Degrees of Boolean Algebras.*

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We discuss conditions under which a Boolean algebra will or will not have a recursive copy.

One of the results is related to work by Feiner, and involves consideration of what types of sets can be coded into recursive Boolean algebras. Here sets are classified according to a measure of complexity defined by Feiner.

The results on low$_1$ and low$_2$ Boolean algebras stated in the titles of [2] and [5] depend on an isomorphism theorem of Remmel which is used to provide conditions under which two linear orders will generate isomorphic interval algebras. The techniques used suggest that a deeper understanding of the relationship between order types and the Boolean algebras they generate may be helpful in determining whether a low$_n$ Boolean algebra has a recursive copy for $n > 2$.

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DOLPH ULRICH, *A four-valued matrix validating the nine-character implication-pure theses.*

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A lemma shows that the matrix

$$\begin{array}{r|cccc}
C & 1 & 2 & 3 & 4 \\
\hline
*1 & 1 & 2 & 3 & 4 \\
2 & 1 & 1 & 3 & 1 \\
3 & 1 & 2 & 1 & 1 \\
4 & 1 & 1 & 1 & 1,
\end{array}$$

whose designated value is marked with an asterisk, includes among its tautologies the theses of the implicational fragment IF of the classical sentential calculus in which at most two distinct sentence letters occur Since the matrix rejects $CCpqCCqrCpr$, there follows as an intermediate corollary the Wajsberg/Diamond-McKinsey theorem for classical implication: with substitution and detachment as rules, every complete set of axioms for IF must include at least one axiom containing occurrences of three or more distinct sentence letters.

Additional argument utilizes the lemma to show that the matrix also validates all pure implicational theses of length nine or less, so that every complete axiom set for IF must include at least one axiom eleven or more characters long. This strengthens a result of Jaśkowski [1] that every such set must contain either one axiom of length at least eleven *or else two axioms each of length at least nine*, and is best possible since $\{CCpqCCqrCpr, CCCpqpp, CpCqp\}$ is known to axiomatize IF.

Though only *classical* implication is explicit throughout, the proofs given and the results obtained hold as well for virtually all logics of pure implication known in the existing literature, including the various logics of pure entailment and relevant implication, the strict-implicational fragments of the standard modal logics, and the relatives of implicational intuitionism and of C-pure logics in the Łukasiewicz family.

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GLEN WHITNEY, *The limits of extensionality in non-deterministic models of recursion*.
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Extending models of computation from deterministic to non-deterministic operations has long posed serious theoretical difficulties, especially when investigating various types of concurrent *merges* of processes. Sets of deterministic behaviors and ordinary functions on these sets most naturally model non-deterministic processes and process transformations, respectively. Yet Brock and Ackermann [1] exhibit computation networks (critically involving *angelic* merge nodes) which cannot be adequately modeled in this fashion, seemingly entailing a loss of *extensionality*. Common responses to such difficulties include disallowing some sets as possible processes [3]; adding new processes not corresponding to any set [4]; or adding additional information to each process [1]. A recent approach by Moschovakis [2] pushes the non-extensionality into the operations; i.e., process transformations are not determined solely by their action on processes, which are simply sets of behaviors. He asks (via a specific open question) whether the non-extensionality can now be eliminated altogether.

The present work yields a negative answer to Moschovakis' question. Beyond that, it provides a broad obstruction to extensional, non-deterministic models of (the standard logic of) recursion, as in the following:

THEOREM 1. *If D is any complete partial order containing a copy of $\omega+1$, then no extensional recursion structure on the powerset of D can assign the set ω as the meaning of some program and interpret some function symbol "or" by binary union so that the following two conditions* *are always satisfied*

$$\begin{aligned} x \textit{ where } [x = f(x) \textit{ or } g(x)] &\supseteq x \textit{ where } [x = f(x)], \\ x \textit{ where } [x = x \textit{ or } c] &= c \cup \{\bot\}. \end{aligned}$$

This improves the Brock-Ackerman counterexample in several important ways:

1. It removes the dependency on the specific construct of angelic merge (itself intrinsically problematic to model), relying only on simple non-deterministic choice ("or").
2. It covers arbitrary structures supporting recursion, not just the "history relations semantics for computing networks" considered in [1].
3. Together with a construction I give of a structure satisfying all but the last clause of the theorem for any D, it helps to delineate the boundary of what can be achieved in modeling recursion and non-determinism while maintaining extensionality.

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DAN E. WILLARD, *Self-verifying axiom systems and the incompleteness theorem*.

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We introduce some weak cut-free First Order axiom systems which can *simultaneously* verify their own consistency and prove more Π_1 theorems than Peano Arithmetic. More formally, let A denote any consistent recursive axiom system whose axioms are indicated by a Δ_0 formula. Define IS(A) to be a semantic tableaux deduction system containing the following three groups of axioms.

1) an initial group defining a preliminary set of functions and relations, such that Addition is the only increasing function (non-zero divison will replace multiplication as a function);
2) the statement that all Π_1 theorems proven by "the inner axiom system" A are valid; and
3) the statement that the full system IS(A), including this third sentence, is consistent.

The third axiom group can be formalized in any of several different manners. These include stating either

i) that there is no semantic tableaux proof from IS(A) of $0 = 1$, or
ii) that the uniform version of IS(A)'s reflection principle is valid for all Σ_1 formulae.

In either case, our main theorem will state that *if A is consistent then it will automatically follow that IS(A) is consistent*. Although our theorems require cut-free deduction, they are also valid in the presence of cuts, if the Addition function in Group-1 is replaced by Subtraction.

Our paper uses an analog of Kleene's Fixed Point Theorem to show that $\mathrm{Prf}_{\mathrm{IS}(A)}(x, y)$ can be formally defined by a Δ_0 formula. It includes negative as well as positive results. The negative results are that IS(A) is inconsistent when A corresponds to Peano Arithmetic and either

a) the Group-3 axioms of IS(A) include the local Π_1 reflection principle, or
b) IS(A)'s Group-1 axioms are expanded to recognize multiplication as a total function and Group-3 recognizes uniform Δ_0 reflection.

Since our paper shows IS(A) is consistent when Group-3 employs merely uniform Δ_0 reflection, the source of Result (b)'s inconsistency must be the multiplication function A surprising aspect of Result (a) is that *although incompatible* with Π_1 reflection, IS(A) (itself!) *can prove that* $\{\exists y \mathrm{Prf}_{\mathrm{IS}(A)}(\ulcorner \forall v \Phi(v) \urcorner, y)\} \subset \Phi(\overline{k})$, for any *fixed constant* $\overline{k}$ and for any Σ_1 formula $\Phi(v)$. This fact implies that while IS(A) does not have access to Π_1 reflection, it does not have access to *a finitized* Π_1 *reflection principle*, indicating that the proof of $\forall v \Phi(v)$ is a vehicle for recognizing the validity of any finite conjunction $\Phi(0) \wedge \Phi(1) \wedge \Phi(2) \wedge \cdots \wedge \Phi(N)$.

ANDRZEJ WIŚNIEWSKI, *Some results on the reducibility of questions.*

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Let L be a semantically interpreted first-order language enriched with questions. Assume that for each question Q of L there exists an at least two-element set dQ of *direct answers* to Q; dQ consists of closed declarative formulas (d-wffs) of L. The sets of direct answers to simple and conditional yes-no questions are of the form $\{A, \neg A\}$ and $\{A \,\&\, B, A \,\&\, \neg B\}$, respectively. Let X, Y stand for sets of d-wffs of L. We say that X *multiple-conclusion entails* (mc-entails) Y iff each normal interpretation of L being a model of X makes true at least one d-wff in Y; X *entails* a d-wff A iff X mc-entails $\{A\}$.

DEFINITION. A question Q is reducible to a non-empty set of questions Φ iff

(i) for each $A \in dQ$, for each $Q^* \in \Phi$: A mc-entails dQ^*, and
(ii) each set made up of direct answers to the questions of Φ which contains exactly one direct answer to each question of Φ entails some direct answer to Q, and
(iii) no question in Φ has more direct answers than Q.

A question Q is *safe* iff $\emptyset$ mc-entails dQ; otherwise Q is *risky*.

THEOREM 1. (1) *Each safe question is reducible to some set of questions made up of simple yes-no questions.* (2) *If entailment is compact, then each safe question is reducible to a finite set of questions made up of simple yes-no questions.*

$A \in \mathrm{Pres}Q$ iff for each $B \in dQ$, B entails A; the elements of Pres$Q$ are called *presuppositions* of Q. A question Q is *normal* iff Pres$Q$ mc-entails dQ; Q is *regular* iff for some $A \in \mathrm{Pres}Q$, A mc-entails dQ.

THEOREM 2. (1) *Each regular risky question is reducible to some set of questions made up of conditional yes-no questions.* (2) *If entailment is compact, then each normal risky question is reducible to a finite set of questions made up of conditional yes-no questions.*

Moreover, if dQ is a finite set, then Q is reducible to a finite set of questions (made up of simple yes-no questions if Q is safe, and of conditional yes-no questions otherwise).

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JIRI WITZANY, *Any behavior of the Mitchell ordering of normal measure is possible.*

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Let U_0, U_1 be two normal measures on κ. We say that U_0 is in the Mitchell ordering less then U_1, $U_0 \lhd U_1$, if $U_0 \in \mathrm{Ult}(V, U_1)$. The ordering is well-known to be transitive and well-founded. It has been an open problem to find a model where $\lhd$ embeds the four-element poset | |. I show that in the Kunen-Paris extension all well-founded posets are embeddable into $\lhd$ on normal measures over κ (assuming κ is highly measurable in the ground model). Hence there is no structural restriction on the Mitchell ordering. Moreover I show that it is possible to have two $\lhd$-incomparable measures that extend in a generic extension into two $\lhd$-comparable measures.

VLADMIR YAKHNIS, *A simplified way to complement Rabin's tree automata within Gurevich-Harrington framework.*

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Gurevich and Harrington (GH) gave a remarkably transparent procedure for complementation of Rabin's tree automata (R-automata). GH introduced slightly different tree automata (GH-automata) which are equivalent to R-automata. To every GH-automata $\mathcal{G}$ and every Σ-valued tree v GH assigned a two player game (GH-game) where the players are called "automaton" and "pathfinder." $\mathcal{G}$ accepts v iff "automata" wins the game. It follows from the GH Restricted Memory Determinacy Theorem (RMDT) that if the tree is not acceptable, then a winning strategy for "pathfinder" can be represented as a Δ-valued tree, where Δ is a special finite alphabet.

Given an R-automaton $\mathcal{A}$, the first three steps of the GH construction may be summarized as follows. (1) Construct a GH automaton $\mathcal{G}$ that accepts the same set of Σ-valued trees as $\mathcal{A}$ does; (2) Convert the sentence "(μ is an infinite $\{0,1\} \times \Sigma \times \Delta$-sequence) and (if we regard the $\{0,1\} \times \Sigma$-projection of μ as a path η in a Σ-valued tree, there is a run of $\mathcal{G}$ on η such that ("automaton" wins the respective play of the GH-game and "pathfinder" in this play follows the strategy represented by the Δ-projection of η))" into an $S1S$ formula; (3) construct a Buchi ω-automaton equivalent to the formula.

Monk simplified steps 2 and 3 by eliminating references to $S1S$ and by constructing a Buchi automaton from the sentence in step 2. Since Monk's construction required building a Buchi automaton for union and intersection of sets of sequences, it is still somewhat complicated.

We introduce tree automata that are slightly different from either R- or GH-automata. Our automata enjoy the same relationship with GH games as GH-automata do, and they are equivalent to R-automata as well. By replacing the GH-automata by our automata in the above construction, we remove all the complexity from steps 2 and 3, since our automata allows one to directly construct a nondeterministic Muller automaton accepting the sequences described in step 2.

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ANDRZEJ M. ZARACH, *What is a proper class?*.

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In GB, or ZF with additional predicates and the Collection Scheme for the extended language,

$$X \text{ is a proper class iff } \forall\alpha\exists x \in X(\mathrm{rank}(x) > \alpha).$$

Thus, the concept of power classes is ZF (GB)-absolute.

If the Power Set Axiom is replaced by "$\aleph(x)$ is bound for every x" where

$$\aleph(x) = \{\alpha \mid \exists f\,(f \text{ is one-to-one function from } \alpha \text{ into } x)\},$$

then the theory is denoted by ZFH (H stands for Hartogs' Number).

In ZFH, L = the class of all constructible sets is a model for ZF. In general, if x is a set, then $L[x] \models$ the Power Set Axiom.

METATHEOREM: *The concept of proper class is not ZFH-absolute.*

Construction of a counterexample: Let $On(C)$ = the weak product (only finite functions) of On copies of C, where C = Cohen's Notion of Forcing; $\omega(C)$ = the weak product of ω copies of C. Let G be $On(C)$-generic over M, where M is a standard, transitive model for ZFC.

Fact 1. $\langle M[G], M, G\rangle \models$ ZFH (since the cofinality is preserved).

In this structure we define **A** = the class of all axes of G, and CollA = the class of all finite functions from ω into **A**. Let B be CollA-generic over $\langle M[G], M, G\rangle$, so $\bigcup B$ is a one-to-one function from ω onto **A**.

Let $b = \{f \in \omega(C) \mid \exists F \in B(\mathrm{dom}(f) \subseteq \mathrm{dom}(F) \,\&\, \forall i \in \mathrm{dom}(f)(f(i) \in F(i)))\}$. Then the nth axis of b is $\bigcup B(n)$, so **A** = the set of all axes of b.

THEOREM: *Let* $V_1 = \langle M[G], M, G\rangle$, $V_2 = \langle M[G][B], M, G, M[G]\rangle$, *and* $V_3 = M[b]$. *Then, A is definable in each of these structures, and*

(i) $V_1 \models ZFH +$ *for every cardinal* κ, *there is some one-to-one function from* κ *into* A $+$ $\forall a \in A(\mathit{rank}(a) = \omega)$,

(ii) $V_2 \models ZF^- +$ *every infinite set is countable* $+$ A *is not a set,*

(iii) $V_3 \models ZFC+$ A *is a countable set.*

Remarks. V_1 and V_3 have the same cardinals, $V_1 \subseteq V_2$ and there is no inclusion for V_1, V_3 and V_2, V_3.

B and G are not simultaneous classes in any universe of set theory. Otherwise,

$$\{(n,\alpha) \mid \text{ the } n\text{th term of } \bigcup B = \text{the } \alpha\text{th axis of } G\}$$

would define a one-to-one function from ω onto On.

Abstracts of papers submitted by title

GUNWON LEE, *HOP-Model (study plan)*.

Department of Mathematics, Seoul National University, Korea.

In principle, the evaluation statements can be derived from the given descriptive statements as the is-ought inference in John R. Searle, with the human hiatus which makes the connections flexible.

However, we try to view it in historical perspective. If the agent a had to do x without any exceptions in a given situation, such as a had to go out through the door d if there is only one door d for the room, HOP-model, aPh; the choice of the agent a, the prescription P in that particular historical situation h, can be formulated as the historical obligatory prescription model such as the "historical definition" of Park Jong-Hong. However in practice, a lot of open questions remain

GREGORY MCCOLM, *Dimension and connectivity.*

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E-mail: `mccolm@math.usf.edu`.

In Fixed Point logic, one sets up an induction by getting a system of formulas

$$\phi_i(x_i, S_0, S_1, \dots, S_\nu), \qquad i = 0, 1, \dots, \nu$$

(where the length of x_i equals the arity of S_i for each i), and running an induction

$$\phi_i^{n+1}(u) \equiv \phi_i(u, \phi_0^n, \dots, \phi_\nu^n)$$

in the hope of reaching a fixed point $\phi_0^N, \dots, \phi_\nu^N$, where $\phi_i^{N+1} = \phi_i^N$ for all i. Fixed point logic consists of the formulas $\phi_0, \dots, \phi_\nu$ producing these fixed points.

We look at two popular complexity measures on fixed point logic. "Number of variables" is the maximum arity of any subformula of any of the ϕ_i: this measure is associated with the logic $L_{\infty\omega}(\omega)$ and is used a lot in finite model theory. "Dimension" is the maximum (first order) arity of any of the ϕ_i, and is more difficult to deal with. We will discuss the problem of determining the number of variables and dimension of given (least) fixed point queries using pebble games.

REFERENCES

[1] GREGORY MCCOLM, *Dimension vs. number of variables—and connectivity, too*, ***Mathematical Logic Quarterly***, to appear.

DAN TALAYCO, *Set theory, cohomology and the combinatorial snipe.*

Department of Mathematics, Boston College, Chestnut Hill, MA 02167, USA.

After a brief introduction to the cohomology of Hausdorff gaps, I will consider some more complex combinatorial objects which might exist. These would arise from higher cohomology groups associated with Hausdorff gaps.

GEORGE WEAVER AND DAVID LIPPEL, *The existence of finitely axiomatizable $\aleph_0$-categorical proper class II theories.*

Bryn Mawr College, 101 N. Merion Avenue, Bryn Mawr, PA 19010-2899, USA.
E-mail: `gweaver@cc.bryn mawr.edu`.
Haverford College.

Let K be a finite similarity type and $\mathbf{A}$ be a countably infinite interpretation of type K whose complete theory is $\aleph_0$-categorical. $l_A : \{n : n \geq 1\} \to \omega$ such that $l_A(n) = m$ provided m is the least member of ω such that for all $a_1, \dots, a_n$ in A (the domain of $\mathbf{A}$) and all $f : \{a_1, \dots, a_n\} \to A$ if $(\mathbf{A}a_1 \dots a_n)$ and $(\mathbf{A}f(a_1) \dots f(a_n))$ are m-equivalent in the sense that they agree on all sentences in m or fewer bound variables, then f can be extended to an automorphism on $\mathbf{A}$. l_A is an increasing function.

The complete theory of $\mathbf{A}$ is a *class I theory* iff for all n, $l_A(n) = 0$; a *class II theory* iff there is $m > 0$ and $t \geq 1$ such that for all $n \geq t$, $l_A(n) = m$; and a *class III theory* provided for all $m \geq 0$, there is $n \geq 1$ such that $l_A(n) > m$. Some class II theories can be extended to class I theories by the addition of finitely many definitions of new predicate constants. Such class II theories are called *pseudo class I theories*. Class II theories which are not pseudo class I are classed *proper class II theories*.

For each $n \geq 0$, T_n is the theory of infinite Boolean algebras with exactly n-atoms formulated in a first order language whose non-logical vocabulary contains only individual constants and functional constants. T_n^* is the theory of infinite Boolean algebras with exactly n-atoms formulated in a first order language whose non-logical vocabulary contains only individual constants and relational constants.

THEOREM: (1) T_0 *is a finitely axiomatizable class I theory,* (2) *for all* $n \geq 1$, T_n *is a finitely axiomatizable proper class II theory;* and (3) *for all* $n \geq 0$, T_n^* *is a finitely axiomatizable proper class II theory*

XUNWEI ZHOU, *Set theory in geometrical logic.*

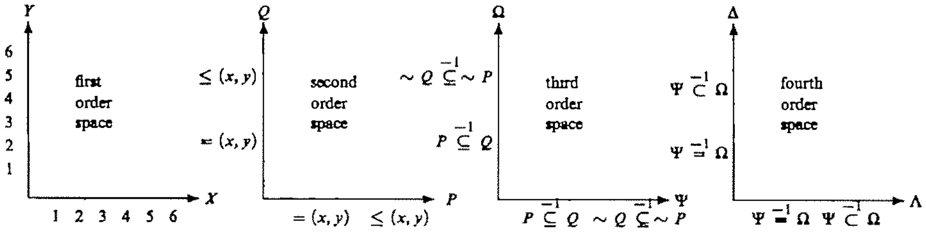
Department of Computer and Automation, College of Automation Engineering, Beijing Union University, Beijing 100009, P. R. China.

Geometrical logic is constructed by the author, it is a geometrical approach to logic. In geometrical logic there is no propositional calculus, and predicate calculus is treated as the same discipline as set theory. In geometrical logic there are 5 operators: $\sim, \sqcap, \sqcup, \oplus, \otimes$ and 7 connectives: $\overset{-1}{\sqcap}$, $\overset{-1}{\sqcup}$, $\overset{-1}{\subseteq}$, $\overset{-1}{=}$, $\overset{-1}{\subset}$, $\overset{-1}{X}$, $\overset{-1}{\oplus}$. $\sim$ corresponds to $\sim$ in naive set theory and $\neg$ in classical logic, $\overset{-1}{\sqcap}$ to $\cap$ in naive set theory and $\wedge$ in classical logic, $\overset{-1}{\sqcup}$ to $\cup$ in naive set theory and $\vee$ in classical logic, $\overset{-1}{\subseteq}$ to $\subseteq$ in naive set theory and $\rightarrow$ in classical logic, $\overset{-1}{=}$ to $=$ in naive set theory and $\leftrightarrow$ in classical logic, $\overset{-1}{\subset}$ to $\subset$ in naive set theory and $\rightarrow$ in classical logic.

In geometrical logic, knowledge is divided into 4 leels, term level, fact level, empirical connection level, and logical connection level. Geometrical logic employs hierarchical axes shown below. The term level knoweldge is represented by the first order space of the hierarchical axes, fact by the second order space, empirical connection by the third order space, logical connection by the forth order space.

x, y in the first order space of the hierarchical axes are term variables varying through elementary zeroth order terms such as $1, 2, 3$; P, Q in the second order space are fact variables varying through elementary first order facts such as $= (x, y)$, $\leq (x, y)$; Ψ, Ω in the third order space are empirical connection such as $P \overset{-1}{\subseteq} Q$, $\sim Q \overset{-1}{\subseteq} \sim P$. Λ, Δ in the fourth order space are logical connection variables varying through elementary third order logical connections such as $\Psi \overset{-1}{=} \Omega$, $\Psi \overset{-1}{\subset} \Omega$.

A point in the first order space such as $(2, 2)$ is a first order element, a point in the P axis of the second order space such as $= (x, y)$ is a first order set, and we say that $(2, 2)$ belongs to $= (x, y)$, or that $= (x[2], y[2])$ is true. A point in the second order space such as $(= (x, y), \leq (x, y))$ is a second order element, a point on the Ψ axis of the third order space such as $P \overset{-1}{\subseteq} Q$ is a second order set, and we say that $(= (x, y), \leq (x, y))$ belongs to $P \overset{-1}{\subseteq} Q$, or that $P[= (x, y)] \overset{-1}{\subseteq} Q[\leq (x, y)]$ is true. A point in the third order space such as $(P \overset{-1}{\subseteq} Q, \sim Q \overset{-1}{\subseteq} \sim P)$ is a third order element, a point in the Λ axis of the fourth order space such as $\Psi \overset{-1}{=} \Omega$ is a third order set, and we say that $(P \overset{-1}{\subseteq} Q, \sim Q \overset{-1}{\subseteq} \sim P)$ belongs to $\Psi \overset{-1}{=} \Omega$, or that $\Psi[P \overset{-1}{\subseteq} Q] \overset{-1}{=} \Omega[\sim Q \overset{-1}{\subseteq} \sim P]$ is true.



THE BULLETIN OF SYMBOLIC LOGIC
Volume 1, Number 1, March 1995

NOTICES

IN MEMORIAM: Paul Lorenzen died in Göttingen, Germany, on October 1, 1994. Lorenzen was born in 1915 in Kiel, Germany, and received his doctorate in 1938 with Helmut Hasse. He became Professor of Philosophy in Kiel in 1956, and in Erlangen in 1962; he retired in 1980. He was a member of the Institute for Advanced Study in Princeton during 1957–58, and was a member of the Göttingen Academy of Sciences since 1960.

IN MEMORIAM: Karl Popper died on September 17, 1994, at the age of 92. Born in Austria, he went to New Zealand shortly before World War II, and then went to England in 1945. He was Professor of Logic and Scientific Method at the London School of Economics from 1949 to 1969. He presented his ideas about science in his first book, *The Logic of Scientific Discovery*, published in 1934. His book *The Open Society and its Enemies*, published in 1945, has been called one of the most influential books of the century. Popper was a long-time member of the Association.

The 1995 ASL EUROPEAN SUMMER MEETING (LOGIC COLLOQUIUM '95) will be held August 9–17, 1995, in Haifa, Israel. The meeting will focus mainly, but not exclusively, on Set Theory, Model Theory (including Finite Model Theory), Recursion Theory, Proof Theory, Logical Aspects of Computer Science, and Linguistics. Tutorials will be given by several people, including S. Goldwasser (Interactive Proofs), K. Compton (0–1 Laws), T. Slaman (Recursion Theory), and D. Marker (Model Theory). Invited speakers will include W. Buchholz, Z. Chatzidakis, A. Dawar, I. Herzog, E. Hrushovski, J. Lynch, P. Maddy, D. A. Martin, G. Moore, M. Rathjen, F. Pereira, K. Peterzil, A. Pillay, S. Shelah, L. Stokup, and A. Stolboushkin. The Local Organizing Committee includes A. Aharoni, A. Avron, N. Francez, V. Harnik, D. Lascar, A. Levy, J. A. Makowsky, Chair, and M. Rubin. The Program Committee consists of J. Barwise, E. Bouscaren, E. Grädel, A. Hajnal, G. Jäger, M. Magidor, Chair, J. A. Makowsky, D. Mundici, R. Shore, M. A. Taitslin, and M. Ziegler. The deadline for submitting abstracts of contributed papers is April 30, 1995. For further information write to the following address (preferably by email): Logic Colloquium '95, Yvonne Sagi, Department of Computer Science, Technion – Israel Institute of Technology, 32000 Haifa, Israel; email: `logic95@cs.technion.ac.il`.

THE TENTH ANNUAL IEEE SYMPOSIUM ON LOGIC IN COMPUTER SCIENCE (LICS95) will be held June 26–29, 1995, at the University of California at San Diego. Further information may be obtained from the Publicity Co-chairs, Amy Felty and Douglas Howe, AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974; email: `felty@research.att.com`, `howe@research.att.com`. (ASL sponsored meeting.)

The 1995–96 ASL WINTER MEETING will be held in conjunction with the annual meeting of the American Mathematical Society, during January 10–13, 1996, in Orlando, Florida.

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THE 1995–96 ASL ANNUAL MEETING will be held March 9–12, 1996, in Madison, Wisconsin. Chair of the local organizing committee is Steffen Lempp, Mathematics Department, University of Wisconsin, Madison, Wisconsin 53706; email: `lempp@math.wisc.edu`.

AN ELECTRONIC VERSION OF THE ASL NEWSLETTER, with quite a lot of additional material about meetings of interest to logicians, may be retrieved using anonymous ftp from the ASL ftp site at the University of California at Los Angeles. To obtain a copy of this document, issue the command `ftp math.ucla.edu` and login as `anonymous`, using your email address as password. Then issue the command `cd pub/asl/news` to change directories and the command `get nov1994.news` to retrieve the electronic newsletter document. This is a text file with some embedded LaTeX commands, which can either be processed using LaTeX or printed directly A similar version of each ASL Newsletter will be posted in this directory from now on. In addition, a printed copy of the Newsletter will continue to be mailed directly to all members of the Association.

THE INTEREST GROUP IN PURE AND APPLIED LOGIC (IGPL) AND THE EUROPEAN FOUNDATION OF LOGIC, LANGUAGE, AND INFORMATION (FoLLI) jointly offer an annual prize of $1,000 for "the best idea of the year in the area of pure and applied logic." Nominations for the initial prize should be sent before the deadline of April 30, 1995, to the Chairman of FoLLI: J. F. A. K. van Benthem, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands; telephone: +31-20-525-5807; Fax: +31-20-525-5101; email: `johan@fwi.uva.nl`.

A SPECIAL YEAR ON LOGIC AND ALGORITHMS will be held at the DIMACS Center for Discrete Mathematics and Theoretical Computer Science during 1995–96. The DIMACS participating institutions are Rutgers University, Princeton University, AT&T Bell Laboratories, and Bellcore. The Special Year will focus on Computer Aided Verification, Finite Model Theory, and Proof Complexity. Information may be obtained from: DIMACS Center, Rutgers University, P. O. Box 1179, Piscataway, New Jersey 08855-1179; telephone: 908-445-5928; Fax: 908-445-5932; email: `center@dimacs.rutgers.edu`.

THE FIFTH INTERNATIONAL WORKSHOP IN ANALYSIS AND ITS APPLICATIONS (IMA) will be held May 17–27, 1995, at the University of Missouri at Rolla. The Program Committee consists of M. Insall, Chair, A. S. Kechris, Č. V. Stanojević, G. Takeuti, W. O. Bray, and A. R. Blass. Further information may be obtained from: Continuing Education, University of Missouri-Rolla, Rolla, Missouri 65401-0241; telephone: 314-341-4200; Fax: 314-341-4992, or from the Program Committee Chair: Dr. Matt Insall, Mathematics Department, University of Missouri-Rolla, Rolla, Missouri 65401-0249; telephone 314-341-4901; Fax: 314-341-4741; email: `insall@umr.edu`.

A WORKSHOP ON LOGIC, DOMAINS, AND PROGRAMMING LANGUAGES will be held May 24–27, 1995, in Darmstadt, Germany. The organizers are K. H. Hofmann, M. Huth, A. Jung, and K. Keimel. Invited speakers include S. Abramsky, O. Danvy, P. O'Hearn, F. Pfenning, G. D. Plotkin, H. Schwichtenberg, D. S. Scott, and B. Thomsen. Further information may be requested from: Dr. M. Huth, AG 1, Fachbereich Mathematik, Technische Hochschule Darmstadt, D-64289 Darmstadt, Germany; email: `ldp195@mathematik.th-darmstadt.de`.

THE SILVER JUBILEE MEETING OF THE SOCIETY FOR EXACT PHILOSOPHY will be held May 25–28, 1995, at the University of Calgary, Canada. Information may be obtained from B. Chellas, Department of Philosophy, University of Calgary, Calgary, Alberta, T2N 1N4 Canada; email: `chellas@acs.ucalgary.ca`.

THE GREATER BOSTON LOGIC CONFERENCE will be held May 27–30, 1995, at the Massachusetts Institute of Technology. The organizing committee consists of E. Schimmerling, E. Hrushovski, A. Kanamori, S. Homer, and S. Friedman. Some financial support may become available for graduate students to attend the conference, with women and minority students especially encouraged to apply. To receive information contact: Ernest Schimmerling, Department of Mathematics, M.I.T., Room 2-587, Cambridge, Massachusetts 02139; email: `ernest@math.mit.edu`.

AN INTERNATIONAL CONFERENCE ON THE INTERACTION BETWEEN ORDER, CONVEXITY, AND MODEL THEORY, IN ANALYSIS will be held June 25–July 1, 1995, at the North Ossetian State University, Vladikavkaz, Russia. Topics of the meeting include ordered vector spaces, convexity and optimization, and applicable model theory. The organizing committee consists of A. E. Gutman, Secretary, O. S. Khatsaev, A. G. Kusraev, and V. G. Sozanov. An International Advisory Board consists of S. S. Kutateladze and E. I. Gordon, Co-chairs, Y. A. Abramovich, C D. Aliprantis, J. M. Borwein, A. V. Bukhvalov, S. Dolecki, C. W Henson, A. D. Ioffe, P. A. Loeb, W. A. J. Luxemburg, G. Takeuti, V. M. Tikhomirov, and M. P. H. Wolff. Registration forms should be sent before April 15, 1995; abstracts of contributed papers should be sent before May 15, 1995. Payment of the fees (not expected to exceed $400 for registration, meals, and accommodation in Vladikavkaz) must be made by June 1, 1995. All correspondence should be sent by email: `nosu@nosu.vladikavkaz.su` (for financial matters) and `gutman@math.nsk.su`. Contact may also be made by telephone to Novosibirsk (383-2)-32-47-63, between 20:00 and 23:00 local time (17:00 and 20:00 Moscow time); this is the telephone number of the Secretary of the meeting, A. Gutman.

A BIBLIOGRAPHY IN FINITE MODEL THEORY has been established at the University of California at San Diego. It may be consulted and updated by anonymous ftp in the directory pub at the address `cs.ucsd.edu`. The bibliography is in the file `finitebib.bib` in this directory Further information may be obtained from Victor Vianu at the email address `vianu@cs.ucsd.edu`.

THE BULLETIN OF SYMBOLIC LOGIC
Volume 1, Number 2, June 1995

HERMANN WEYL'S INTUITIONISTIC MATHEMATICS

DIRK VAN DALEN

Dedicated to Dana Scott on his sixtieth birthday.

It is common knowledge that for a short while Hermann Weyl joined Brouwer in his pursuit of a revision of mathematics according to intuitionistic principles. There is, however, little in the literature that sheds light on Weyl's role and in particular on Brouwer's reaction to Weyl's allegiance to the cause of intuitionism. This short episode certainly raises a number of questions: what made Weyl give up his own program, spelled out in "*Das Kontinuum*", how did Weyl come to be so well-informed about Brouwer's new intuitionism, in what respect did Weyl's intuitionism differ from Brouwer's intuitionism, what did Brouwer think of Weyl's views, ... ? To some of these questions at least partial answers can be put forward on the basis of some of the available correspondence and notes. The present paper will concentrate mostly on the historical issues of the intuitionistic episode in Weyl's career.

Weyl entered the foundational controversy with a bang in 1920 with his sensational paper "On the new foundational crisis in mathematics".[1] He had already made a name for himself in the foundations of mathematics in 1918 with his monograph "The Continuum" [18]; this contained in addition to a technical logical-mathematical construction of the continuum, a fairly extensive discussion of the shortcomings of the traditional construction of the continuum on the basis of arbitrary—and hence also impredicative—Dedekind cuts.

This book did not cause much of a stir in mathematics; that is to say, it was ritually quoted in the literature but, probably, little understood. It could not be properly appreciated until the phenomenon of impredicativity was better understood.[2] The paper "On the new foundational crisis in mathematics" had a totally different effect; it was the proverbial stone thrown into the quiet pond of mathematics. Weyl characterized it in retrospect with the somewhat apologetic words:

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[1] The page numbers quoted below are from the original 1921 paper [19]. Additional remarks and explanations are inserted within square brackets

[2] Cf. [5].

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> Only with some hesitation do I acknowledge these lectures, which reflect in their style, which was here and there really bombastic, the mood of excited times—the times immediately following the First World War.[3]

Indeed, Weyl's "New Crisis" reads like a manifesto to the mathematical community; it uses an evocative language with a good many explicit references to the political and economic turmoil of the post-war period. The opening sentences castigate the mathematical community for its complacency, for its failure to pay any serious attention to the potential dangers of the various paradoxes of Cantor, Russell, Richard, ... :

> The antinomies of set theory are usually regarded as border skirmishes that concern only the remotest provinces of the mathematical empire and that can in no way imperil the inner solidity and security of the empire itself or of its genuine central areas. Almost all the explanations given by highly placed sources for these disturbances (with the intention of denying them or smoothing them over), however, lack the character of a clear, self-evident conviction, born of totally transparent evidence, but belong to that sort of half to three-quarters honest attempts at self-deception that one so frequently encounters in political and philosophical thought. Indeed, every earnest and honest reflection must lead to the realization that the troubles in the borderland of mathematics must be judged as symptoms, in which what lies hidden at the center of the superficially glittering and smooth activity comes to light—namely the inner instability of the foundations upon which the structure of the empire rests.

The readers of 1920 were all too familiar with the phenomenon; the German State with its skirmishes in the Baltic and its political instability was a painful reminder of the self-deception of politics. Weyl used the political metaphor with great dexterity, comparing for example the classical use of existential statements with the use of paper money:

> The point of view sketched above[4] only expresses the meaning which the general and existential propositions in fact have for us. In its light mathematics appears as a tremendous "paper economy". Real value, comparable to that of food products in the national economy, attaches only to the direct, simple singular; general and existential statements participate only indirectly. And yet we mathematicians seldom think of cashing in this "paper money"! The existence theorem is not the valuable thing, but the construction carried out in the

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[3] See [24].

[4] [i.e., the construction of a specific object versus its non-effective existence]

> proof. Mathematics is, as Brouwer sometimes says, more activity than theory (*mehr ein Tun als eine Lehre*).

And the final clarion blast of Weyl, one that fired the imagination and fed the wrath of many a practicing mathematician, rang through the next decade: "For this order can not be maintained in itself, as I have now convinced myself, *and Brouwer—that is the revolution!*"

Before analyzing Weyl's views on intuitionistic mathematics, let us see how Weyl got involved with this particular branch of constructive mathematics. Weyl was one of the outstanding mathematicians of his time. He was Brouwer's junior by 4 years, but he was already making a name for himself when Brouwer was still struggling for international recognition. In 1913 Weyl published one of the all-time successes of mathematics, his "The idea of Riemann surfaces" which established his name for good. The relation between Brouwer and Weyl dated back to at the latest 1912, as is documented in a letter from Weyl to Klein (16.5.1912).[5] With Brouwer attending conferences in Germany and regularly visiting Göttingen, they could hardly have missed each other. It seems most likely that the two, who shared a vivid interest in the foundations of mathematics, discussed the various issues in private, probably already before the war. The first concrete evidence of Brouwer's influence is to be found in a letter from Brouwer to Fraenkel, in which he mentioned that during a stay in the summer vacation of 1919 in Engadin in Switzerland, he had a number of conversations with Weyl. This piece of information solves much of the riddle of Weyl's early expertise in Brouwer's new intuitionism that went in an essential way beyond Brouwer's early foundational program, and which was the result of Brouwer's wartime research, carried out in isolation. Indeed, the intuitionism of Brouwer's dissertation, or of the inaugural address differs in essential points from the post-war intuitionism, as expounded to Weyl in the Swiss mountains. Brouwer gave a one-man course to Weyl on his new insights that went back to 1916 and that were published in communications of the Royal Academy in Amsterdam from 1918 onward. Weyl quickly grasped the main points of Brouwer's arguments and started to work out the consequences for himself. In May 1920 the paper on "the New Crisis" was finished and a copy was sent to Brouwer:

> Zürich, 6.5.'20
>
> Dear Brouwer,
>
> Finally I have sent what I have long promised you. It should not be viewed as a scientific publication, but as a propaganda pamphlet; thence the size. I hope that you will find it suitable for this purpose, and moreover suited to rouse the sleepers; that is why I want to

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[5] In this letter Weyl sums up the various points of the Brouwer-Koebe controversy on the theory of automorphic functions.

> publish it. I would be grateful for your criticism and comments. Did I enclose everything that you let me have only as a loan? If not, please reclaim it; the lecture on Formalism and Intuitionism[6] was already in my possession in the old days; at that time I did not pay attention to it or understand it ...

One can imagine how pleased Brouwer was with Weyl's conversion; here was one of the foremost members of the new generation of mathematicians, who was not only a master in almost all traditional topics in mathematics, but who had an exceptional insight in philosophical and foundational matters. The fact that Weyl was Hilbert's favorite son, and was viewed as the crown prince of mathematics in Göttingen, may have contributed to the importance of this first convert, but one should keep in mind that the Grundlagenstreit had not yet started and that the relations between Brouwer and Hilbert were still free from personal and scientific antagonism. As the enclosed manuscript showed, Weyl wholeheartedly gave up his own foundational program, trading his arithmetically definable sequences for Brouwer's choice sequences, and rejecting the principle of the excluded middle (*PEM*). Only a rough draft of Brouwer's reaction, full of crossed-out parts, has survived. Nonetheless, one can fairly well guess Brouwer's views from this draft and from a few remarks, inserted in pencil in the margin of Weyl's manuscript. The draft contains a number of comments on Weyl's manuscript; we will quote the various parts when discussing Weyl's paper. Here we translate the first few lines:

> Your wholehearted scientific support has given me infinite pleasure. The reading of your ms. was a continuous delight and your explanation, it seems to me, will also be clear and convincing for the public .... That we judge some side issues differently will only stimulate the reader. However, you are completely right in the formulation of these differences of opinion; in the restriction of the objects of mathematics you are in fact more radical than I am; however, one cannot argue about this, these matters can only be decided by individual concentration.

Brouwer's notes in the margin of the manuscript are mostly short comments on specific claims or assertions of Weyl concerning technical matters. There is one spot, however, where Brouwer thought his new disciple guilty of a strategic blunder: after introducing the functio continua, Weyl, somewhat apologetically, remarked that "It should be stressed once more that certain individual functions of that kind occur occasionally in mathematics, that general theorems are, however, never asserted about them. The general formulation of these notions is therefore required only if one is giving a justification of the meaning and methods of mathematics; for mathematics itself, the subject matter of its theorems, it is never considered at all". Weyl's view can be paraphrased as "in mathematical practice one does not quantify over functiones continuae". We will return to the matter after the discussion of the functio continua and just give Brouwer's comments from the draft:

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[6] See [1], [2].

> It seems to me that the whole purpose of your paper is endangered by the end of the second paragraph of page 34.[7] After you have roused the sleeper, he will say to himself: "So the author admits that the real mathematical theorems are not affected by his considerations? Then he should no longer disturb me!" and turns away and sleeps on. Thereby you do our cause injustice, for with the existence theorem of the accumulation point of an infinite point set, many a classical existence theorem of a minimal function, as well as the existence theorems for geodesics without the second differentiability condition, loses its justification!

In this respect Weyl showed himself to be less radical than Brouwer, that is to say Brouwer did not fear the confrontation with the traditional practice of mathematics, where Weyl considered his fundamental considerations as somewhat less relevant for everyday mathematics. It is not implausible that Weyl's position was dictated by his views on quantification in general (cf. below).

There are a few striking aspects to Weyl's version of intuitionism. One of them is the nature of the continuum and another the role of logic. Weyl distinguished two distinct views of the continuum, the atomistic one and the continuous one. In the first version the continuum is made up of individual real numbers which can be sharply distinguished. Weyl, in his program of "The Continuum", allowed only specific (arithmetic) relations on natural numbers, and as a consequence, also a restricted class of Dedekind cuts. For this particular class of reals Weyl accepted existence problems involving reals as meaningful:

> Only if we conceive the notion [of the continuum] in this way, which fixes and demarcates its extent, do questions of existence concerning real numbers become meaningful. By this restriction of the notion, a bunch of individual points is picked from the flowing mash. The continuum is smashed to isolated elements, and the blending of all its parts is replaced by certain conceptual relations, based on "greater – smaller", between these elements. Therefore I speak of an atomic conception of the continuum.

The arithmetically definable continuum of "The Continuum" has this atomistic character.

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[7] See [19, p. 66].

Almost immediately Weyl added, however, that "It has never been my opinion that the continuum given by intuition is a number system of Weyl; rather that analysis needs only such a system for its constructions and does not have to worry about the "continuum" poured in between". (p. 47). Indeed, already in "*Das Kontinuum,* Chapter II, §6, Intuitive and mathematical continuum", Weyl pointed out that his atomic version of analysis represented "a *theory of the continuum* which has (over and above its logical correctness) to prove itself correct, just as a physical theory does".

After the discussion of "Weyl's continuum", the "New Crisis"-paper continues with an exposition of Brouwer's choice sequences as a basis for the continuum. Real numbers are given by Weyl as infinite sequences of shrinking intervals, and thus the need arises to further specify the notion of sequence. Weyl allowed two sorts of sequences: those given by a law—standing for the individual points of the continuum, and the free choice sequences—which determine a continuum of "becoming", or emerging, sequences (*werdende Folgen,*[8]) i.e., sequences of intervals that are freely chosen and hence cannot in a predetermined way point at an element of the continuum. In the most literal sense of the word, these sequences remain 'in statu nascendi' forever. In Weyl's words:

> It is a first basic insight of Brouwer, that the sequence which is emerging (*werdend*) through free choice acts, is a possible object of mathematical concept formation. Where the law φ, which determines a sequence out to infinity, represents the single real number, the choice sequence, restricted by no law in its freedom of development, represents the continuum.
>
> Brouwer's remark is simple but deep: here a "continuum" arises in which, indeed, the single real numbers fit, but which itself does not by any means resolve into a set of completed existing (*fertig seiender*) real numbers; rather it is a medium of free becoming. (p. 50.)

It should be mentioned here that Weyl had briefly mentioned choices in "*Das Kontinuum*", but only to dismiss them out of hand. He regarded as nonsense the conception of a set as a collection of infinitely many arbitrary choices, which is recognized as a whole. Here one hears an echo of the discussion of the axiom of choice after Zermelo's proof of the wellordering theorem.

Weyl's evocative description of the "flowing" continuum is an almost artistic act; he dared to put into writing what Brouwer would only describe in discussions and lectures. The image of a continuum in a continuous state of creation (becoming) was a decisive step forward in the quest for something truly "continuous", i.e., something that cannot be cut or broken into pieces, let alone be built up from the atomic particles called points!

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[8] The translation of the German "*werdende Folge*" is somewhat problematic. Various adjectives, such as 'becoming', 'developing', 'emerging', 'in statu nascendi' have been used in the literature.

There are basically two fundamental issues in Weyl's exposition of intuitionism:

- the properties of choice sequences,
- the consequences for logic.

As to the first issue, let us compare Brouwer's notion and Weyl's side by side:

> A spread (*Menge*) is a law, according to which, if over and over again a numeral of the sequence ζ [i.e., N] is chosen, each of these choices generates either a certain sign or nothing, or, on the other hand, the stopping of the process and the definite destruction of its result, where for each $n > 1$ after each non-stopped sequence of $n-1$ choices at least one numeral can be indicated, which, when chosen as the n-th numeral does not stop the process. Each sequence of signs generated in this way by an unlimited choice sequence, (which thus in general cannot be presented as finished) is called an *element of the spread* [3].[9]

In the next line the phrase "choice sequence" is introduced for the numerical sequences that generate the elements.

Weyl did not *give* a definition of 'choice sequence' but rather introduced the notion "in passing". A few quotations are listed below:

(1) The difficulty is located in the notion of sequence. If any viewpoint at all is at the basis of contemporary analysis, from which its propositions and proofs can be understood, then it is this: a sequence results from the successive arbitrary choices of its individual numbers; the result of these infinitely many choice acts is available in finished form, and with respect to the finished infinite sequence I can ask e.g., if 1 occurs among its numbers. But this standpoint is absurd and untenable; for inexhaustibility is inherent to the essence of the infinite. A single *determined* (and determined out to infinity) sequence can only be defined by a *law*. When on the contrary a sequence is created step by step by free acts of choice, then it should be considered as "becoming" (growing), and only those properties can be meaningfully attributed to a becoming choice sequence for which the decision "yes or no" (...) is already reached when the sequence has reached a certain point .... (p. 50).

(2) *a sequence in which each choice is totally free* (p. 52).

(3) a law which generates a natural number n from an emerging sequence, depending on the result of the choices, is necessarily such that the

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[9] The correction of 1919 has been inserted.

> number n is determined as soon as a certain finite segment of the choice sequence is given in finished form, and it remains the same, no matter how the choice sequence may develop further. (p. 51.)

A methodological difference between the approaches of Brouwer and Weyl is that Brouwer does not elaborate the choice process and concentrates on the concept of spread, whereas Weyl does not consider spread at all, but tries to get the choice process right. One gets the impression that Weyl hesitates to attribute to choice sequences the sort of individuality that a natural number or a lawlike sequence possesses. E.g., "where a law φ which determines a sequence all the way to infinity, represents a single *real number*, a choice sequence, restricted by no law in its freedom of development represents the continuum" (p. 50). In particular, Weyl allows the passage from universal statements to particular ones only for lawlike sequences (objects), so from $\forall\alpha A(\alpha)$ one can infer that $A(a)$ holds for a lawlike a.

One should note that general statements of the form $\forall\alpha A(\alpha)$ may be true on the grounds that "it is part of the essence of emerging sequences that a sequence, in which every single choice is totally free", has a certain property A (p. 52). Whereas for Brouwer a choice sequence is an individual (think of "the sequence that I am creating now"), Weyl apparently views only lawlike sequences as such: "a single particular (and determined out to infinity) sequence can only be defined by a law".

At the end of Part II, Section 1, another argument for his view is given: "Every application of mathematics must depart from certain objects, intended for mathematical treatment, which can be distinguished by a number of characters; the characters are natural numbers. By means of the symbolic process, which replaces objects by their characters, the connection with pure mathematics and its constructions is obtained". One has to keep in mind that Weyl uses the name "application of mathematics" rather generously, e.g., the point geometry on a line is an application of mathematics, where the coded basic objects are rational segments. Weyl's view is certainly compatible with Brouwer's early philosophical views, and—in a technical sense—with Brouwer's spread definition, where the "generated signs" can be thought of as objects coded in natural numbers.

At the time that Brouwer and Weyl were discussing the matter, the ideas concerning choice sequences were rather implicit, and in view of Brouwer's use of (and examples of) spreads we may conclude that already at the level of choice sequences both parties diverged. Brouwer allowed choice sequences, given by a law (e.g., in the spread of the natural numbers [3], p. 3, or finite spreads), whereas Weyl excluded lawlike sequences from the domain of choice sequences. At a first reading (after [12]) one might think that Weyl had lawless sequences in mind, with an absolute freedom of choice; both (2) and (3) seem to point that way. However, he also allowed choice sequences built from other choice sequences, e.g., the sequence $m_1 + m_2 + \cdots + m_n$ obtained from the sequence m_n (p. 50). The reader will note that (3) looks very much like the principle of open data (for a function value), since it discusses the output of a function for a single choice sequence, and hence it points towards a strong degree of lawlessness. We know now that this would yield the wrong class of choice sequences, but it would be highly unjust to demand the present day insights in a pioneering paper in 1920. In comparison, Brouwer too would allow all choice sequences over the universal spread (tree), but he would not restrict the notion of "choice", i.e., for him all paths, no matter how chosen, were eligible. It would be tempting to ascribe a similar view to Weyl, but at various places in the paper it is made clear that Weyl actually means something close to our present day 'lawless', e.g., in his summing up of the differences with Brouwer's views. It seems most plausible that the finer distinctions were not quite fathomed by Weyl. It even is hard to guess to what extent Brouwer knew exactly which pitfalls were to be found on his path; nonetheless it is remarkable that he (almost) always formulated the right concepts and principles. The miracle therefore is not that Weyl's conception of choice sequence was somewhat awkward and cumbersome, but that Brouwer got his notion right! One is tempted to conclude from Weyl's paper that after Brouwer's tutorial, he set out to reduce the Brouwerian notion of choice sequence to something more basic.

Note that in the case of (3) above, Brouwer had formulated the correct principle already in his 1918 paper:

> (4) A law that assigns to each element g of C [the universal spread] an element h of A [the natural numbers], must have determined the element h completely after a certain initial segment α of the sequence of numerals of g. (p. 13.)

The crux is that Brouwer considered *all* choice sequences of C, instead of a single choice sequence, as in (3). It should be added that Brouwer did not further justify this principle, but it shows that he realized that the universe of choice sequences was the home of a rich variety of sequences, ranging from lawlike to lawless, with all kinds of versions in between.

Brouwer's spreads may be thought of as trees with natural numbers at their nodes (a subset of the set of all finite sequences of natural numbers), where some particular nodes were forbidden (and hence also all nodes below them). The choice sequences could be visualized as infinite paths through the tree. In fact the choices made were in practice from a given domain, thus real numbers appeared as choice sequences of rational numbers, or of rational intervals. But the basic function of a spread was to regulate the possible choices, a matter not left till later. In Weyl's approach, all natural numbers were eligible, and only afterwards other sequences were generated by suitable mappings. From a topological point of view one could say that

Weyl got the desired choice sequences by means of continuous mappings acting on Baire space. Hence there was no need for Weyl to restrict the possible choices in advance, the mappings would take care of the need for specific sequences. Hence also Weyl's preference for unrestricted choices.

Given the natural numbers and the choice sequences, one wants to have functions on these domains in order to carry out a reasonable amount of mathematics. Weyl introduced for that purpose three classes of functions:

(1) *functio discreta*: a law assigning numbers to numbers (a lawlike function from N to N)
(2) *functio mixta*: a lawlike function from numbers to lawlike functions $(N \rightarrow (N \rightarrow N))$, or a law assigning numbers to choice sequences, $((N \rightarrow N) \rightarrow N)$.
(3) *functio continua*: a lawlike function from choice sequences to choice sequences $((N \rightarrow N) \rightarrow (N \rightarrow N))$.

The functio discreta is introduced by means of the un-analyzed notion of a law. The functio mixta for the first case is given by a double sequence, which is essentially a functio discreta, and for the second case it is regulated by the following *essential insight* (*Wesenseinsicht*) (p. 64):

> *According to this law there always is a moment for an emerging sequence, no matter how it develops, when it gives birth to a number.*

As a matter of fact, Weyl listed two generating principles for the functio mixta, but he explicitly warned the reader that he did not claim these principles to yield all cases of functio mixta.

The clauses for Weyl's functio mixta are in modern formulation:

(1) F, with $F(\alpha) = f(\overline{\alpha}k)$ is a functio mixta, where $f : N \rightarrow N$ is a lawlike function, $\overline{\alpha}k$ is the (coded) initial segment of length k of the choice sequence α.
(2) if H and G are functiones mixtae then F with $F(\alpha) = H(G(\alpha), \alpha)$ is a functio mixta.

The class of functions given by the generating principles is evidently a proper subclass of the class K of Brouwer operations.[10] In the margin of the manuscript Brouwer noted: "this rule is indeed included in my notion of spread".

As in Brouwer's spread, Weyl allows partial functions, i.e., the function may be undefined for certain choice sequences, but from the functio mixta law it must follow for each α that "at a certain argument (depending on α) either the generated number is available or the certainty exists that the sequence under consideration is a deaf sequence,[11] which will be barren for all eternity". Brouwer's comments on the concept of 'functio mixta' are a bit cryptic. With respect to the functions given by the two generating principles, he remarked "this rule is indeed covered by my concept of spread", and with respect to the partial functio mixta, "this is not necessary from my point of view".

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[10] cf. [14], Ch. 4, Section 8.

[11] i.e., a sequence not in the domain of the functio mixta.

What is the relation between 'functio mixta' and a spread? Let us note that Weyl used the functio mixta and functio continua to obtain suitable choice sequences; so a functio mixta operating on "all basic choice sequences" produces natural numbers as outputs. Brouwer's remark should be understood as saying that the range of a functio mixta can also be obtained by a spread law. This is easily seen as follows, consider the choice sequence $a_0, a_1, a_2, a_3, \dots$ assign nothing to the initial segments until the functio mixta F gives an output n, then assign n, and assign nothing to all longer initial segments. A similar trick works for the partial functio mixta. Conversely, of course, a spread of natural numbers can be mimicked by a suitably chosen functio mixta.

The functio continua takes choice sequences to sequences, and it is defined by Weyl as a continuous operation:

> A functio continua is a law, according to which every step which adds another term to a sequence, emerging through acts of free choice, generates a certain number or nothing. (p. 65.)

Since this definition does not guarantee that the resulting sequence will proceed indefinitely, Weyl explicitly stipulated that eventually a new number will be generated. Here Brouwer commented, "superfluous from my point of view, it excludes unnecessarily discontinuous functions" (p. 65). This seems curious in view of Brouwer's later theorem that functions on the continuum (and also Baire space, i.e., the set of choice sequences) are continuous; the condition of Weyl, however, excluded partial functions, and they could easily be discontinuous.

Weyl obtained discontinuities by mimicking the destruction clause from Brouwer's spread definition. He had, however, to add a complicated clause to ensure closure under substitution. Brouwer again declared this clause superfluous for spreads. So much for Weyl's view of choice sequences.

Weyl seemed to consider first choice sequences over all possible number-choices (all lawless paths through the universal tree), and only afterwards to use continuous operations in order to get other families of choice sequences. This is in a way a plausible and even modern view; in the recent past these techniques have been used by Troelstra–van Dalen and Van der Hoeven to obtain suitable choice universes as "projections of lawless sequences" [15], [16]. Needless to say that in 1920 one would not even dream of such matters.

Weyl's second personal innovation in intuitionistic mathematics attracted a certain amount of attention. It was his reconsideration of the role of existential and general statements. In Part II Section 1, Weyl described his own "conversion" to Brouwer's rejection of the principle of the excluded third. He started by pointing out that for propositions involving choice sequences he had no difficulty accepting Brouwer's view, but even here his arguments do not follow the plausible pattern of the Brouwerian counterexamples. Brouwer's simple counterexamples are based on such existential statements as "the decimal expansion of π contains a sequence 0123456789", and hence do not even make use of the choice concept! Weyl's reason for rejecting the principle of the excluded third rested on an analysis of the meaning of $\exists\alpha A(\alpha)$ and $\forall\alpha A(\alpha)$ (where α from now on ranges over choice sequences). $\exists\alpha A(\alpha)$ meant for him that there is an individual sequence with the property, but on his interpretation of "individual sequence" this had to be a lawlike sequence. In the system of "The Continuum", the laws determining sequences were inductively given, and—according to Weyl's pre-intuitionistic interpretation—either there was such a law or not, but under the present circumstances

> the notion of law is no longer put on the Procrustes bed of the construction principles. Rather, if in any non-circular way, no matter how, a construction of a law of the desired kind has been successful, then we are justified in asserting that such a law *exists*. Thus, here it is definitely no longer a matter of the *possibility* of the construction, but rather we make such an existential statement in reference to the *successful construction*, the *proof actually carried out*. The negative statement that such a proof does not exist remains, of course, devoid of any meaning.

Here Weyl argues that with respect to an open-ended notion, such as 'law,' the negation makes no sense. Or in his terminology, the negation of an "*umfangs-definite*" property need not be "*umfangs-definit*"—the analogue of 'the complement of a set in an open-ended universe need not be a set.' Weyl then goes on to consider the positive version of the above negation:

> We can, however, interpret it positively: every sequence has the property $\neg A$, and now it [i.e., the negation of the existential statement] gets a meaning, provided we mean by 'sequence' not the law but, ..., the sequence emerging through free choice acts.

Weyl paraphrased the above as " "there exists" ties us to Being and Law, "every" puts us into Becoming and Freedom".[12] Since the interpretation of the bound variable in $\exists \alpha A\alpha$ and $\forall \alpha \neg A\alpha$ was completely different, "it would be absurd to think here of a complete disjunction". Summing up: on Weyl's interpretation of the quantification over sequences, there was not a shred of evidence for the principle of the excluded third. The matter was not so simple for quantification over natural numbers; for such statements $A(n), \exists n A(n)$ is established if one finds a number n, during the consecutive testing of $A(0), A(1), A(2), \ldots$. The negation would then come to "after running through all natural numbers no instance $A(n)$ has been established", but Weyl correctly rejects this as nonsense, $\forall n \neg A(n)$ can only be true because it is part of the essence of the notion natural number. He concluded that these statements do not act as each other's negation. So this pleads against *PEM*. But, he continued:

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[12] *Das "es gibt" verhaftet uns dem Sein und dem Gesetz, das "Jeder" stellt uns ins Werden und die Freiheit.*

> Although this speaks in favor of Brouwer, I was always thrown back to my old standpoint by the thought: if I run through the sequence of natural numbers and break off when I find a number with the property A, then this eventual breaking off does or does not occur; it is or is not the case, without wavering and without a third possibility [...] Finally I found for myself the magic word. An existential statement—e.g., "there is an even number"—is not a judgment[13] in the proper sense at all, which states a state of affairs; existential states of affairs are an empty invention of logicians. (p. 54.)

Weyl termed such pseudo-statements "judgement abstracts", to be compared to "a piece of paper which announces the presence of a treasure, without divulging its location". Similarly, general statements are only hints at judgments. For universal statements Weyl uses the following metaphor: "To use another picture: if I compare knowledge with a fruit and the discerning realization of the knowledge with the consumption of the fruit, then a universal statement is to be compared to a hard shell filled with fruit. It certainly has its value, but not the shell by itself, rather because of its contents consisting of fruit; it is no use to me as long as I do not open it and actually take out a fruit and eat it". Here Brouwer jotted in the margin: "The Invariant theorem of Hilbert without a method of computation was only the certainty that this shell does not contain a fruit, this kind of existence theorem does not seem to be discussed here at all."[14]

Since, in Weyl's opinion, existential statements and general statements are not proper statements, they cannot be negated, and "the possibility to formulate an 'axiom of the excluded third' with respect to them has disappeared". Brouwer's view of logic, later made explicit in the proof interpretation, had the advantage of allowing full iteration of all logical connectives, Weyl stopped at the quantifiers. Both approaches found a systematic treatment—Brouwer's logic in [7] and [11]—Weyl's logic in Kleene's realizability. Kleene went even further than Weyl (and Hilbert-Bernays) by considering all, say, arithmetical statements as incomplete, and hence in need of completion, cf. [10].

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[13] Weyl's use of judgment is not in agreement with present day use, which follows Frege. It is rather a proposition, cf. [18], p. 1.

[14] The invariant theorem referred to is the famous finiteness theorem of the theory of invariants, nowadays known as Hilbert's basis theorem. Brouwer views it as a $\neg\forall\neg\neg$ statement.

Readers of the foundational controversy between intuitionism and formalism will have noted that Hilbert lumped Brouwer and Weyl together; in his early papers (after [19]) Weyl is mentioned explicitly, but in the later ones only Brouwer is mentioned, and some of Weyl's views are conveniently attributed to Brouwer. E.g., in [8], Brouwer is charged: "To prohibit existence statements and *PEM* is tantamount to relinquishing the science of mathematics altogether". The charge perfectly fits Weyl's position, but does not do full justice to Brouwer. After all Brouwer had no quarrel with existential statements, he just interpreted them in a constructive manner! Hilbert's example shows that even the leading proof theorist was not able to distinguish properly between the brands of intuitionism of Weyl and Brouwer (although Weyl explicitly enumerated the points of difference!); how much less could one expect the general reader to spot the distinctions.

Weyl's conception of existential and general statements has influenced the treatment of the finitary viewpoint in arithmetic.[15] The finitary meaning of $\exists x A(x)$ is given as a "partial assertion",— i.e., "an incomplete communication of a more precisely determined statement, which either consists of the direct indication of a natural number with the property $A(n)$, or the indication of a procedure for the extraction of such a number ... " A general statement has a hypothetical finitary meaning, "such an assertion states a law which must verify itself in each particular instance". Hilbert and Bernays follow Weyl fairly faithfully (intentionally or not—Weyl is not quoted), in the passages concerning finitary consequence, including the *PEM*, "The complicated situation, which is found here with respect to the negation of judgments from the finitary viewpoint, corresponds to Brouwer's thesis that *PEM* is not valid for infinite collections. This non-validity consists from the finitary standpoint in this: that one does not succeed in finding a negation with finitary content which satisfies *PEM*".[16]

There is a certain temporal aspect to Weyl's rejection of *PEM*, but it is rather indirect; the undeterminedness of choice sequences with respect to future choices is of course at the basis of Weyl's considerations, but he added another major aspect. Namely that of the meaning of statements with choice quantifiers; it was on the basis of this difference in meaning of $\exists\alpha$ and $\forall\alpha$

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[15] See [9], Section 2b.

[16] See [9], p. 34.

that he came to the conclusion that the unrestricted 'excluded middle' made no sense.

The combination of criticism of quantified statements and insistence on "free choice sequences" brought considerable complications. In II, Section 3, Weyl looked at the consequences for the simplest kind of statements and properties. Since free choice sequences could not satisfy mild conditions, such as "all values are even", he was faced with the problem what to make of relativized statements (or implications). In modern notation, what does "$\forall\alpha(A(\alpha) \rightarrow B(\alpha))$ mean? Let us, for convenience, follow Weyl's example: $A(\alpha) := \forall n E(\alpha(n))$, where $E(x)$ stands for "x is even". Consider $\forall\alpha(\forall n E(a(n)) \rightarrow B(\alpha))$; we can strengthen the statement (constructively) by pulling out the $\forall n$, $\forall\alpha\exists n(E(\alpha(n)) \rightarrow B(\alpha))$. By continuous choice we get $\forall\alpha(E(\alpha(f(\alpha)) \rightarrow B(\alpha))$ for a functio mixta f. Now we have succeeded in giving a constructive meaning to the problematic relativized general quantifier. This is roughly the escape route Weyl indicated.

Surprisingly, Weyl denied the possibility to interpret a relativized statement of the same sort, only with choice quantifiers instead of numerical quantifiers (p. 71). Brouwer, in the draft of the letter, pointed out that this conflicted with his views:

> Your assertion of p. 37, lines 3–6, which contradicts my view (as you, by the way, know) should be explained a bit further. It seems to me that the reader who has followed you so far will have trouble with this passage.

It is not hard to see that indeed the job done by the functio mixta, could just as well be done by a functio continua.

The above gives an impression of the complications inherent in Weyl's rejection of quantified statements in general.[17]

At the end of Section 3, "Mathematical propositions, properties, and sets", Weyl summed up the main foundational points:

> I adopt from Brouwer 1. the basis, which is in all respects the essential thing, namely the idea of emerging sequences and the doubt of the principium tertii exclusi, 2. the idea of the functio continua. For the functio mixta and the viewpoint, which I summarize in the following three theses, I am responsible:
>
> (1). The notion of sequence oscillates, depending on the logical context in which it occurs, between "law" and "choice", "being" and "becoming".
>
> (2). The general and existential statements are no judgments in the proper sense, they do not state states of affairs, but are judgment indications, resp. judgment abstracts.

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[17] Note the similarity with the position of finitism with respect to quantifiers

> (3). Arithmetic and analysis only contain general statements about numbers and freely emerging sequences; no general function and set theory of independent content.

There are two comments of Brouwer in the margin:—with respect to (1),

> for me "emerging sequence" is neither one; one considers the sequences from the stand-point of a helpless spectator, who does not know at all in how far the completion has been free

and with respect to (2) and (3),

> I do, however, not agree with (2) and (3).

Brouwer's comment on Weyl's notion of choice sequence is consistent with the few remarks one finds in his publications. Apparently he wished to allow for choice sequences also higher order restrictions of choices, so that during the choice process one might decide at any moment that all future choices (of the particular sequence that one is generating) will be made according to a law. Hence the powerless spectator.

The "New Crisis" paper is extraordinarily rich in ideas and claims; the last section "The Continuum" contains a few gems that anticipate some developments of the twenties. One of them explicitly states the connectedness of the continuum:

> A true continuum is simply something connected in itself and cannot be split into separated pieces; that contradicts its nature.

This is the first enunciation of the unsplittability of the continuum. However, Weyl does not prove the principle, but rather asserts its plausibility on conceptual grounds. Similarly, the prize-winner—the continuity of real functions—is announced without a proof:

> Above all, however, there can be no other functions at all on a continuum than continuous functions (p. 76).

Did Weyl thus establish the continuity theorem years before Brouwer? The answer is 'no' for a number of reasons. In the first place Weyl defined real functions via mappings of the intervals of the choice sequence determining the argument to intervals of the image sequence. Hence such a function is continuous by definition. Weyl's argument for this definition is on the whole plausible: one should be able to find effectively approximations to the output from approximations to the input. Brouwer's continuity theorem, however, goes one step further, it establishes the continuity of a function from choice sequences to choice sequences or from reals to reals. So, where Weyl reduced the type of a function from R to R in advance to that of a function from N to N (initial segments to initial segments), Brouwer showed that this by necessity followed from the intuitionistic principles and the functional character (i.e., the $\forall\exists!$-combination). It should be pointed out that Weyl's assertion of the continuity of real functions was more or less dictated by his refusal to grant choice sequences individually, i.e., the combination $\forall\alpha\exists!\beta$ could not be handled on the basis of the principle "existence entails lawlikeness and generality entails choice"; the only way to handle such combinations was to invoke a functio continua. $\forall\alpha\exists!\beta A(\alpha, \beta)$ could not even be viewed as an "instruction for a statement", but $\forall\alpha A(\alpha, f(\alpha))$ for a continuous f, would do nicely.

Furthermore, there is ample evidence that Weyl was, also in this case, inspired by Brouwer. Brouwer had already quite early reached the conclusion that there could not be any full (i.e., total) discontinuous functions on R [18]: "Through this above mentioned theorem, which is an immediate consequence of the intuitionistic stand-point, and which has since 1918 often been mentioned by me in lectures and in conversation, ... "; and he was already looking out for a natural notion of a discontinuous function on (a subset of) R in 1920. This is substantiated by the draft mentioned above:

> With reference to your considerations on the notion of continuous function I would like to draw your attention to my notion of the fully defined (*volldefinierte*) function of the continuum. I mean by that a law which assigns to every point of a set equivalent to the continuum (*örtlich übereinstimmenden Punktspezies*) a further point of the continuum. Such a function can very well be discontinuous without being generated by a conjoining of continuous functions on separated continua; one can, by the way, operate with them in many ways (one can e.g., in certain cases integrate them without having information on their continuity or discontinuity).

After pointing out the absence of discontinuous functions on R, Weyl went on to say that:

> When the old analysis allowed the formation of discontinuous functions, it thereby showed most clearly how far it is from grasping the essence of the continuum. What one calls nowadays a discontinuous function, consists in fact (and this also is basically a return to older intuitions) of a number of functions on separated continua.

Brouwer added in the margin the remark:

> better: is a not everywhere defined function. The definition of the function does not have to be split in cases, as the pointwise discontinuous function shows.

Weyl illustrated his statement by the piecewise defined function $f(x)$ which is x for $x > 0$ and $-x$ for $x < 0$. This partial function, he remarked can be extended to the total function $|x|$, but the function g with $g(x) = 1$ for $x > 0$ and -1 for $x < 0$ cannot be extended to a total function. Brouwer noted, with respect to the latter part "very true! To be underlined, for this is what matters".

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[18] See [4], p 62

The conjecture of the continuity of all real functions was, according to Brouwer's letter to Fraenkel (28.1.1927), brought up in 1919: "In the summer of 1919 I have in the course of private discussions with Weyl in Engadin, as a result of which he was converted to my views, once in connection with the definition of continuous functions in Section 1 of the first part of my "Founding set theory independent from the logical proposition of the excluded third", expressed and motivated the conjecture that these functions are the only ones existing in the full continuum [. . .]. It can be only on the basis on this utterance of a conjecture (and others, which were half-understood by Weyl) by me, that since then the legend has been spread by Weyl, that it is self-evident that in Brouwer's analysis none but continuous functions can exist".

In [4] there is a reference to the conjecture: "By this theorem,[19] which is an immediate consequence of the intuitionistic position, and which I repeatedly mentioned in discussions and in lectures, theorem 3 below[20] which asserts much more, and which I managed to prove only much later, is suggested".

The above comment of Brouwer has a somewhat bitter ring, for reasons that have nothing to do with Weyl or his papers,—at the time of writing Brouwer had the (not wholly unjustified) impression that the credits in the foundations of mathematics were distributed in a rather off-handed way. But at least it makes it quite clear that Weyl had not quite grasped Brouwer's arguments about the continuous functions.

Not surprisingly, the available information on Brouwer's reaction raises as many questions as it provides answers. It makes clear that Brouwer did not share Weyl's radical rejection of existential and general statements, and it also formulates the objections against Weyl's notion of choice sequence. The question remains, however, why did Brouwer not give in the letter to Weyl an exposition of his own views on these topics? Since we only have the draft of the letter, it cannot be excluded that the letter itself contained more extensive arguments. It seems more likely, however, that Brouwer did not wish to repeat his arguments of the Engadin discussion in 1919, and that he wanted to respect Weyl's version, as appears from the first paragraph of the draft:

> these matters can only be decided by individual concentration.

This probably explains why Brouwer did not provide more detailed comments on Weyl's wording of the continuity principle; for in Weyl's version of choice sequence, his "open data"-like formulation—"for a single choice sequence" rather than "for all choice sequences"—was plausible indeed. It would also have been interesting to have Brouwer's view on the meaning of the quantifiers in writing; here also he respected Weyl's view and was content to just register his disagreement.

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[19] the negative continuity of all real functions.

[20] the positive locally uniform continuity of all real functions.

Apart from the "propaganda pamphlet" of 1920, Weyl figures only twice in the history of intuitionism. After this short episode Weyl practiced intuitionism only once more; in 1924 he gave a constructive proof of the fundamental theorem of algebra.[21] One should not get the impression that Weyl's commitment to intuitionism was a mere infatuation; in a letter to the president of the *Schulrat* in Switzerland[22] Weyl reported the offer of a chair in Amsterdam, which rather appealed to him, as "... the close relation I have to Brouwer—we cooperate in revolutionizing and re-founding analysis—, disposes me to consider the offer seriously". As a matter of fact, the offer from Amsterdam was eventually turned down, to the bitter disappointment of Brouwer. Weyl remained a lifelong admirer of Brouwer and his intuitionism, but he dissociated himself from the actual practice and also from the foundational debate. His influence on the ideas of Hilbert and Bernays should by no means be underestimated. We have already seen that the treatment of finitary statements and of quantifiers by Hilbert and Bernays was very similar to Weyl's treatment of meaningful statements (in particular the quantifier combinations, e.g., $\forall\exists$, are handled very much in the same manner). As a matter of fact, there had been discussions of foundational matters between Weyl and Bernays, of which at least one is documented: Bernays reported in a letter to Hilbert (25.10.25)that he met Weyl in Zürich, where "in particular we discussed your foundational theory, to which, as you know, Weyl is now no longer opposed". Even more important, it is mentioned in the same letter that Bernays just had observed a certain distinction between the finitary viewpoint and that of Brouwer—at that time a far from common insight! It is not far fetched to assume that this insight was a result of the above mentioned discussion with Weyl.

When the Grundlagenstreit, which ensued from his action in 1920, was well under way, Weyl remained rather aloof. Only in 1928, when Hilbert attacked intuitionism during a seminar talk in Hamburg, Weyl came once more to the defense of Brouwer.[23] Following Hilbert's talk, Weyl took it upon himself to straighten out a number of points, in particular the role of Brouwer's intuitionism in connection with Hilbert's program. He pointed out that Brouwer had systematically pursued the investigation of the extent of the "contentual thought".[24]

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[21] See [20].

[22] 27.9.1920, [6] p. 49.

[23] [22]

[24] *inhaltliches Denken.*

> Brouwer, like everybody else, required of mathematics that its theorems be (in Hilbert's terminology) "real propositions", meaningful truths. But he was the first to see exactly and to the full extent how it had in fact everywhere far transcended the limits of contentual thought. I think that we owe him gratitude for this recognition of the limits of contentual thought.

Hilbert, he said, had respected as a matter of course in his metamathematical arguments these limits; contrary to earlier remarks of Hilbert,[25] Brouwer's insistence on constructive content could thus not be construed as "artificial prohibitions". Weyl went on to express his admiration for Hilbert's "radical re-interpretation" of mathematics, and to point out that even an intuitionist could not object to the passage through ideal statements, i.e., the conservative extension property of, say, full arithmetic with respect to finitary mathematics.[26] Brouwer's view on conservative extension results (e.g., as resulting from Gödel's translation) is not known, but from a constructive viewpoint, there is little one can object to. Hence, in principle, Weyl was quite right in his assessment. He fully acknowledged the situation in his conclusion. "Also in the epistemological evaluation of the new situation thus created, nothing separates me any longer now from Hilbert". From the following passages, it appears that Weyl had Hilbert's conservative extension program (or "real interpretation of ideal statements") in mind. He compared the situation with that in physics, where the (mathematical) theories were no longer in immediate contact with the phenomenal physical world, and quoted Hilbert's justification of the theory with an appeal to its success. Although Weyl recognized the potential effects of Hilbert's program, which was still a real option at the time, he somewhat sadly reflected that "if a triumph of formalism over intuitionism were to be the case – as it seems to all appearances , then I see in that a decisive defeat of the philosophical approach of a phenomenology, which proves itself insufficient for the understanding of creative science, on the most primitive and most accessible area of knowledge, mathematics". It is a bit hard to draw the right conclusion from this contribution. On the one hand Weyl felt the need to stand up for Brouwer, on the other hand he no longer felt compelled to stick to the intuitionist views on the role of formalism. One is tempted to view Weyl's statement as rather ahead of his time, i.e., his acceptance of the results of proof theory as a theoretical enterprise based on intuitionistic principles. This view certainly is compatible with the paper and also with later ones.

In his influential book "Philosophy of Mathematics and Natural Science" (1949), which grew out of an early essay "The present state of knowledge in mathematics" ([21]) and its subsequent extension *"Philosophie der Mathematik und Naturwissenschaft"* (1927), Weyl presented a penetrating survey of the foundational situation in mathematics, in which he discussed both intuitionism and formalism. The 1925 version still witnesses the enormous relief of having found a faithful theory of the continuum. The section on "Brouwer's intuitive mathematics" opens with the magisterial sentence

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[25] cf. [8], p 80

[26] which had not yet been settled in the negative by Gödel's theorems

> The ice field was broken into floes, and now the element of the flowing will soon completely be master over the rigid. L. E. J. Brouwer designs—and this is an accomplishment of the greatest epistemological importance—an exact mathematical theory of the continuum, which conceives it not as a rigid being, but as a medium of free becoming.[27]

The same paper contains also the first sign of his despair in view of the ultimate success:

> Mathematics attains with Brouwer the highest intuitive clarity; his doctrine is idealism in mathematics thought through to the end. But with pain the mathematician sees the larger part of his towering theories fall apart.

Two years later Weyl expressed himself even more pessimistically:

> Mathematics attains with Brouwer the highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural way, retaining much closer contact with intuition than before. But one cannot deny that, when one progresses to higher and more general theories, the fact that the simple principles of classical logic are not applicable finally results in a hardly bearable awkwardness. And with pain the mathematician sees that the larger part of his tower, which he thought to be built from strong blocks, dissolves in smoke.[28]

These statements are in curious contrast to the—almost dual—one in the "New Crisis"-paper, where Weyl drew a diametrically opposed conclusion:

> The new conception, as one sees, brings very far-reaching restrictions in comparison with the generality that fades into vagueness, to which the analysis developed so far has made us accustomed in the last few decades. We must again learn modesty. We wanted to storm the heavens and we have only piled fog on fog that cannot support anybody who tries in earnest to stand on them. What remains tenable could at first sight appear so meager that even the possibility

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[27] *Die Eisdecke war in Schollen zerborsten, und jetzt ward das Element des Fließenden bald vollendes Herr über das Feste. L. E. J. Brouwer entwirft—und das ist eine Leistung von der größten Erkenntnistheoretischen Tragweite—eine strenge mathematische Theorie des Kontinuums, die es nicht als starres Sein, sondern als Medium freien Werdens faßt* [21].

[28] See [22].

> of analysis is questionable; this pessimism is, however, unfounded, as the next section will show.

We now, of course, know that Weyl's later pessimism was somewhat exaggerated; things were not as bad as they seemed. In a sense Brouwer was himself to be blamed for the lack of popularity of his intuitionistic practice. Instead of polishing up the presentation of important parts of mathematics, say differential and integral calculus, he preferred to concentrate on the basic phenomena of intuitionistic mathematics. This strategy was not designed to win over the young practicing mathematicians but at best the isolated devotee of foundational research. It was not before Bishop's book in 1967, that a convincing basic text became available. Weyl, in his postscript of 1955 ([24]), had already somewhat modified his view; he hailed the (just published) book of Lorenzen, [13], as a big step forward: "The operations of the formal calculus are here intertwined in a fruitful and relaxed way with conceptual considerations on their products; the discovery of Gödel thereby loses all of its disturbing aspects". An additional factor for Weyl's high opinion of Lorenzen's book may have been Lorenzen's predicative rather than intuitionistic concerns.

Weyl never could quite part with his earlier intuitionistic past. He remained fair with respect to Brouwer's intuitionism in a period where intuitionism had become a minor curiosity in the larger context of mathematics. In a larger review paper in 1946, he summed up his feelings:

> This history should make one thing clear: we are less certain than ever about the ultimate foundations of (logic and) mathematics, like everybody and everything in the world to-day, we have our "crisis". We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe", and it has been a constant drain on the enthusiasm and determination with which I pursued my research work. The experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavors mean in the context of man's whole caring and knowing, suffering and creative existence in the world. [23]

Weyl may have been right, maybe he was not alone in taking the "crisis in mathematics" seriously in a personal way. He certainly was the most prominent mathematician with a conscience sensitive to the cracks in the foundations of the building of mathematics. The story had a sequel,— Brouwer visited Weyl at the end of his life in Zürich and the two had a long talk. At the parting of the two old friends, Weyl sadly remarked: "*Brouwer, es ist alles wieder schwankend geworden*".

Appendix. Draft Brouwer to Weyl – 1920 (undated). Ihre rückhaltslose wissenschaftliche Handreichung hat mir eine unendliche Freude bereitet. Die Lektüre Ihres M.S. war mir ein fortwährender Genuss und Ihre Auseinandersetzung scheint mir - auch für das Publ. klar und überzeugend wird sein [. . .] Dasz wir beide über einige Nebensachen verschieden urteilen, wird auf die Leser nur anregend wirken. Allerdings haben Sie mit Ihrer Formulierung dieser Meinungsverschiedenheiten volkommen recht; in der Einschränkung des Objektes der Mathematik sind Sie tatsächlich radikaler als ich; es läszt sich aber darüber nicht diskutieren, diese Sachen sind nur durch individuelle Konzentration zu entscheiden. Im Anschlusz an Ihre Darlegungen über den Begriff der stetigen Funktion möchte ich Sie auf meinen Begriff einer volldefinierten Funktion des Kontinuums aufmerksam machen. Ich verstehe darunter ein Gesetz, das jedem Punkte einer mit den Kontinuum örtlich übereinstimmenden Punktspezies einen weiteren Punkt des Kontinuums zuordnet. Eine solche Funktion kann sehr gut unstetig sein, ohne irgendwie durch Zusammensetzung von stetigen Funktionen in getrennten Kontinuen erzeugt zu werden; man kann übrigens mit ihnen in mannigfacher Weise operieren. [man kann sie z.B. in gewissen Fallen integrieren ohne hinsichtlich ihrer Stetigkeit oder Unstetigkeit Aufschlusz zu haben]. Abgesehen von unseren Differenzpunkten hatte ich noch folgende Bemerkungen: Den Non-existential Sätzen (wozu z.B. die Mächtigkeitssätze auf S. 134 u. 43 des ersten Teiles meiner Abhandlung und auch der Hilbertschen Endlichkeitssatz des vollen Invariantensystems in seiner ersten Herleitung gehören) räumen Sie bei Ihrer Aufzählung der mathematischen Urteile gar keinen Platz ein. S. 3, Z. 8 (ebenso wie an der analogen Stelle auf S. 13 von "Das Kontinuum") ist mir der Sinn des Wortes "Sachkenntnissen" dunkel. Durch den Schluss des zweiten Absatzes von S. 34 scheint mir der ganze Zweck Ihrer Schrift gefährdet zu werden. Nachdem sie eben den Schlafenden aufgerüttelt haben, sagt er sich hier: "Also der Verf. gibt zu, dass die wirklichen mathematischen Sätze von seinen Darlegungen nicht berührt werden? Dann soll er mich weiter nicht stören!" und wendet sich ab und schläft weiter. Damit tun Sie aber unserer Sache Unrecht, denn mit dem Existenzsatz des Verdichtungspunktes einer unendlichen Punktmenge wird doch gleichzeitig manchem klassischen Existenz-theorem einer Minimalfunktion wie auch dem Existenzialsatze der geodätischen Linie bei Fortlassung der zweiten Differenzierbarkeitsbedingungen der Boden entzogen! Ihre S. 37 Z.3–6 aufgestellte, übrigens wie Sie wissen, meiner Auffassung widersprechende Behauptung sollte, wie ich glaube, etwas näher erklärt werden. Es scheint mir, dass auch der Leser, der Ihnen bis dahin genau gefolgt ist, mit diesem Passus Mühe haben wird. Ihr Funktio diskreta und Funktio mixta scheinen mir eben so gut wie Ihre Funktio kontinua in meinem Mengenbegriff enthalten zu sein. Mein Mengengesetz kann ja sehr gut von vornherein für jede Wahlfolge Sicherheit geben, dass nachdem einmal im

Zeichen erzeugt ist, weiterhin immer wieder nichts erzeugt wird. Auf Ihre Entscheidung zwischen Göttingen, Berlin und Zürich bin ich ungeheuer gespannt. Mögen Sie klar sehen, und die richtige Wahl treffen! Leicht wird Ihnen das nicht fallen! Den gesandten Abzug Ihres Manuskripts darf ich wohl behalten? Sie brauchen mir nichts mehr zurückzusenden. Weil inzwischen einige meiner älteren Separata neu abgedruckt worden sind, so möchte ich Sie um Mitteilung bitten, welche von meinen folgenden Publikationen Sie augenblicklich besitzen:

(1) Intuitionisme en formalisme (Holländisch)
(2) Intuitionism and formalism (Englisch)
(3) De onbetrouwbaarheid der logische principes (Holländisch)
(4) Het wezen der meetkunde (Holländisch)

Ich kann Ihnen das eventuell Fehlende jetzt ergänzen. Nochmals aufrichtigen Dank für die Freude und die Satisfaktion, die Ihre Schrift mir bereitet hat, herzliche Grüsse auch an Ihre Frau und auf Wiedersehen!

Ihr Egbertus Brouwer.

Acknowledgements. The first referee pointed out a number of improvements in the translation of the German quotations. He also put forward a number of critical questions concerning the relation between Brouwer, Hilbert and Weyl with respect to the actual beginning of the *Grundlagenstreit*. The matter will be taken up in the forthcoming biography of Brouwer; so far all available material seems to confirm the view that before the New Crisis paper no hostility had arisen. As the referee remarked, Hilbert's philosophical views are relevant to the course of the *Grundlagenstreit*; this is, however, a matter which deserves an independent treatment.

I am indebted to Andreas Blass for the improvements and corrections he suggested in the translations of the quotations.

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MEASURES: BACK AND FORTH BETWEEN POINT SETS AND LARGE SETS

NOA GOLDRING

It was questions about points on the real line that initiated the study of set theory. Points paved the way to point sets and these to ever more abstract sets. And there was more: Reflection on structural properties of point sets not only *initiated* the study of ordinary sets; it also supplied blueprints for defining extra-ordinary, "large" sets, transcending those provided by standard set theory. In return, the existence of such large sets turned out critical to settling open conjectures about point sets.

How to explain such action at a distance between the very large and the rather small? Rather than having an air of magic, could these results rest on deep structural similarities between the two superficially distant species of sets?

In this essay I dissect one group of such two-way results. Their linchpin is the notion of *measure*.[1]

§1. Vitali's impossibility result. Our starting point is a problem in measure theory regarding the notion of "Lebesgue measure." Before presenting the problem, I would like to review the notion of Lebesgue measure. Rather than listing its main properties, I would like to show how Lebesgue measure is born out of an attempt to generalize the notion of the length of an interval to arbitrary sets of reals. One tries to approximate arbitrary sets of reals by intervals, in the hope that the lengths of the intervals will induce a measure on these sets. The simplest case of such an approximation is that

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[1] Throughout this paper I avoid the question of whether large sets, usually referred to as "large cardinals," "really exist." I substitute the "whether" with a "where": Bracketing away the "philosophical" question—*whether* large cardinals exist—I rather focus on the mathematical question: *Where* exactly, in mathematical practice, does one encounter such sets? It is the main point of what follows that one does not have to be a specialist of abstract set theory to get involved with such "exotica": Led by mundane questions of measure theory and analysis, the most hum-drum mathematician is destined to encounter such "large" cardinals. I also believe that viewing large cardinals axioms as formulating *mathematical* (rather than distinctly set-theoretical) hypotheses takes away some of the mystery related to their "action at a distance" mentioned above. I relate to the more "philosophical" questions in a twin paper, "Set theory as a branch of mathematics."

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of approximating a point (on the real line) by intervals; since a point can be covered by intervals of arbitrary small length, its measure (or, to be precise, the measure of the singleton containing this point) ought to be zero. Thus the purported measure will be one that *vanishes on points*. If we let m denote our measure, then for every real number r, $m(\{r\}) = 0$.

We should note here that in the above example we were relying on our intuition that our notion of measure ought to be *monotone*: If one set is contained in another, then the "smaller" set should have measure less than or equal to that of the "larger" one.

Some sets cannot be so readily approximated by (a sequence of) *single* intervals. The simplest example is the disjoint union of two intervals. Here we need a principle that will help us combine the measures of the smaller sets to gauge the measure of the larger set. With the above example, it seems clear that we should opt for the principle that says that the measure of the union of the intervals is the sum of their measures. In fact, that should hold for the disjoint union of a pair of *arbitrary* sets. Using the induction principle, we can show that this requirement is equivalent to that of *finite additivity*: For any finite sequence of pairwise disjoint sets $\langle A_i : i < n\rangle$, we would like to have that $m(\bigcup_{i<n} A_i) = \sum_{i<n} m(A_i)$.

Should our measure abide by stronger principles of additivity? A principle which says that the measure of *any* disjoint union of sets is the sum of the measures of the individual sets is plainly false: *Any* set can be written as the disjoint union of the singletons of its members. Thus the interval $(0, 1)$ is the union of singleton r's, for reals r in $(0, 1)$; the entire real line is the disjoint union of all singletons of real numbers. Clearly, even though the measure of each singleton is zero, the unit interval or the real line will have positive measure. The problem lies with the *size* of the union: A *large* union of small sets may be large. In the above examples the union was definitely "large," being the size of the continuum. On the other hand, the measure on intervals is *countably additive*: If an interval is decomposed into countably many disjoint intervals, then the length of the original interval is the (infinite) sum of the lengths of the "component" intervals. This suggests that we may consider countable unions as "small." Thus we get the requirement of *countable additivity*: If $\langle A_i : i \in \omega\rangle$ is a countable sequence of disjoint sets, then $m(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} m(A_i)$.

Finally, the property of *translation invariance* suggests itself. Informally put: The measure of a given set is independent of its location on the real line. We can "move it around" without affecting its measure. Formally, we define the notion of a "shift" of a set as follows: A set Y is a shift of the set X (by the real number d) if Y is the "result" of adding d to every number in X (in symbols: $Y = X + d$, where $X + d$ is the set of all y's such that for some x in X, $y = x + d$.) *Translation invariance* says that if Y is a shift of X then X and Y have the same measure. This property is true for intervals measured by their lengths; and it ought to be true for the notion of measure on arbitrary sets of reals. It ought to be true for two reasons: First, our project is to approximate arbitrary sets by intervals, and it seems that translation invariance should be "passed on" from the measure on the intervals to the measure on arbitrary sets. But also, this requirement seems a precondition on any reasonable notion of measure on subsets of the *real line*: The notion of "moving a set around" the real line seems better understood than the notion of measure for an arbitrary set. How could the *size* of a subset of the real line change if all we did was move each point in it by some fixed number of units to the right (or to the left)?

We now know what principles we would like our measure to satisfy and (roughly) how we should set about finding the measure of an arbitrary set of reals. The question is: Is there such a measure on the real line, measuring *every* subset of it? Lebesgue measure is a measure on sets of reals which is obtained precisely by following the guidelines mentioned above.[2] Is Lebesgue measure total?

The turn of the century mathematician, Vitali, was the first to realize it was not. He relied on the axiom of choice in constructing a non-measurable set. In fact, his proof shows not only that Lebesgue measure is not total, but that *no* measure satisfying the above mentioned properties can be total. Let me first list these properties, and then review Vitali's proof.

Denote by m the Lebesgue measure on the real line (R).

We say that a subset X of R is *measurable* iff it belongs to the domain of m. Then m has the following properties:

(1) The measure of any finite interval is its length. E.g., if $J = (a,b)$, then $m(J) = b - a$. Infinite intervals have infinite measure.
(2) Non-triviality: For any real number r, $m(\{r\}) = 0$ and the measure of the entire real line is non-zero: $m(R) \neq 0$.
(3) Monotonicity: If X and Y are measurable subsets of R such that X is a subset of Y then $m(X) \leq m(Y)$.
(4) Countable additivity: If $\langle X_n : n \in \omega \rangle$ is a pairwise disjoint family of measurable subsets of the real line then $\bigcup_{n\in\omega} X_n$ is measurable and $m(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} m(X_n)$.
(5) Translation invariance: If X and Y are subsets of the real line and d is a real number such that $Y = X + d$, then Y is measurable iff X is and (if they are both measurable) $m(Y) = m(X)$.[3]

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[2] Perhaps this can best be seen if one defines the Lebesgue measure of a set to be its *outer* measure if it (the outer measure) agrees with the inner measure. See [2] for a definition of inner and outer measures.

[3] For a definition of Lebesgue measure see [11, chapter 3]. In general, by a *measure* on a set X I will mean a function h from (all or some) subsets of X to the real line plus an

I will now review Vitali's proof: Consider the equivalence relation on the reals in the (closed) unit interval induced by: $x * y$ iff $x - y$ is rational $(0 \leq x, y \leq 1)$. Using the axiom of choice, pick a member from each equivalence class. The resulting set S is non-measurable: The unit interval can be covered by countably many sets, each a "shift" by a rational number (between -1 and 1) of the original set. By translation invariance all these "shifts" have the same measure (if they are measurable at all); thus either all have measure zero, which is impossible, since their countable union covers the unit interval,or else all have a fixed, positive, measure, equally impossible, for their union, which is contained in the interval $[-1, 2]$, would have infinite measure. Thus S is not measurable.[4]

The proof above does not rely on the specific definition of Lebesgue measure. It follows from properties 1.– 5. of Lebesgue measure listed above (and, in fact, does not depend on 1.: As long as finite intervals get finite, positive measures, the proof would still go through). Thus Vitali's construction of the non-measurable set not only shows that Lebesgue measure is not total; it establishes that there is no way of extending it to (or replacing it by) a total measure, if the extension (or replacement) is to assign finite positive measures to finite intervals and to be countably additive and translation invariant (recall that any measure satisfies, by stipulation, requirements 2.–3.). For any such measure admits the construction of a non-measurable set in the same manner described above. We thus get Vitali's impossibility result, (VI):

> (VI) There is no non-trivial, translation invariant, countably additive *total* measure on the real line (assigning finite measure to finite intervals).

I would like to emphasize that Vitali's result really stops us in our tracks. For it not only shows that some technical construction is impossible; rather it shows that the natural notion of measure on the real line cannot measure *all* subsets of the real line.

1.1. In the wake of Vitali's result. We now face the task of picking up the pieces. One path open to us is to turn our backs on the iceberg and sail back to the safe waters ruled by Lebesgue measure.

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additional point ("infinity"), satisfying (the analogs of) 2–3 above and *finite* additivity, i.e., if $\langle X_i \quad i < n\rangle$ is a finite sequence of subsets of X belonging to the domain of h, then $\bigcup_{i<n} X_i$ belongs to the domain of h and $h(\bigcup_{i<n} X_i) = \sum_{i<n} h(X_i)$

Lebesgue measure is an example where the range of the function h is maximal, i.e., every point on the real line (plus "infinity") is the measure of some subset of the real line. Later we will discuss measures at the other extreme, where the range of the function h will consist only of two points.

[4] For details of this construction see [11], pp. 52–54.

The non-measurable set constructed is far from being a "natural" set of reals. It is a set specifically constructed to serve as a counter-example. It is built from "above," by using the "logical" operation of choosing representatives from equivalence classes. It contrasts with e.g., the *Borel* sets. These are built from "below": They are built from topologically simple sets (open intervals) by closing under simple "mathematical" operations (complementations, countable unions and countable intersections). The Borel sets of reals are Lebesgue measurable. Can one chart the navigable waters up to the non-measurable sets?

I mention two results that seek such cartography. Back in 1917, Lusin (in [6]) showed that all *analytic* sets of reals were Lebesgue measurable. Since the class of analytic sets extends the class of Borel sets, Lusin drew a new boundary for the class of Lebesgue measurable sets.[5]

In 1938, Gödel approached the boundary from the other side: Rather than displaying more complicated measurable sets, he came up with a simpler, *definable*, non-measurable set. But Gödel did not *prove* that there had to be such a set. Rather, he displayed a *model* of set theory where such a set existed.

Thus Gödel departed from the path taken by Vitali and Lusin in the *methods* he used to tackle the problem. He introduced *meta*-mathematical techniques in the hunt. Whereas Vitali and Lusin were directly analyzing the real numbers, Gödel was also analyzing the "proof system" employed by them. By displaying a model where all the assumptions used by Vitali and Lusin were verified (as well as preserving truth under their rules of inference) where a definable non-measurable set existed, Gödel was able to show that you cannot *prove*, using the same hypotheses as Vitali and Lusin, that all definable sets were Lebesgue measurable.

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[5] One can appreciate Lusin's result and the discussion that follows by introducing a scale measuring the complexity of the definitions (using formulas of first order set theory) of various sets of reals. On the two opposite ends, we have the Borel sets and Vitali's non-measurable set. *Borel* sets are all quite simply defined: These are the sets obtained from open intervals (themselves easily definable) by the operations of complementation and countable intersection (iterated countably many times) On the other hand, Vitali's set S described above is not definable at all: It contains one member from each equivalence class (of the relation $*$ defined above) but there is no way to define (in ZFC) which member was picked from each equivalence class. Thus we cannot explicitly define the members of this set. (The situation may change if we have a definable well ordering of the reals; see Footnote 6.) In between these extremes are definable sets of reals whose definition becomes more and more complicated The collection of *projective* sets is a natural extension of the Borel sets: The projective sets are those definable in a first-order language, where the quantifiers range over the real numbers and the primitive relations are Borel relations. In fact, one can draw a further distinction *within* the class of projective sets: For each projective set, we count the number of quantifiers (ranging over the real numbers) needed for its definition. The more quantifiers needed to define a given set, the higher its rank in the hierarchy will be. This produces the projective *hierarchy*· The Borel sets form its basis and the other sets are divided along its ranks. The *analytic* sets mentioned above are close to the bottom of this hierarchy: Only one existential quantifier ranging over the real numbers is needed to define them. The set constructed by Gödel, to be mentioned in the main text below, is "almost" one rank higher· It requires two "real" quantifiers for its definition, but it has two definitions; one with an existential quantifier preceding a general quantifier, and one with a general quantifier preceding an existential quantifier. Such sets are standardly referred to as Δ^1_2 sets. I note that the set constructed by Gödel does not require any real parameters for its definition, even if we write out explicitly the formula for the Borel relation. See [3] and [18] for Gödel's result.

Gödel's model was a model of first order set theory, Zermelo-Fraenkel set theory (with choice), henceforth abbreviated ZFC. His result then showed that one cannot prove in ZFC that all definable sets are Lebesgue measurable. The principles used by Vitali and Lusin can be cast within ZFC but by no means do their principles exhaust ZFC. ZFC is a theory which describes many sets, of which sets of reals are only a very small fragment. Yet Gödel showed that even this theory could not prove that all definable sets of reals were Lebesgue measurable.

In fact, the non-measurable set in Gödel's model was a rather *simply defined* set of reals (not much more complicated than the analytic sets). Thus we get an even stronger result: One cannot prove, in ZFC, (much) more than what Lusin proved; viz., that all analytic sets were Lebesgue measurable. Gödel showed in effect that Lusin had hit a barrier, henceforth, "Lusin's Barrier," as to what was provable, in ZFC, about the extent of the Lebesgue measurable sets.[6,7]

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[6] Gödel constructed his non-measurable set in L, the model of constructible sets. In this model, there is a *definable* well ordering of the reals. It is the definability of the well ordering which enables one to define non-measurable sets. It is not clear what specific example (of a non-measurable set) Gödel had in mind (see [18]). But one such example (of the complexity asserted by Godel) can be obtained by carrying out Vitali's own construction inside L, replacing his arbitrary choices of members from the equivalence classes by definable choices (namely, the least element from each equivalence class with respect to the well ordering mentioned above, which is itself definable). See also the previous footnote.

Gödel's result is considered again in Section 1.2. That section also contains a more detailed discussion of obtaining meta-mathematical results by using models. For more on ZFC and the sets it describes see Section 1.3. Gödel's model L is discussed in Section 3.

[7] I say in the text that Gödel's result shows that Lusin had hit a "barrier" as to the extent of the sets that could be shown to be measurable in ZFC. To be precise, there are classes of sets of reals which do extend the analytic sets (which were shown by Lusin to be Lebesgue measurable) which can be shown, in ZFC, to contain only Lebesgue measurable sets. In fact, Lusin himself, in a joint result with Sierpinski, ([7]), showed the Lebesgue measurable sets were closed under the Suslin operation $\mathcal{A}$ (See [9, p. 72], for the definition of the Suslin operation.) The closure of the analytic sets under this operation yields an example of a class of sets of reals extending the class of analytic sets in which all sets are Lebesgue measurable. The strongest result in this vein is an unpublished result of Solovay, proving (in ZFC) that if a set can be proved in ZFC to have a Δ^1_2 definition then it is Lebesgue measurable. The sets considered by Lusin and Solovay are all Δ^1_2; by Gödel's result, one *cannot* show (in ZFC) that *all* Δ^1_2 sets are Lebesgue measurable. Thus in terms of levels of the *projective hierarchy*, Gödel's result shows that Lusin's result is the best possible. Henceforth, whenever I talk about

We are now confronted by what I shall call the Vitali-Gödel Straits. Vitali presented us with a non-measurable set. This set was carefully constructed to serve as a counter-example and was not built up from below from simpler sets. On the other hand, Lusin presented us with a (relatively small) class of measurable sets, the analytic sets, all rather simply defined, and Gödel showed that this was roughly the best one could do (while working within ZFC set theory). But "most" sets of reals do not fall in either category: They are neither "fabricated" sets of reals nor are they as simple as the analytic sets. For any such set A (which is neither "fabricated" nor analytic), we naturally wonder:

Is A measurable or not?

The question seems simple enough. However, the above discussion shows that its solution cannot be so simple. Many possible routes are blocked even before any attempt to follow them has been made. We may be very ambitious and try to show *all* sets are Lebesgue measurable. Vitali's result puts an end to that quest. Or we might be less ambitious and just try to show that more sets (more than the analytic sets) are Lebesgue measurable. This time it is Gödel's result that stops us in our tracks, at least as long as we are operating within ZFC set theory.

Perhaps what the Vitali-Gödel results tell us is that ZFC does not exhaust the basic truths about sets. Might the true situation regarding the extent of Lebesgue measure be better understood if we use stronger (true) set theoretical principles than those described by the axioms of ZFC? In particular, can we hope to show, using a "strengthened" theory, that *more* (i.e., more than the analytic) sets are Lebesgue measurable?

Strengthening our hypotheses is one way of trying to get more information about our original problem. Another is to weaken our goals, by looking at a revised notion of measure. The "real" measure for sets of reals is Lebesgue measure, but it cannot be total. What if we look, e.g., at measures which are not translation invariant? They are not as intuitive as Lebesgue measure, but they pose an interesting question: Can such measures be total? Answering this question might not provide us directly with new information about the original problem; but indirectly, we might gain new insight as to the source of the malaise.

Caught between the Scylla and Charybdis of the Gödel-Vitali results, we may confront the Gödel end by strengthening our theory and hoping to prove more sets (more than the analytic sets) are Lebesgue measurable. Alternatively, we may bypass Vitali's trap by pursuing a revised notion of measure.

"more sets (more than the analytic) being Lebesgue measurable," I mean "higher levels of the projective hierarchy containing only Lebesgue measurable sets."

But we must take notice of the different kind of resolution involved. Whereas confronting Gödel's end leaves us with a *truth of the matter* oriented search conducted at the (real) universe of sets, Vitali's obstacle leads us to investigate *consistency* results and *models* of set theory.[8]

Our search for escape routes will thus bifurcate into truth of the matter results (involving Lebesgue measure) and consistency results (involving revised measure). I see the distinction between *truth of the matter* results and *consistency* results as a fundamental one and so will pause to expound it before detailing the two escape routes from the Vitali-Gödel straits.

1.2. Truth vs. consistency. Truth oriented results *describe* the reality of sets. They record, depict if you will, the way things really are in the one and only universe of sets.

Lusin's result is an example of the kind. Lusin proved, using the axioms of ZFC, that all analytic sets were Lebesgue measurable. Thus his result will hold in any model of ZFC. In particular, it must be true in the real universe of sets. It is this last which is the real force of Lusin's result: Regarding the *real* sets of reals, all analytic sets are Lebesgue measurable.

In contrast, the main goal of consistency results is to craft a model of set theory with some desired combinatorial properties in order to witness the consistency of certain propositions. The model is not the universe of sets but rather a consistency verifying artifact. We have just encountered one such result, Gödel's, about the existence, in a model, of a definable non-measurable set. Gödel's model is the class of constructible sets (mentioned if Footnote 6, see also Section 3). Gödel's result does not show that there exists a simple, definable, non-measurable set, only that the existence of such sets is consistent with ZFC, i.e., that one cannot prove, in ZFC, that such sets do not exist.

One final observation. Whether our end-results pertain to truth or consistency, we confront the question: what *means* are needed to reach those ends? When truth is our focus, we are not constructing models but investigating the universe of sets directly. We ask: What information about this universe is indispensable for ascertaining a *specific* result? E.g., to go back to Lusin's result, all that was needed to prove that the analytic sets were Lebesgue measurable were the axioms of ZFC. Thus looking at the universe of sets itself, we know that one need not exhaust all its riches to ascertain the truth of Lusin's result: Those structural features of the universe, which are revealed by the axioms of ZFC, suffice to ground his result. However, if we are to show that larger fragments of sets (i.e., more than the analytic sets) are measurable in the real universe of sets, stronger set theoretic principles will have to be used. This much is made clear by Gödel's result, which set limitations on what can be done when we are confined to the axioms of ZFC. In the results described below, the additional set theoretic principles strengthening our theory will all assert the existence of some *large cardinals*. By exploiting the fact that such large cardinals exist in the universe of sets, we will be able to show more sets of reals are Lebesgue measurable (i.e., more than the analytic sets).

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[8] The claim about the need to examine consistency results in order to bypass Vitali's trap is not obvious: Might we not be investigating the real universe of sets in order to determine whether, in it, there exists e.g., a non-translation invariant measure which is total? Indeed, there is a way to weaken the notion of measure which does allow for the construction, in the real universe of sets, of a total measure. I will mention this below (in Footnote 11). But the weakened notion of measure I will pursue will turn out to involve, and essentially so, discussion of consistency results See Footnote 18.

The question of the means being used also arises in the context of consistency results. Here the question usually takes the form: What assumptions are needed in order to construct the model we are after? In Gödel's example of a non-measurable set mentioned above nothing more than ZFC was needed to construct the model. As we shall see, in order to construct models which will help us steer away from the Vitali-Gödel straits, principles stronger than those of ZFC, principles securing the existence of more sets, will be called for. Again it will be necessary to assume the existence of large cardinals. We must bear in mind, however, that the question thus posed is not whether there really exists this or that large cardinal in the universe of sets; it is rather what price must be paid to prove the consistency of ("provide a model for") a certain claim.

The rest of this paper is devoted to both the consistency and truth oriented escape routes from the Vitali-Gödel straits.[9]

1.3. Intermezzo: Large cardinals. I have mentioned the notion of a large cardinal and will use it extensively below. What are large cardinals? Rather than attempt to define them let me try to explain why they are called *large*. The "universe of sets" can be thought of in the following way: We start with the empty set and construct more and more sets. If we are given the ordinals, we can describe the construction using two operations: Power set and union. At successor stages we take the power set of the previous stage, at limit stages the union of the preceding stages.

Large cardinals are those (uncountable) cardinals which cannot be reached from "below" in this manner. The smallest large cardinal, the *inaccessible* cardinal, sets a good example. The inaccessible cardinals are defined as the regular strong limit cardinals: If κ is an inaccessible cardinal, κ is not the union of less than κ sets each of size less than κ. Also, the power set of any cardinal less than κ must always itself be less than κ. Any cardinal satisfying these last two requirements is an inaccessible cardinal.

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[9] Formally speaking, in consistency results we do not need to assume the existence of large cardinals, only that their existence is consistent with ZFC As for the existence of large cardinals in the universe: Most set theorists believe, or feign to believe, that large cardinals exist. As noted above, I do not wish to take a stand on this issue. I would, however, like to draw the distinction between the use of large cardinals in truth of the matter results and in consistency results. In the former, the presence of the large cardinal sheds light on the very universe of sets; in the latter, the large cardinals merely serve as tools for constructing consistency-verifying models.

The inaccessible cardinals also exemplify another property which is common to all large cardinals and witnesses their size: If there is an inaccessible cardinal then the hierarchy described above, taken up to the first inaccessible cardinal, forms a model of ZFC set theory. In particular, the consistency strength of "ZFC + 'there exists an inaccessible cardinal' " is greater than that of the theory ZFC alone. This is typical of all large cardinals. As they cannot be reached using the operations of ZFC from "below," assuming their existence provides a way of forming a model of the theory of ZFC together with whichever cardinals (large or small) exist below them; and the assumption that such a cardinal exists always adds consistency strength to the theory.[10]

§2. **The consistency path—revising the notion of measure.** We first try to regain total measure by generalizing our notion of measure. From the various properties of Lebesgue measure, it is natural to ponder the subtraction of either countable additivity or translation invariance. Dispensing with translation invariance turns out to be more relevant to the theme of this paper, so I follow this path.[11] In fact, it will be useful at this point to dispense with both translation invariance and the requirement that the measure of finite intervals is their length. With both these requirements gone, we leave behind us the original, "concrete," geometrical form of measure. The only distinguishing mark of the measure space now is its cardinality; the remaining properties of measures do not tell apart different measure spaces of the same cardinality. Since a cardinal κ is, itself, a set of cardinality κ, we might as well think of the measures as being on the cardinals themselves. The original question about measures on the real line is thus transformed into: Is there a total, countably additive measure on the cardinal of the real numbers, $2^{\aleph_0}$? Before answering this question, we might ask: Are there *any* cardinals carrying total, countably additive measures? Ulam (in [19]) proved that the first cardinal κ with such a (total) measure has the property that all (total) measures on it are κ additive.[12] Hence in order to see whether there is any cardinal κ with a total, countably additive measure on it, we might as well see whether there is any cardinal κ with a κ additive measure on it. This search bifurcates into a search for *atomic* and *non-atomic* measures. I will consider both in this order. We shall see that large cardinals are intrinsic to both.[13]

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[10] For more on large cardinals, see [5].

[11] It is possible to regain total measure if one drops the requirement of countable additivity (while holding on to the translation-invariance requirement). In fact, one can prove, in ZFC, that such a measure exists on the real line. The first to realize this was Banach. See [20, p. 156], for a discussion of this result.

[12] A measure on κ is κ additive, or κ complete, if whenever $\langle A_\alpha | \alpha < \beta\rangle$ is a sequence of pairwise disjoint subsets of κ of length *less than* κ (i.e., $\beta < \kappa$) the measure of the union of

2.1. Atomic measures. An atom is a set in the measure space with positive measure which is, as its name suggests, unsplittable. Any attempt to "break" an atom into two will result in one of the two "pieces" having measure zero. In other words, any subset of an atom either has measure zero or has measure equal to that of the original set. We can restrict ourselves to one such atom, i.e., consider the atom, say A, as our new measure space, where the only sets to be measured are the subsets of A.

Since A was an atom, every subset of A has either measure zero or the same measure as A itself. Further, if A was a subset of κ, and the original measure on κ was κ additive, then A has cardinality κ. We thus face the notion of a *two valued* measure, i.e., a measure which takes only two values. Putting all this together, we arrive at a definition of a measurable cardinal:

> (MC) A cardinal κ is *measurable* iff there exists a non-trivial, two valued, κ additive (or: κ complete) total measure on κ.

It is customary to assume that the measure assigned to κ by a two valued measure on κ is one. So if κ is a measurable cardinal, every subset of it gets assigned either zero or one by the measure. Note that of the five characteristic properties of Lebesgue measure mentioned in Section 2, it is the second, third and fourth that appear in the definition of this new notion: 2–3 are left intact (replacing R by κ) and 4 is strengthened—instead of demanding countable additivity we now require κ additivity. The new notion of measure is indeed a generalization of the original notion of Lebesgue measure.

How does the discovery of this new notion bear on our original quest for a total measure on the real line? At first glimpse, it seems not to. For measurable cardinals turn out to be *large* cardinals. They are inaccessible cardinals, and this already suffices to rule out the possibility that the cardinal of the real numbers, $2^{\aleph_0}$, will be measurable (it can be reached by the power set operation from the set of natural numbers), or that the existence of *any* measurable cardinal could be proved in ZFC. In fact, measurable cardinals are much larger than inaccessible cardinals: Below the first measurable cardinal κ there are κ inaccessible cardinals. Thus inasmuch as there emerged a concept of total measure, it is one that only applies to large cardinals. In

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the A_α's is the sum of their measures. According to this terminology, countably additive (or countably complete) measures are $\aleph_1$ additive ($\aleph_1$ complete).

[13] When opting to consider non-translation-invariant measures, we also decided to part with the requirement that finite intervals be assigned their lengths by the measure. The decision about the intervals is independent from the one about translation invariance, as can be seen from Solovay's work discussed later in this paper. See Section 2.3 and Footnote 15.

contrast, the cardinal of the set of real numbers, $2^{\aleph_0}$, is "small" and cannot carry such a measure. The measurable cardinals offer no immediate end to our quest for a measure on the reals; however, they do supply a critical clue for the construction of a suitable measure on the set of real numbers itself.

2.2. Non-atomic measures. The two valued measures were generated out of atomic measures. By focusing on non-atomic measures, we encounter a concept of measure which *can* sometimes apply to $2^{\aleph_0}$, that of *real valued* measure.

If a total, κ-complete measure on a cardinal κ has no atoms, then we can split any positive measured set into two smaller positive measured sets. This results in the measure having a multitude of values, in fact, as many values as the real numbers, as opposed to the two valued measures encountered before. Such measures are called "real valued measures." Cardinals carrying such measures are called "real valued measurable cardinals":

> (RVMC) A cardinal κ is a *real valued measurable cardinal* iff there exists a non-trivial, κ additive (or: κ complete) measure on κ.

Although $2^{\aleph_0}$ cannot be a (two-valued) measurable cardinal, it can sometimes be a *real valued* measurable cardinal. At the end of the sixties, R. Solovay constructed a model for:

> $2^{\aleph_0}$ is a real valued measurable cardinal.[14]

From measures on large cardinals we are now down to measures on $2^{\aleph_0}$, which is the cardinality of the set of reals.

This is much closer to our original question, which concerned measures on the real line. Can we get even closer to our original target? Yes: Let $2^{\aleph_0}$ be a real valued measurable cardinal. One can now construct a *total* countably additive measure, m^*, on the real line which extends the *Lebesgue* measure: Any set which is Lebesgue measurable, will be assigned by m^* its Lebesgue measure. But, unlike Lebesgue measure, m^* will assign *any* subset of the real line *some* value.

Relying on m^*, Solovay constructed a model of

> ZFC + "Lebesgue measure has a countably additive (non-translation invariant) extension to all subsets of the unit interval"

Solovay thus found one escape route out of the Vitali-Gödel straits: It *is* possible to generalize the notion of Lebesgue measure in a way that makes total measure on the reals compatible with the axioms of ZFC. It is possible, but what are the "costs?" What assumptions were made by Solovay in constructing the above described model?[15]

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[14] See [17]. The assumptions needed to construct this model will be discussed in the next section (2.3).

[15] Solovay's model demonstrates that one cannot, in general, deduce that a countably additive measure on the real line is translation invariant just because finite intervals are

2.3. Measurable cardinals and measuring point sets. Solovay started out with a model of set theory, alias *the initial model.* He transformed it into *the final model* where $2^{\aleph_0}$ was a real valued measurable cardinal. Solovay assumed that the initial model, besides being a model of ZFC, contained a *measurable cardinal.*

In fact, it is the measurable cardinal itself, κ, which becomes at once $2^{\aleph_0}$ and the real valued measurable cardinal in Solovay's *final* model. The two valued measure on κ is no longer total in the final model, but Solovay showed that it could be extended to a total *real valued* measure on κ, i.e., on $2^{\aleph_0}$. This real valued measure, in turn, enabled Solovay to construct a countably additive extension of *Lebesgue measure* to all subsets of the real line.

The presence of a measurable cardinal is thus *sufficient* for the construction of a model where Lebesgue measure has a countably additive extension to all subsets of the real line. Solovay showed it was also *necessary* for the construction, in the following sense: Starting with a model where Lebesgue measure had a countably additive extension to all subsets of the real line, Solovay constructed a model with a measurable cardinal. Thus the three theories:

- ZFC + "Lebesgue measure has a countably additive extension to all sets of reals"
- ZFC + "$2^{\aleph_0}$ is a real valued measurable cardinal"
- ZFC + "there exists a measurable cardinal"

were shown to be equi-consistent.[16]

The above results show that the measurable cardinal is essentially connected to the measure problem. This is no mere coincidence: Solovay exploited the structural similarities between the different notions of measures: On the one hand, a total, countably additive extension of Lebesgue measure on the reals and on the other, an atomic, κ-additive, non-trivial measure on a cardinal κ. Solovay showed how to transform one into the other. The size of the large cardinals may be very different from that of sets of reals; however, the measures on them bring out their structural similarities.[17,18]

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assigned their length as their measure. Since m^* extends Lebesgue measure, finite intervals are assigned their length by m^*. But m^* is not translation invariant, for if it were, it would contradict Vitali's impossibility result (VI) (since m^* is total and countably additive).

[16] Solovay's result is an equi-consistency result. Thus one has to be careful when stating the "necessity" of the existence of a measurable cardinal in the initial model. What is necessary is that the initial model be a model of a theory of the same "consistency strength" as that of ZFC + "a measurable cardinal exists."

[17] What is the route from a total, countably additive, non-trivial measure on the real line to a measurable cardinal? First, by Ulam's result quoted earlier in the paper, if there exists a non-trivial, countably additive, total measure on the real line then there exists a non-trivial, κ-additive measure on some cardinal κ which is less than or equal to $2^{\aleph_0}$. In other words, a real valued measurable cardinal exists (that the measure has no atoms follows from the fact

§3. Large cardinals and Lebesgue measure: The truth of the matter. We have seen the impact of large cardinals on Vitali's impossibility result (VI) in the realm of "consistency results." It is consistent that total measure can be regained *if* the existence of some large cardinal is consistent. But do large cardinals have any impact on the *truth* of the matter, consistency results notwithstanding?

Recall that Gödel constructed a model where there was a non-measurable set which was not much more complicated than the analytic sets. If we are to prove that more sets than the analytic sets are measurable, we had better see what it is, exactly, in Gödel's model that we are to avoid.

Gödel's model of constructible sets, L, is constructed by recursion on the ordinals, where at each stage α one adds a "minimal amount" of sets: only those definable over the part of the model already constructed. The result is a very structured but minimal model of set theory (ZFC). One indication of the "structure" of L is that it has a definable well-ordering: There is a formula of set theory which determines a well-ordering of all the constructible sets. In particular, there is a definable well-ordering of the constructible reals (i.e., the real numbers which are present in Gödel's model). It is this definable well-ordering which enabled Gödel to construct his definable, non-measurable set.

The minimality of Gödel's model is manifested in the fact that it is contained in any transitive (class) model of set theory. One can carry out

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that κ has size less than or equal to the continuum). The remaining question therefore is how to construct a model with a measurable cardinal provided there exists a model with a real valued measurable cardinal (I assume all models are models of ZFC). This can be done by relativizing Gödel's model of the constructible sets so that it also includes sets "constructed from" the family of those sets assigned measure zero by the real valued measure in question. See Section 3 for more on Gödel's model. Note that real valued measurable cardinals and non-trivial, countably complete, total measures on the real line are very closely related. One exists if and only if the other exists. This shows that when we decided to dispense with the length of finite intervals as their measures in addition to dispensing with translation invariance, we were not abstracting "too much": as long as the measures in question are non-atomic (and countably complete), given that one exists, a measure will exist on the real line which does assign finite intervals their lengths.

[18] Let me justify my earlier claim that the escape route mentioned in the text (constructing countably additive total measures which are not translation invariant) necessitates investigation of consistency results. That the existence of a countably additive total measure (which is not translation invariant) cannot be proven in ZFC follows from the fact, mentioned in the text, that the consistency strength of such a claim is stronger than that of ZFC (assuming ZFC is consistent). Might we hope to prove the existence of such a measure if we assume the existence of some large cardinals? The answer is still negative. The existence of such a (total) measure on the real line violates the continuum hypothesis, i.e., it implies that the cardinality of the set of reals is not the first uncountable cardinal. However, the existence of large cardinals does not contradict the continuum hypothesis, and therefore, one cannot hope that their existence will imply the existence of measures described above. See [5, pp. 298–303].

Gödel's construction in any such model and, since the construction itself is absolute, i.e., does not depend on the specifics of the model it is being carried out in, the result is always the same, namely, Gödel's model of the constructible sets. Thus no matter which model we set out to work in, Gödel's model will always be present as a submodel. If we are to succeed in constructing a model of set theory where more sets (more than the analytic sets) are Lebesgue measurable, we have to guarantee that our model is "sufficiently different from," i.e., "sufficiently larger than," Gödel's (minimal) model.

Large cardinals provide the means of achieving this. For example, if a measurable cardinal exists, then the first uncountable cardinal ($\aleph_1$) is itself a large cardinal in L: It is inaccessible in L. Thus small uncountable cardinals of the universe may become large in the minimal model L: It lacks the sets needed to demonstrate that the sets in question are really "small." Turning this fact around, we see that cardinals which are "declared" large by L may really be small. E.g., we just saw that an inaccessible cardinal of L could be the first uncountable cardinal of the universe. Referring still to the same example, all the many cardinals of L which are smaller than the above mentioned inaccessible cardinal are smaller than the "real" first uncountable cardinal, i.e., are "merely" countable ordinals in the universe. Thus in the presence of a measurable cardinal, there is indeed a big gulf between the universe of sets and Gödel's model of constructible sets, L.

It was once again Solovay who first showed how one could use this gulf in order to establish: If a measurable cardinal existed, more sets than the analytic sets were Lebesgue measurable. Solovay showed that if a measurable cardinal existed, sets belonging to the next level of the projective hierarchy beyond the analytic sets (i.e., Σ^1_2 and Π^1_2 sets) had to be Lebesgue measurable. This truth of the matter result is related to a consistency result proved by Solovay: Assuming an inaccessible cardinal existed (to be precise: Assuming the consistency of ZFC plus the assertion that an inaccessible cardinal exists), Solovay constructed a model of ZFC where *all* projective sets were Lebesgue measurable. Now truth in a model of set theory does not, in general, imply truth in the universe of sets. However, in this instance, Solovay was able to show that the existence of a *measurable* cardinal in the universe created certain similarities between the universe and his consistency-verifying model. Solovay exploited the gulf between the universe and L (in the presence of a measurable cardinal) alluded to above together with Shoenfield's absoluteness result ([14]; see Footnote 20) to prove that if a measurable cardinal existed then the sets directly beyond the analytic sets in the projective hierarchy, the Σ^1_2 and Π^1_2 sets, were Lebesgue measurable in the universe of sets.

What light do Solovay's results shed on Lusin's barrier? First, Solovay's consistency result: Solovay constructed a model of ZFC where all projective sets were Lebesgue measurable (assuming the consistency of an inaccessible cardinal). This complements Gödel's result which says that there is a model of ZFC where a set belonging to the level beyond the analytic sets in the projective hierarchy is *not* Lebesgue measurable. Thus Lusin's barrier can really be seen as a ZFC imposed barrier—this last cannot settle the measurability status of sets in the projective hierarchy, one way or another.

But next comes Solovay's *truth of the matter* result:

> (STM) If a measurable cardinal exists, then all Σ_2^1 (and Π_2^1) sets are Lebesgue measurable, i.e., all sets of reals belonging to the second level of the projective hierarchy are Lebesgue measurable.

This result actually topples Lusin's barrier. The presence of a large cardinal guarantees that sets which were lying *outside* the barrier are now in fact Lebesgue measurable. The barrier has been passed.[19]

Is the barrier removed, or was it just placed one level higher in the projective hierarchy? Can we show *more* sets (e.g., all projective sets) are Lebesgue measurable, if we assume *more* large cardinals exist? This was unknown for many years, until Woodin and Shelah came up with an important extension of Solovay's result: If one assumes the existence of sufficiently many large cardinals then one can show that all sets in the projective hierarchy are Lebesgue measurable. The large cardinals in question are called "Woodin cardinals." Shelah and Woodin's *truth of the matter result* is:

> (SWTM) If infinitely many Woodin cardinals exist, then all projective sets are Lebesgue measurable.

We have now come full circle trying to come to terms with Vitali's impossibility result, (VI). Whereas in ZFC, one cannot even show that sets belonging to the second level of the projective hierarchy are Lebesgue measurable, in the presence of large cardinals one can show that sets belonging to the entire projective hierarchy are Lebesgue measurable. And this result is no mere consistency result, it is a truth of the matter result, concerning the universe of sets and the "real" sets of reals.

This last result is interesting in its own right, describing modern developments following Vitali's impossibility result, (VI). But are the large cardinals involved in (SWTM) also relevant to the present essay's theme, viz., are they connected to the original measure problem for sets of reals? The answer is: Yes. Woodin cardinals are a generalization of measurable cardinals. The exact definition is rather elaborate, but the essence of a Woodin cardinal is still the concept of measure. This time it is not a measure measuring the subsets of the Woodin cardinal itself, but rather a whole system of measures, measuring subsets of smaller cardinals. Rather than having one measure on one cardinal, we now have many cardinals, each with many measures on it, where all these measures, viewed as one system, obey certain technical "coherence" criteria. Woodin cardinals are a *generalization* of measurable cardinals but they are a generalization of *measurable* cardinals. Since the measure on measurable cardinals is, itself, a generalization of Lebesgue measure, the measures associated with the Woodin cardinals are still linked to Lebesgue measure. Once again, the large cardinals may be far away from the sets of reals they affect; their structure, however, is quite similar.[20,21]

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[19] See [15] for Solovay's "truth of the matter result." In the text Solovay's result was presented as "If a measurable cardinal exists then all Σ_2^1 and Π_2^1 sets are Lebesgue measurable." It is possible to weaken the hypothesis slightly (by assuming the existence of a smaller large cardinal). The assumption that an inaccessible cardinal exists is *not* sufficient for Solovay's truth of the matter result. Solovay's model where all projective sets are Lebesgue measurable is presented in [16]. That an inaccessible cardinal was unavoidable for constructing that model was shown by Shelah. See [12] and [10]. Solovay's proof also establishes the necessity of relying on the axiom of choice when constructing non-measurable sets

§4. Epilogue. In Shelah and Woodin's "truth of the matter" theorem we find an overall synthesis: The reunion of the real universe of sets and the original notion of Lebesgue measure, a sort of pre-1905 harmony restored. And with things thus restored, we have come round to our starting point. Conflicting dramatic philosophical presentations notwithstanding, point sets and large sets do not lead a segregated existence in the real universe of sets. Recall that the origin of our tale was the naive question regarding how many point sets are Lebesgue measurable. Catalyzed by the Vitali-Gödel result, the idea

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[20] One way of viewing Shelah and Woodin's proof is the following. Solovay had already provided us with a model of ZFC where all projective sets were Lebesgue measurable. Our problem was that we could not transfer his result to the universe of sets, except for the case where the sets in question were defined in a very simple way, i.e., in a Σ^1_2 (or Π^1_2) way The key to Solovay's success with these sets (but not with sets appearing higher up in the projective hierarchy) was Shoenfield's *absoluteness* result (in [14]; see also [1], p.163) which concerns transitive models of ZFC and Σ^1_2 formulas: Such a formula is true in a transitive model of ZFC if and only if it is true in the real universe. In general, one cannot improve this absoluteness result to more complicated formulas (i.e., to Σ^1_n formulas, where n is greater than 2). However, unpublished results of Woodin show how to construct models with "more absoluteness": The more large cardinals present in the model, the more complicated the formulas which the model is correct about. Woodin's absoluteness results enable us to show that if Solovay's techniques are applied in the presence of many large cardinals, then one can indeed deduce the Lebesgue measurability of all projective sets in the universe from their measurability in Solovay's model.

[21] For a definition of Woodin cardinals see [8]. For Shelah and Woodin's result, see [13] Regarding the hypotheses used by Shelah and Woodin: It is known that one cannot prove all projective sets are Lebesgue measurable if one only assumes the existence of finitely many Woodin cardinals. Results showing the Lebesgue measurability of even more sets, assuming the existence of more large cardinals, are also known. See [21].

of a measure on point sets quickly led to generalized measures and in its wake measurable cardinals, large sets indeed. Soon enough results flowed in the reverse direction—just how many point sets are Lebesgue measurable turned out to depend on the existence of sets as large as the measurable cardinal. By (SWTM), yet larger sets, still bred by measures, are part of the full and on-going tale.[22]

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A GENERAL FRAMEWORK FOR PRIORITY ARGUMENTS

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The degrees of unsolvability were introduced in the ground-breaking papers of Post [20] and Kleene and Post [7] as an attempt to measure the *information content* of sets of natural numbers. Kleene and Post were interested in the relative complexity of decision problems arising naturally in mathematics; in particular, they wished to know when a solution to one decision problem contained the information necessary to solve a second decision problem. As decision problems can be coded by sets of natural numbers, this question is equivalent to: Given a computer with access to an oracle which will answer membership questions about a set A, can a program (allowing questions to the oracle) be written which will correctly compute the answers to all membership questions about a set B? If the answer is yes, then we say that B is *Turing reducible* to A and write $B \leq_T A$. We say that $B \equiv_T A$ if $B \leq_T A$ and $A \leq_T B$. $\equiv_T$ is an equivalence relation, and $\leq_T$ induces a partial ordering on the corresponding equivalence classes; the poset obtained in this way is called the *degrees of unsolvability*, and elements of this poset are called *degrees*.

Post was particularly interested in computability from sets which are partially generated by a computer, namely, those for which the elements of the set can be enumerated by a computer. These sets are called (*recursively*) *enumerable*, as are their degrees. He showed [20] that the enumerable degrees have a least and greatest element, and asked whether there were other enumerable degrees. This problem, which became known as *Post's Problem*, was solved a decade later by Friedberg [5] and Mučnik [19], and their solutions introduced a new technique, the *priority method*, which is the subject of this paper.

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The degrees of unsolvability form an algebraic structure which is induced by the information content of sets. One motivation for studying this structure is to see how closely this algebraic structure recaptures information content. Thus we would like to know if sets with different information content give rise to degrees which look different inside the structure, i.e., is it the case that the degrees have no non-trivial automorphisms? Many people have worked on this problem, and combined results of Shore [24], Slaman and Woodin [25], and Cooper [4], [3], show that there are few, if any, non-trivial automorphisms, and any such automorphism must move degrees to nearby degrees. Other research has pursued answers to similar questions about substructures of the degrees such as the enumerable degrees, and has investigated the relationship of these substructures to arithmetic. Attempts at answering these and other global questions about degree structures have required a careful analysis of the algebraic structure, and one of the key techniques for obtaining structure theorems, especially for the enumerable degrees, has been the priority method. As the complexity of the structural questions increased, more powerful variants of the priority method were developed to try to answer these questions. The framework which we will describe in this paper encompasses the techniques used to prove the individual structure theorems.

The priority method has been applied outside degree theory, e.g. in the study of the lattice of enumerable sets by Soare, Maass, Harrington, and many others; and in effective model theory by Nerode, Remmel, Ash, Knight, and others. There have been several important applications outside computability theory. Among these are Martin's [18] original proof of the Axiom of Borel Determinacy, Solovay's [27] characterization of the degrees of models of true arithmetic, and Slaman's result that ACA_0 is not conservative over RCA_0+ Ramsey's Theorem for Pairs. It is our hope that the framework which we present will provide a deeper understanding of the nature of the priority method; and that this understanding will aid in identifying classes of problems for which the priority method is useful for finding solutions. (We envision this to be similar to the way forcing provides a very useful framework for set theorists.)

The priority method can be viewed as an effective version of the Cantor diagonal argument. One wishes to carry out an effective construction whose result is the satisfaction of an infinite list $\{R_i : i \in N\}$ of requirements. A typical requirement will have the form $(\varphi \rightarrow \psi) \,\&\, (\neg\varphi \rightarrow \theta)$. We call φ the *directing sentence*. If φ is true, then we want to carry out *validated action* ψ, and if φ is false, then we want to carry out *activated action* θ. The need for effectiveness precludes the use of a Cantor diagonal argument, since we may not be able to effectively determine the truth of the directing sentence for a given requirement. The priority method provides schemes for carrying out action based on the truth of effectively generated guesses about the directing sentence. As action for one requirement may conflict with the ultimate truth of the action already carried out for another requirement, there is also a need to organize the construction to resolve such conflicts. The priority method produces ways of assigning priority to attempts at satisfying requirements, and having this priority organize the action carried out for the guesses at the truth of directing sentences.

The simplest priority method is called the *finite injury priority method*, and is the method discovered by Friedberg [5] and Mučnik [19]. In this case, the portion, σ, of the directing sentence determining the type of action to be taken is existential, so corresponds to an enumerable condition. While waiting to discover the truth of σ for a requirement R of high priority, action is taken for lower priority requirements under the assumption that σ is not true. Upon discovery of the truth of σ, action is taken for R which may *injure* the ultimate truth of the directing sentence or action already taken for requirements of lower (but not higher) priority than R, and new attempts at satisfying the lower priority requirements are begun which are compatible with the preservation of the truth of the directing sentence and the action taken for R.

As directing sentences become more complicated, their ultimate truth cannot be discovered in an enumerable fashion. What is required is a decomposition of the sentence into fragments of lower complexity, and a use of the truth of such fragments as a way both to generate action, and to guess at the truth of the full sentence. At the next level of quantifier complexity, the corresponding method has been called *infinite injury*, and was discovered by Sacks [21], Shoenfield [22], and Yates [28] and developed primarily by Sacks. The level after that was initially called *monstrous injury*, and was discovered by Lachlan [14]. Harrington introduced a nicer classification of the levels of complexity of priority arguments in terms of the degree of the oracle needed to determine how each requirement is satisfied; thus for each $n \in \mathbf{N}$, the $\mathbf{0}^{(n)}$*-priority method* is one which requires a $\mathbf{0}^{(n)}$-oracle to unravel the construction in this way. (So a finite injury argument is a $\mathbf{0}'$-argument, an infinite injury argument is a $\mathbf{0}''$-argument, etc.)

The need to use priority arguments of higher levels of complexity to answer structural questions about the enumerable degrees becomes evident as the quantifier complexity of the question increases. Such questions naturally occur when they involve iterates of the jump operator. The development of the framework presented in this paper was motivated by our proof of the decidability of the existential theory of the enumerable degrees in the language of least and greatest element and predicates $\leq_n$ for $n \in \mathbf{N}$, where $\mathbf{a} \leq_n \mathbf{b}$ iff $\mathbf{a}^{(n)} \leq \mathbf{b}^{(n)}$. The decision procedure required us to show that a sentence of this language is true of the enumerable degrees iff it is consistent with the poset axioms, and the strictly increasing and order-preserving properties of the jump operator. The proof of the latter result uses the $\mathbf{0}^{(n)}$-priority method for all n.

Attempts at finding frameworks for the priority method, or fragments thereof, were undertaken early in the development of the method. Each framework was viewed as a way to capture the common combinatorics of a given class of priority arguments, and so avoid redundancy in proofs using such arguments. Sacks [21] provided such a framework for finite injury, and Lachlan made some early attempts for infinite injury as well, taking both a game-theoretic approach [12], and an effective Baire category approach [13]. Yates [29] developed an approach for combining infinite injury arguments with effective perfect closed set forcing in the setting of the degrees below $\mathbf{0}'$, using Banach-Mazur games to model the constructions. Subsequent frameworks for priority arguments of restricted complexity have been developed by Shoenfield [23] using a tree of strategies, and Kontostathis [10], [11], and [9] using an effective Baire category approach. Harrington was the first to find a general way to approach priority arguments at all arithmetical levels, and accomplished this by combining the tree of strategies approach with the use of the Kleene Fixed-Point Theorem. Ash [1], [2], and Knight [8] have developed various frameworks which apply to restricted classes of problems in effective model theory using trees of enumerations, but cover all hyperarithmetic levels of complexity of the priority argument. Another approach, using trees of trees, has been introduced by Groszek and Slaman [6]. The latter two approaches have influenced the development of our framework, which is substantially developed in [15], and will be fully developed in [16]. This framework takes an inductive approach and uses a separate tree of strategies for each level of the induction. While it has been developed only for finite levels of priority through $\mathbf{0}^{(\omega)}$, it seems amenable to extension to hyperarithmetic levels.

The framework which we have developed [15], [16] is built on the *tree of strategies* approach to priority arguments which was introduced by Lachlan [14], developed by Harrington, and popularized by Soare [26]. However, we use a sequence of trees of strategies rather than a single tree. We begin by assigning requirements of high quantifier complexity to a tree T^n of level n for some n. We then require a level-by-level decomposition of the requirement into fragments which are assigned to nodes of lower level trees. This procedure ends when we reach T^0, the level at which an effective construction takes place. It must be shown that the satisfaction of the fragments of the requirements which trace their heredity to the true paths through all trees will ensure the satisfaction of the high-level requirement assigned to a node of T^n.

$$\begin{tikzcd}
\langle P_0, \leq_0\rangle \arrow[r, "f_1"] \arrow[d, "\equiv"'] & \langle P_1, \leq_1\rangle \arrow[r, "f_2"] \arrow[d, "\equiv"'] & \cdots\cdots \arrow[r, "f_3"] & \langle P_m, \leq_m\rangle \arrow[d, "\equiv"'] \\
\langle G_0, \leq\rangle \arrow[r, "'"] \arrow[d, "\subseteq"'] & \langle G_1, \leq\rangle \arrow[r, "'"] \arrow[d, "\subseteq"'] & \cdots\cdots \arrow[r, "'"] & \langle G_m, \leq\rangle \arrow[d, "\subseteq"'] \\
\mathcal{R} \arrow[r, "'"] & \mathcal{R}(\mathbf{0}') \arrow[r, "'"] & \cdots\cdots \arrow[r, "'"] & \mathcal{R}(\mathbf{0}^{(m)})
\end{tikzcd}$$

FIGURE 1

The paper is organized as follows. In Section 1, we illustrate the basic features of the framework using the example of the above-mentioned decidability result [15]. In Section 2, we indicate how these ideas can be used to formulate a general framework and discuss the ways in which we expect the framework to be useful for proving additional results.

§1. **An example.** We illustrate the major ideas of our framework using the example of the above-mentioned decidability result by the authors [15]. We begin with some definitions. Let $\mathcal{R} = \langle \mathbf{R}, \leq, 0, 0'\rangle$ be the poset of enumerable degrees with least element $\mathbf{0}$ and greatest element $\mathbf{0}'$. For each $m \in \mathbf{N}$ and $\mathbf{a}, \mathbf{b} \in \mathbf{R}$, we define

$$\mathbf{a} \leq_m \mathbf{b} \Leftrightarrow \mathbf{a}^{(m)} \leq \mathbf{b}^{(m)}.$$

A *jump poset* is a 5-tuple $\langle P, \leq, P', \leq', f\rangle$, such that $\langle P, \leq\rangle$ and $\langle P', \leq'\rangle$ are posets of cardinality ≥ 2 with least and greatest elements, and f is an order-preserving map from P onto P'. An *m-jump poset* is a structure

$$\langle P_0, \leq_0, P_1, \leq_1, f_1, \dots, P_m, \leq_m, f_m\rangle$$

such that for each $k < m$, $\langle P_k, \leq_k, P_{k+1}, \leq_{k+1}, f_{k+1}\rangle$ is a jump poset. The following structure theorem is proved in [15]. Since this theorem implies that any existential statement not excluded by the trivial properties of Turing reducibility and Turing jump can be realized, it follows easily that the existential theory of the enumerable degrees in the language of least and greatest element and the predicates $\leq_n$ is decidable.

THEOREM. *Fix $n \in \mathbf{N}$, and let $\langle P_0, \leq_0, P_1, \leq_1, f_1, \dots, P_m, \leq_m, f_m\rangle$ be a finite m-jump poset such that P_0 has least element* 0 *and greatest element* 1. *Then there is a finite set $\mathbf{G}_0$ of enumerable degrees, and there are finite* *sets* $\mathbf{G}_k = \{\mathbf{d} : \exists \mathbf{a} \in \mathbf{G}_0(\mathbf{a}^{(k)} = \mathbf{d})\}$ *for each* $k \in [1, m]$ *such that the diagram in Figure* 1 *commutes. Furthermore, the embedding maps* $0 \in P_0$ *to* $\mathbf{0}$ *and* $1 \in P_0$ *to* $\mathbf{0}'$. (In fact, the proof of this theorem can easily be extended to all countable $< \omega$-jump posets.)

The proof of the theorem uses a $\mathbf{0}^{(n)}$-priority argument to show that a certain set of requirements can be satisfied. The embedding maps $c \in P_0$ to the degree of the enumerable set A_c (where we set $A_0 = \emptyset$ and $A_1 = \emptyset'$ for the least and greatest element 0 and 1 of P_0). Inductively define the function g_k by: g_0 is the identity, and for $k > 0$, $g_k = f_k \circ g_{k-1}$. There are three types of requirements to be satisfied.

Incomparability requirements ensure that if $g_k(c) \not\leq g_k(b)$ then $A_c^{(k)} \not\leq_T A_b^{(k)}$. Using an iterated version of Shoenfield's Limit Lemma, we will build a functional Δ computable in A_c whose k-fold iterated limit does not equal the k-fold iterated limit of any functional Φ_e computable in A_b. To make this precise, we must ensure

$$(1) \qquad \Delta(A_c) \text{ is total } \& \ \forall x(\lim_{u_1} \ldots \lim_{u_k} \Delta(A_c; u_1, \ldots, u_k, x)\downarrow), \text{ and}$$

$$(2) \quad \exists x(\lim_{u_1} \ldots \lim_{u_k} (\Delta(A_c; u_1, \ldots, u_k, x) \neq \lim_{v_1} \ldots \lim_{v_k} \Phi_e(A_b; v_1, \ldots, v_k, x))$$

for all e. (1) will follow from (2) and properties of the construction; so we will not have explicit strategies for it on our trees. To satisfy (2), we fix a distinct x whenever this requirement is assigned to a node $\alpha^k \in T^k$.

The basic module for satisfying (2) (for $k = 0$) is just the Friedberg-Mučnik argument, which we recast as follows in order to allow later generalizations: We first fix a diagonalization witness x. The directing sentence is

$$(3) \quad \exists s \forall t \geq s(\Phi_e(A_b; x)[s]\downarrow = 0 \ \& \ A_b \upharpoonright u(A_b; e, x)[s] = A_{b,t} \upharpoonright (u(A_b; e, x)[s])).$$

(Formally, this is a Σ_2-sentence; however, the inner universal quantifier over stages can be ignored by preserving the set A_b once a computation has been found, so we will treat (3) as a Σ_1-sentence.)

We now define $\Delta(A_c; x) = 0$ with some big use $\delta(x)$. As long as the directing sentence appears false (i.e., as long as no witness s has been found for (3)), we are done. Once (3) appears true, we reset $\Delta(A_c; x) = 1$ (after enumerating $\delta(x)$ into A_c) and try to preserve the truth of the directing sentence by restraining A_b.

The basic module for $k = 1$ is now essentially an ω-sequence of basic modules for $k = 0$: The directing sentence will be

$$(4) \qquad \exists^\infty v \exists s \forall t \geq s(\Phi_e(A_b; v, x)[t]\downarrow = 0),$$

which is equivalent to

$$(5)\qquad \forall v \exists v' \geq v \exists s \forall t \geq s (\Phi_e(A_b; v', x)[t]\downarrow = 0).$$

(By the same remark as for (3), we can treat this as a Π_2-sentence.) We can now split up (5) into directing subsentences by bounding the outermost quantifier on v by some v_0, say. A substrategy working with this directing subsentence will perform subaction by first defining $\Delta(A_c; u, x) = 0$ for all $u \leq$ some u_0, and later, when the directing subsentence appears true, if ever, resetting $\Delta(A_c; u, x) = 1$ for these u. There are now two possibilities for satisfying this requirement. If (5) really holds then every substrategy will eventually reset $\Delta(A_c; u, x) = 1$ for its values u; otherwise, there will be some last substrategy (working with a sufficiently large v_0) that never finds a pair of witnesses (v', s) for its directing subsentence, and this substrategy will define $\Delta(A_c; u, x) = 0$ for almost all u.

The cases $k > 1$ are now treated similarly, using an inductive argument.

The second type of requirement which must be satisfied ensures comparability of kth jumps (for $k > 0$) if the smaller of the kth jumps is not to be the kth jump of $\mathbf{0}'$, i.e., $\mathbf{0}^{(k+1)}$. (We do not consider the case $k = 0$ here since it can be satisfied by direct coding. We treat the kth jump of $\mathbf{0}'$ separately, in the third type of requirement below, since we cannot restrain the set $\emptyset'$, and so the complexity of the directing sentence has to be computed differently.) The comparability requirements thus ensure that if $g_k(b) \leq g_k(c)$, then $A_b^{(k)} \leq_T A_c^{(k)}$. Again using an iterated version of Shoenfield's Limit Lemma, we will build a functional Δ computable in A_c whose k-fold iterated limit equals $A_b^{(k)}$. Thus we must ensure (1) and, for all e,

$$(6)\qquad \lim_{u_1} \ldots \lim_{u_k} \Delta(A_c; u_1, \ldots, u_k, e) = \begin{cases} A_b^{(k)}(e) & \text{if } k \text{ is odd,} \\ \overline{A_b^{(k)}(e)} & \text{if } k \text{ is even.} \end{cases}$$

The basic module for $k = 1$ has directing sentence $e \in A_b'$, i.e.,

$$(7)\qquad \exists s \forall t \geq s (\Phi_e(A_b; e)[s]\downarrow),$$

which we may (again by preservation) treat as a Σ_1-sentence. We now define $\Delta(A_c; u, e) = 0$ for larger and larger u until, if ever, (7) appears true, at which time we start setting $\Delta(A_c; u, e) = 1$ for all subsequent u. The crucial point is that if (7) later becomes false (due to injury) then, almost always, the number entering A_b will also enter A_c and thus allow us to reset $\Delta(A_c; u, e)$ from 1 back to 0. (We call this feature *automatic correction.*)

The cases $k > 1$ are again handled by induction as for the incomparability requirements.

The third type of requirement which must be satisfied preserves comparability of kth jumps if the smaller of the kth jumps is the kth jump of $\mathbf{0}'$, i.e., $\mathbf{0}^{(k+1)}$. These highness requirements ensure that if $g_k(1) \leq g_k(c)$, then $\emptyset^{(k+1)} \leq_T A_c^{(k)}$. As for the second type of requirement, we thus must satisfy (1) and, for all e,

$$\lim_{u_1} \dots \lim_{u_k} \Delta(A_c; u_1, \dots, u_k, e) = \begin{cases} \emptyset^{(k+1)}(e) & \text{if } k \text{ is even,} \\ \overline{\emptyset^{(k+1)}}(e) & \text{if } k \text{ is odd.} \end{cases} \tag{8}$$

The basic module for satisfying this requirement is the same for incomparability requirements, except that we now do not have the ability to preserve computations relative to $A_b = \emptyset'$; so we lose the ability to preserve computations, and (7), for example, has to be treated as a Σ_2-sentence.

We define the *dimension* of a requirement as the quantifier-complexity of its directing sentence (modulo preservability as mentioned above). So incomparability and highness requirements have dimension $k+1$ while comparability requirements have dimension k.

Our construction takes place on a finite number of *trees of strategies*, which we define by setting

$$T^0 = \{0, \infty\}^{<\omega}, \text{ and} \tag{9}$$

$$T^{k+1} = \{\sigma \in (T^k)^{<\omega} \mid \forall i, j < \mathrm{lh}(\sigma)(i < j \rightarrow \sigma(i) \subset \sigma(j))\}. \tag{10}$$

The intuition is the following: On T_0, a node can have outcome 0 (denoting that the directing subsentence was found by the node to be false) and ∞ (denoting that it was found to be true). On T^{k+1}, the outcome $\xi^k \in T^k$ of a node $\sigma^{k+1} \in T^{k+1}$ denotes that $\sigma^k = (\xi^k)^-$ is a substrategy of the strategy σ^{k+1} and that the outcome of σ^k also gives the "final" outcome of σ^{k+1}, namely, that either σ^k has found the witness to the directing sentence of σ^{k+1}, or that σ^k is a node of minimal length working for σ^{k+1} and that neither σ^k nor any of its extensions working for σ^{k+1} find a witness for σ^{k+1}. (We denote by $\mathrm{up}(\sigma^k) = \sigma^{k+1}$ the fact that σ^k is a substrategy of σ^{k+1}.)

We next define a function $\lambda : T^k \rightarrow T^{k+1}$, denoting that a node $\eta^k \in T^k$ guesses that the true path (coding the correct outcomes) through T^{k+1} extends the node $\lambda(\eta^k)$. The node $\lambda(\eta^k)$ is defined by induction as follows: Given that $\sigma^{k+1} \subseteq \lambda(\eta^k)$ (where $|\sigma^{k+1}| = n$, say), we specify

$$(\lambda(\eta^k))(n) = \begin{cases} \xi^k & \text{if } (\xi^k)^- \text{ finds a witness for } \sigma^{k+1} \\ & \qquad \text{and } \xi^k \subseteq \eta^k \text{ codes this outcome,} \\ \mu^k & \text{if } \mu^k \text{ is the least } \sigma^{k+1}\text{-substrategy} \subset \eta^k, \\ & \qquad \text{but none finds a witness,} \\ \uparrow & \text{if there is no substrategy} \subset \eta^k \\ & \qquad \text{working for } \sigma^{k+1}. \end{cases} \tag{11}$$

Once the third clause of (11) applies, the definition of the node $\lambda(\eta^k)$ is complete. The λ-function naturally extends to a function $\lambda : T^k \rightarrow T^{k+1}$

on infinite paths. (We will denote by Λ^k the *true path* of the construction on T^k, which will compute the outcome of each node along it correctly according to the truth of the directing sentences. The true path Λ^0 of T^0 will be computable, while the paths $\Lambda^{k+1} = \lambda(\Lambda^k)$ will be computable in $\emptyset^{(k+1)}$.)

We will always maintain the condition

$$\text{(12)} \qquad \operatorname{up}(\eta^k) \subseteq \lambda(\eta^k),$$

i.e., any substrategy on T^k works for a node on T^{k+1} which it believes to lie along the true path. This is part of our condition of *consistency* on the way we determine $\operatorname{up}(\eta^k)$. We omit the formal definition of this notion; besides (12), consistency requires essentially that we have not yet found a witness for $\operatorname{up}(\eta^k)$ along η^k; and that $\operatorname{up}(\eta^k)$ not be *restrained by a* $\lambda(\eta^k)$*-link* $[\mu^{k+1}, \tau^{k+1}]$, i.e., we do not want

$$\mu^{k+1} \subseteq \operatorname{up}(\eta^k) \subset \tau^{k+1} \subset \lambda(\eta^k),$$

where τ^{k+1} has a different guess about the outcome of a node on a higher tree than μ^{k+1}. (Links are a very general concept, unifying the notions of initialization, links in the sense of Soare [26, Ch. XIV], and several others into one. It is possible to show that the absence of a link around $\operatorname{up}(\eta^k)$ actually implies (12).)

The definition of our functionals Δ is now determined by a notion of *control,* deciding which strategy is allowed to define a functional at which arguments. The definition is rather straightforward for the incomparability requirements, since each strategy at the top level will work with a distinct diagonalization witness. For the comparability and highness requirements, however, we have, even at the top level, many incomparable strategies, all competing to define the same functional according to their (possibly conflicting) guesses about the truth of their directing sentences (as well as the effect of other strategies). The notion of control alone is not able to handle these conflicts on T^{k-1}, i.e., one level below the dimension of the requirement. Instead, we introduce the notion of *implication chain* to resolve these conflicts.

We illustrate the implication chain machinery in the following example. Suppose that we have $\sigma^{k-1} \subset \hat{\sigma}^{k-1} \subset \Lambda^{k-1} \in [T^{k-1}]$ such that σ^{k-1} and $\hat{\sigma}^{k-1}$ are trying to define the same functional on possibly the same argument. A problem will occur if there is no way to determine which of σ^{k-1} and $\hat{\sigma}^{k-1}$ is derived from nodes along the true paths for the construction (although $\hat{\sigma}^{k-1}$ seems to have this property based on the current approximation to the true paths), and σ^{k-1} and $\hat{\sigma}^{k-1}$ wish to make definitions with different values. We then delay this definition and investigate the process of returning σ^{k-1} and its antiderivatives to a later approximation to the true paths. There will be a minimum set of nodes of the top tree whose outcomes will have to be (forcibly) switched by switching the outcomes of some of their derivatives, regardless of the truth of their directing sentences. We determine whether such switches would injure the oracle set for any axioms we would currently declare, if we were to declare axioms. If the answer is yes (the *non-amenable* case), then it is safe to declare axioms; if σ^{k-1} returns to the true path, these axioms will not reflect computation from the final oracle, so σ^{k-1} will have the ability to declare new axioms. Otherwise (the *amenable* case), we start building an *implication chain* to resolve the conflicts. We first go through the process of returning σ^{k-1} to the true path without any functional definitions, and determining an outcome for the new derivative of either σ^{k-1} or $\mathrm{up}(\sigma^{k-1})$ obtained in this manner. If the outcome differs from that for σ^{k-1}, then we will have resolved our conflict on the new path, and can proceed to define axioms. Otherwise, we will have replicated our starting situation on T^{k-2}, and can now repeat the procedure described above. We continue until we either resolve the conflict by changing the outcome of one of the nodes, or reach T^0. The process is arranged to preserve implications between directing sentences in such a way that the outcomes are contradictory (i.e., one node claims to have found a witness below a certain bound while the other claims there is none below a higher bound). Once we reach T^0, this contradiction cannot occur (as the outcomes are now computable and we will have an implication between directing sentences), so the conflict can be resolved by changing the outcome of one of the nodes. Thus the construction must also validate $\hat{\sigma}^0$, resolving the conflict.

In using the implication chain machinery, there is a need to preserve implications between directing sentences, level-by-level, while preventing uncorrectable axioms from being declared. The machinery which ensures that this can be achieved is complex and delicate.

As can be seen from the above very rough description, there will be many nodes on our trees, even along the true path, which are not allowed to act, or determine their outcomes, according to the truth of their directing sentences. In the above, this can happen in two ways: The node may be restrained by a link, or it may be involved in bringing another node back on the truth path as part of the implication chain machinery. We need to verify that enough *critical nodes* remain (i.e., nodes which are allowed to act according to the truth of their directing sentence) so that we can argue that the substrategies of a strategy work together to ensure their strategy's success.

§2. The general framework and the framework theorem. In Section §1, we illustrated some basic features of the framework, and the properties which need to be verified to ensure that all requirements in a given construction are satisfied. The example given in the present paper indicates that such a verification can still be very complicated—in fact, it is much lengthier than the development of the necessary lemmas about the whole framework. However, the notions used for the particular example, namely, *link*, *consistency*, *control*, *implication chains*, and *critical nodes*, seem to be applicable to many constructions, although some modifications to their exact definitions may be necessary. Thus we envision a modular approach to priority arguments. In addition to the properties which are universally required, one can axiomatize notions such as control and implication chains, and the development of their properties will follow from this axiomatization. For other proofs in which these notions are useful, we will merely have to verify that these axioms hold, and then will have available the properties spelled out by the lemmas which follow from the axioms. In this way, we can avoid having to prove the same facts in different situations. Instead, we can appeal to a *Framework Theorem*, which will state that if certain properties of the above notions are satisfied then so are all the requirements. We will show how to use the Framework Theorem to prove a number of structure theorems in [16].

Another feature of the *iterated trees of strategies* approach is that the description of the basic module used to satisfy a requirement is a finite tree rather than a flow chart. This makes the analysis easier, and certainly simpler to describe. The loops which enter into the flow chart description are absorbed into the properties we require of our decompositions of directing sentences and action as we pass from tree to tree.

The usefulness of the iterated trees of strategies increases with the level of complexity of the priority argument. At the lower levels, the combinatorial facts proved within this framework are relatively simple, and their proofs are shorter than the characterization of the decomposition from tree to tree. As the combinatorial interactions become more complicated, the decomposition becomes a smaller part of the proof. The framework is especially useful when there are requirements of the same nature at several levels which can be handled uniformly, or the level of the argument is too high to be easily visualized in one step. (An advantage of the framework in the latter case is that its presentation is as close as possible to the standard presentation, if an inductive component is to be introduced.) In [17], we used the framework to prove a new theorem, the existence of a minimal pair of enumerable degrees whose jumps form a minimal pair over $\mathbf{0}'$. This theorem can probably be generalized to carry the simultaneity of the minimal pairs through all finite levels, and perhaps to decide the existential theory of $\mathcal{R}$ when a relation symbol for meet is added to the language of Section 1. While the two-level proof could have been carried out without the framework, the framework would be a natural way to obtain a proof for infinitely many levels. Many results of this nature should be accessible through the use of iterated trees of strategies.

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THE BULLETIN OF SYMBOLIC LOGIC
Volume 1, Number 2, June 1995

IN MEMORIAM: CHRISTOPHER JOHN ASH

Christopher John Ash died in February, 1995, at the age of 50. Ash was based at Monash University in Australia, throughout his career, having gone there from Oxford with John Crossley, his thesis advisor. Ash held visiting positions in the United States, at the University of Connecticut, the University of Wisconsin, and Notre Dame. He was a Fellow of the Australian Academy of Science.

Within logic, Ash worked on a variety of problems, but over the past 15 years, he focused mainly on a specific program in recursive structure theory. This program, which grew out of joint work with Nerode, involved finding syntactical conditions which correspond to bounds on recursive complexity that persist when a given structure is copied. For a recursive structure $\mathfrak{A}$ and a recursive relation R on $\mathfrak{A}$, Ash and Nerode gave conditions guaranteeing that in all recursive copies of $\mathfrak{A}$, the image of R is r.e. Ash asked Ewan Barker (then working on a master's degree) to generalize this, replacing r.e., by Σ^0_α, for an arbitrary recursive ordinal α. Producing a recursive copy of $\mathfrak{A}$ in which the image of R is not r.e., involved a finite injury priority construction. Producing a recursive copy in which the image of R is not Σ^0_α required a much more complicated construction. Ash formulated in an abstract way the object of a nested priority construction, and he proved a general "metatheorem," listing conditions which guarantee the success of the construction. Barker used Ash's metatheorem. There have been many other applications, and there are variants of the original metatheorem with further applications.

In addition to his work in logic, Ash produced several deep results in algebra. At Monash, colleagues interested in semigroups discussed their work with him, and stated for him some of the important open problems in their field. Ash was able to solve some of these problems. In particular, he proved a long-standing conjecture of John Rhodes.

Chris and I collaborated on 10 papers already published, with others in progress. He was full of ideas, an excellent correspondent, and fun to talk with. I enjoyed every minute I had with him.

JULIA F. KNIGHT

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THE BULLETIN OF SYMBOLIC LOGIC
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1994 EUROPEAN SUMMER MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

LOGIC COLLOQUIUM '94

Clermont-Ferrand, France, July 21–30, 1994

The 1994 European Summer Meeting of the Association was held at the Université d'Auvergne in Clermont-Ferrand, France, July 21–30, 1994.

The following main lectures and tutorials were given at the meeting.

Sanjeev Arora (Berkeley, USA), *Probabilistic checking of proofs and hardness of approximation problems.*

Matthias Baaz (Wien-Austria), *Generalization of proofs and term complexity.*

Lenore Blum (Berkeley-USA), *The theory of computation and complexity over the reals.*

Patrick Dehornoy (Caen-France), *The structure group of an algebraic identity.*

Solomon Feferman (Stanford-USA), *How is it that finitary proof theory became infinitary.*

Moti Gitik (Tel-Aviv-Israel), *The singular cardinal hypothesis: the current stage.*

Erich Grädel and Yuri Gurevich (Aachen-Germany), *Finite model theory and beyond.*

Serge Grigorieff (Paris-France), *Decidability and complexity of logical theories.*

David Harel (Rehovot-Israel), *Towards a theory of recursive structures.*

Richard Kaye (Birmingham-UK), *Models of arithmetic: three perpectives.*

Alexander Kechris (Pasadena-USA), *The descriptive set theory of group actions and equivalence relations.*

Daniel Leivant (Bloomington-USA), *Transfer principles between logic-based approaches to computational complexity.*

Maciej Liśkiewicz and Rüdiger Reischuk (Wrocław-Poland and Darmstadt-Germany), *Lower bounds, hierarchies and closure properties for sublogarithmic space classes.*

H. Dugald Macpherson (London-UK), *Variations of o-minimality.*

Larisa Maksimova (Novosibirsk-Russia), *On some aspects of the history of modal logics.*

Bruno Poizat (Lyon-France), *Géométrie des modèles et théorie algébrique..*

Jean Pierre Ressayre (Paris-France), *Finitistic applications of transfinite power series.*

Simona Ronchi della Rocca and Luca Roversi (Torino-Italy and Pisa-Italy), *Lambda calculus and intuitionistic linear logic.*

Jerzy Tiuryn (Warsaw-Poland), *Simply typed lambda calculus with subtyping – the problem of type reconstruction.*

Johan Van Benthem (Amsterdam-Netherlands), *Modal logics and process theories.*

Moshe Y Vardi (Houston-USA), *Infinitary logic for computer science.*

Boban Velickovic (North-York-Canada), *Forcing and* $0^{\#}$.

Victor Vianu (INRIA-Rocquencourt-France), *Relational complexity.*

Frank Wagner (Oxford-UK), *On the structure of stable groups.*

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The abstracts of the main lectures, tutorials, and contributed talks given by members of the Association for Symbolic Logic follow.

For the Program Committee
DENIS RICHARD

Main lectures and tutorials

► SANJEEV ARORA, *Probabilistic checking of proofs and hardness of approximation problems.*

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The classical theory of NP-completeness, pioneered by Cook, Karp and Levin in the 70s, explains the apparent computational intractability of optimization problems by proving them NP-hard. But its techniques were unable to explain why for many optimization problems, even finding good approximation algorithms is difficult (or at least, why significant amounts of research failed to find such algorithms)

Recent work has yielded a simple explanation: many NP-hard optimization problems are also NP-hard to approximate. Among the problems for which this has been proved are Independent Set, Chromatic Number, Set Cover, Vertex Cover, Max-Cut, Decoding to the nearest codeword, Learning in the presence of Errors, and many others. At the heart of these new results is a new type of NP-completeness reduction which acts globally (as opposed to the classical reductions, which used local transformations involving gadgets etc.). A more interesting way to view this reduction is as a new probabilistic definition of NP. NP is exactly the class of languages for which membership proofs (i.e., certificates) can be checked in polynomial time by using only $O(\log n)$ random bits and examining only $O(1)$ bits in them.

Of further interest to mathematicians will be the result that every mathematical proof can be rewritten in a form in which its correctness can be checked (probabilistically) by examining a constant number of bits it it.

The talk will be a survey of this area. It will be self-contained.

► MATTHIAS BAAZ, *Generalization of proofs and term complexity.*

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We investigate the concept of *short proofs*, i.e., proofs of theorems in a fixed number of steps containing complex terms, in various circumstances, namely relative to different axiom systems and to different formulations of the underlying (classical) logical system. Results about generalizations provide a way to distinguish between different proof systems for one and the same theory, i.e., to distinguish between formulations which are indistinguishable by model theoretic properties.

Usually, generalizations of *theorems* are considered, e.g., *Kreisel's Conjecture* If there is a k s.t. $A(s^n(0))$ is provable in k steps in Peano Arithmetic, then $(\forall x)A(x)$ is provable The important aspect, however, is generalization of *proofs* and the circumstances and conditions under which it is possible. Generalization of theorems, then, can be viewed as a borderline case of generalization of theorems, where every sound proof transformation is allowed.

The strongest kind of generalization is the calculation of a *term basis* (cf. [1])

DEFINITION 1. A finite set of n-tuples of terms $B = \{\langle t_1^i, \ldots, t_n^i \rangle\}_{i=1}^{m}$ is called a *term basis* for $A(x_1, \dots, x_n)$ and a class of proofs Π in a theory T if

(1) $T \vdash A(t^i_1, \dots, t^i_n)$ for $1 \leq i \leq m$,
(2) if $T \overset{\Pi}{\vdash} A(s_1, \dots, s_n)$ (s_j variable free) then there is a substitution σ s.t. for some i ($1 \leq i \leq m$) it holds that $s_j = t^i_j\sigma$ for all j, $1 \leq j \leq n$.

If a formal system has term bases relative to proofs of bounded length, then Kreisel's Conjecture follows from number-theoretic axioms

Term basis exists, e.g., for Π the class of cut-free proofs of bounded length in a finitely axiomatized theory T. It also exists for the class of all proofs of bounded length in T, if either cut-elimination, or replacement of quantifier inferences by inferences acting on *blocks* of like quantifiers in proof descriptions, is an allowed transformation.

The computation of term bases usually proceeds by obtaining a term-minimal proof, e.g., by defining a unification problem over the skeleton of a given proof. This method works for cut-free **LK**, and we show that it also works for general $\mathbf{LK_B}$, a variant of **LK** where quantifier inferences introduce sequences of like quantifiers in one step (cf. [2]); it does not work for general **LK**. If we start from a theory T plus the schema S, and use $\mathbf{LK_B}$ instead of **LK**, the construction of the term-minimal proof transforms single quantifiers in the instances of S into blocks of quantifiers. If these instances are themselves derivable from $T + S$, then term bases exist. This criterion is our main result.

We point out that this criterion holds, e.g., for the schemata of order induction and elementary Dedekind cuts, but of course not for the schema of identity (A result of Yukami shows that there can be no term basis for the schema of identitiy w.r.t. to proofs of bounded length). This also indicates that the character of these results is in no way model theoretic, and hence "genuinely" proof theoretic. Here, the schema of identity is "difficult" while order induction is "easy."

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[2] M. BAAZ and R. ZACH, *Algorithmic structuring of cut-free proofs*, ***Computer Science Logic. Selected papers from CSL '92*** (E. Börger, editor), LNCS, Springer, Berlin, 1993, pp. 29–42.

► LENORE BLUM, *A theory of computation and complexity over the reals.*

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The theory of computation originated in the 1930's with work of logicians interested in resolving questions of decidability and undecidability. This theory was refined and further developed in the 1960's by computer scientists interested in understanding the intrinsic difficulty—or complexity—of solving (solvable) problems.

The classical theory deals with discrete problems, for example over the integers, about graphs, etc. On the other hand, most computational problems that arise in numerical analysis and scientific computation, in optimization theory and more recently in robotics and computational geometry, have as natural domains the reals, or complex numbers. Adaptations of the classical theory to study these problems are problematic.

In 1989, Blum, Shub and Smale introduced a theory of computation and complexity (over a ring) that integrates key ideas from the classical theory in a setting more amenable to problems defined over continuous domains. The theory yields results in the continuous setting analogous to the pivotal classical results over the integers of Gödel (on undecidability) and Cook (on NP-completeness), yet reflecting the special mathematical character of the underlying space. For example, we have that:

(1) the Mandelbrot set as well as most Julia sets are undecidable over the reals, and
(2) the Hilbert Nullstellensatz is NP-complete over the complex numbers.

I will discuss these results as well as recent developments and directions in the field

► PATRICK DEHORNOY, *The structure group of an algebraic identity*

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We propose to associate with every algebraic identity an operator monoid and, in good cases, a derived group, which capture the geometry of that particular identity and may be used to describe the free corresponding objects.

We first concentrate on the case of left selfdistributivity for which the approach was originally created The decidability of the word problem for free left distributive structures has been proved in 1989 by Richard Laver using the hypothesis that there exists an n-huge cardinal for every integer n. It has been subsequently established by the author without any set-theoretical assumption The problem is as follows. Let us call a set equipped with a left (self)distributive law an LD-system. Purely algebraic developments had shown that a sufficient condition for the above word problem to be decidable is the existence of an LD-system satisfying an additional algebraic condition known as irreflexivity, or acyclicity. Then large cardinal hypotheses provide elementary embeddings of (fragments of) models of ZFC from which one easily defines an LD-system which is acyclic, and this is the way large cardinals enter the picture. In order to drop the logical hypothesis, it suffices to construct another example of acyclic LD-system. Unfortunately all standard examples of LD-systems are very far from acyclic, and we are left with the question of constructing new distributive operations, and, because an acyclic LD-system must include free LD-systems with one generator, with the more particular question of defining 'concrete' realizations of the free LD-systems.

Since there are no relevant examples to start with, we try to construct a 'syntactic' LD-system whose domain is (a quotient of) the family of all identities which are consequences of LD, an approach which is reminiscent of Henkin's completeness proof for first-order logic where one constructs a model for a given coherent set of sentences whose domain is (a quotient of) the family of all (existential) sentences. In the present case we observe that two terms say P, Q are equivalent up to LD if and only if the pair (P, Q) is an instance of some consequence of LD which can be taken in a fixed canonical family, if and only if some convenient operator maps P to Q. It is easy to define a monoid structure on these operators using composition (or on the identities using unification), thus obtaining a 'structure monoid' $\mathcal{M}_{\mathrm{LD}}$ for the distributivity identity LD. Technically it is convenient to guess a presentation for this monoid and then to work with the group $\mathcal{G}_{\mathrm{LD}}$ which admits this presentation

The main idea in order to construct a realization of the free LD-system is to associate with every term P a particular operator $\widetilde{P}$ which *constructs* P from some uniform initial term (this is the specific point where the considered identity is really important). The way the operator $\widetilde{P.Q}$ is constructed from $\widetilde{P}$ and $\widetilde{Q}$ gives the receipt for defining an LD-system on the right cosets of $\mathcal{G}_{\mathrm{LD}}$ associated with a certain subgroup H_0, and the proof that this LD-system is acyclic follows from a long but natural study of the group $\mathcal{G}_{\mathrm{LD}}$ using the particular form of its presentation. And because the group $\mathcal{G}_{\mathrm{LD}}$ happens to be an extension of Artin's braid group B_∞ and the associated kernel includes the subgroup H_0, the above LD-system projects to braids. One shows that this projection is injective, so that finally one has obtained a realization of the free LD-system with one generator inside the braid group B_∞ (from this point on one can subsequently deduce new properties of the braids, like the existence of a linear ordering).

Leaving apart the last points which are specific of the case of left distributivity, we think that the above approach can be uniformly applied to a number of algebraic identities. We shall briefly mention the cases of associativity, where the study is easy and gives a new strong form of the so-called MacLane-Stasheff pentagon property, and the case of left distributivity with idempotence where the method could possibly be used to approach recent conjectures by Larue and Kepka.

► SOLOMON FEFERMAN, *How is it that finitary proof theory became infinitary?*

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Proof theory was the technical area of logic instituted by Hilbert in the 1920s in order to carry out his program to establish, by finitary methods, the consistency of axiomatic systems underlying actual mathematics. Hilbert believed that a successful prosecution of his program for number theory, analysis and, eventually, set theory, was necessary in order to justify all manifestations of the infinite (both implicit and explicit) in mathematics. Whatever its merits, Gödel's incompleteness theorems apparently undermined Hilbert's Program as originally conceived; "apparently", because the exact extent of finitary methods were never spelled out, but only understood informally. Nevertheless, the goal to establish the consistency of number theory and stronger systems continued to be pursued, but increasingly with the infusion of infinitary concepts and methods. The first step in that direction was taken by Gentzen in 1936, who used transfinite induction up to the first epsilon number in order to establish the consistency of Peano Arithmetic.

Beginning in the 1950s, proof theory began to employ infinitary methods more frankly, first with the use of infinitary rules of inference (such as the omega rule) and then with the use of infinitely long formulas. At first, these rules, formulas and the resulting derivations were all countable, but eventually proof theory moved on to make use of prima facie uncountably long rules of inference and/or formulas, thence uncountably long derivations. These countably and uncountably long derivations have been applied to treat various formal systems for parts of analysis and set theory, by translating their finite derivations into corresponding infinite ones of a more purely logical character. By then applying the process of cut-elimination or other forms of normalization, the latter are transformed into detour-free derivations whose associated ordinals give a measure of complexity to the systems thus treated. For example, they can be used to classify their provably recursive functions. Cut-free derivations are easily shown not to end in a contradiction, so one also concludes consistency from such analyses.

In general, the more that such infinitary methods were employed, the farther did proof theory depart from its initial aims and methods, and the closer did it come instead to ongoing developments in recursion theory, particularly as generalized to admissible sets; in both one makes use of analogues of regular cardinals, as well as "large" cardinals (inaccessible, Mahlo, etc.). By reference to some chapters at different points in this transformation of our subject, these lectures will explore how and why these remarkable changes took place. We will also take up the question of finding a new and more appropriate rationale for proof theory, given its (all but) whole-hearted embrace of infinitary methods.

Where does that leave finitary proof theory? If time permits, there will be a brief introduction to its unexpected successes in recent years, disjoint from all of the above.

[1] P. Aczel, H. Simmons, and S. Wainer, ***Proof theory***, Cambridge University Press, 1992.

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[3] W. Buchholz, S. Feferman, W. Pohlers, and W. Sieg, ***Iterated inductive definitions and subsystems of analysis: Recent proof-theoretical studies***, Lecture Notes in Mathematics, no. 897, Springer, 1981.

[4] W. Buchholz and K. Schütte, ***Proof theory of impredicative systems***, Bibliopolis, 1977.

[5] S. Feferman, ***Lectures on proof theory***, Lecture Notes in Mathematics, no. 70, Springer, 1968.

[6] G. Gentzen, ***The collected papers of Gerhard Gentzen***, North Holland, 1969.

[7] J. Y. Girard, ***Proof theory and logical complexity***, Bibliopolis, 1987.

[8] D. Hilbert and P. Bernays, ***Grundlagen der Mathematik***, 2nd ed., vol. 1–2, Springer, 1968.

[9] G. Jager, ***Theories for admissible sets. A unifying approach to proof theory***, Bibliopolis, 1986

[10] G. E. Mints, ***Selected papers in proof theory***, Bibliopolis, 1992.

[11] J. Myhill, A. Kino, and R. E. Vesley (editors), ***Intuitionism and proof theory***, North Holland, 1970.

[12] W. Pohlers, ***Proof theory, an introduction***, Lectures Notes in Mathematics, no. 1407, Springer, 1989.

[13] K. Schütte, ***Proof theory***, 2nd ed., North Holland, 1977

► MOTI GITIK, *The singular cardinal hypothesis: the current stage.*

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We describe the current stage of the problem The particular emphasis will be devoted to recent forcing constructions for blowing up the power of a singular cardinal.

► ERICH GRÄDEL AND YURI GUREVICH, *Metafinite model theory.*

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Although questions involving finite structures have always been of interest to logicians, finite model theory has emerged as a separate research area only in the 1970's. Part of the motivation came from computer science, in particular from databases and complexity theory. Finite structures posed a nontrivial challenge for mathematical logic, in particular for model theory; since most of the important classical results and techniques of mathematical logic (such as, e.g., compactness, completeness, the usual preservation theorems) fail when only finite structures are considered.

Even though we believe that finite model theory has been rather successful, the time has come to re-examine the situation. Motivated again by challenges from computer science we feel that the strict adherence to finiteness is too restrictive and suggest to *extend the approach and methods of finite model theory beyond finite models*

We propose a more general class of structures, which we call *metafinite structures* and a number of logics to reason about them. Typical metafinite structures consist of (i) a primary part, which is a finite structure, (ii) a secondary part, which may be finite or infinite, and (iii) a set of "weight" functions from the first part into the second. However, the infinity that we seek is very modest It should not manifest itself too obtrusively, deviating our attention to phenomena that are pertinent to infinite structures only. Therefore our logics of metafinite structures—appropriate modifications of the usual logics of interest to finite model theory, such as first-order logic, fixed point logics or $L^{\omega}_{\infty\omega}$—access the infinite part only in a limited way. An important feature of these logics is that they contain, besides formulae and terms in the usual sense, a calculus of functions from the primary to the secondary part, which we call *weights*.

We investigate the expressive power of these logics and reconsider issues of finite model theory, like descriptive complexity, asymptotic probabilities, back and forth systems, in this extended framework. In many cases the metafinite framework turns out to be more natural and convenient.

Applications of metafinite model theory arise, for example, in databases, optimization, complexity theory and discrete dynamic systems. In particular, metafinite model theory provides a proper setting for a descriptive complexity theory over the real numbers with respect to the model of Blum, Shub and Smale.

► SERGE GRIGORIEFF, *Decidability and complexity of logical theories.*

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The *first talk* will present the decision problem in a historical perspective and give the most salient results obtained in the last 60 years:

- its emergence from Leibniz to Frege,
- first positive results
  - propositional calculus : Post, 1921;
  - fragments of predicate calculus · Bernays & Schonfinkel, 1928, Ackermann, 1928, Gödel, 1930;
  - real algebra : Tarski, 1931,
  - additive number theory Presburger, 1930,
- the general negative solution
  - predicate calculus and number theory : Church, Turing, 1936, based on Gödel, 1931),
- subsequent positive results
  - boolean algebras : Tarski, 1949,
  - abelian groups . Szmielew, 1949,
  - monadic theory of one successor : Büchi, 1962;
  - case of two successors : Doner, 1965, Rabin, 1969
- the intrinsic complexity results (Meyer, 1972; Fischer & Rabin, 1974).

In the *second talk*, methods will be presented to obtain decidability and upper complexity bounds.

The *last talk* will be devoted to undecidability and lower bounds.

[1] E. Börger, *Decision problems in predicate logic*, ***Logic Colloquium '82***, North-Holland, 1984, pp. 263–301.

[2] K. Compton and C. W. Henson, *A uniform method for proving lower bounds on the computational complexity of logical theories*, ***Annals of Pure and Applied Logic***, vol. 48 (1990), pp. 1–79.

[3] B. Dreben and W. Goldfarb, ***The decision problem : solvable classes of quantificational formulas***, Addison-Wesley, 1979

[4] J. Ferrante and C. Rackhoff, ***The computational complexity of logical theories***, Lecture Notes in Mathematics, no. 718, Springer, 1979.

[5] S. Grigorieff, *Décidabilité et complexité des théories logiques*, ***Logique et informatique: une introduction*** (B. Courcelle, editor), INRIA, 1991, pp. 7–97.

[6] Y. Gurevich, *Monadic theories*, ***Abstract model theory and stronger logics*** (J. Barwise and S. Feferman, editors), Springer, 1984, pp. 479–506.

[7] H. Lewis, ***Unsolvable classes of quantificational formulas***, Addison-Wesley, 1979

[8] M. O. Rabin, *Decidable theories*, ***Handbook of mathematical logic*** (J. Barwise, editor), North-Holland, 1977, pp 595–629.

► DAVID HAREL, *Towards a theory of recursive structures.*

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In computer science, one is interested mainly in finite objects. Insofar as *infinite* objects are of interest, they must be computable, i.e., recursive, thus admitting an effective finite representation. This leads to the notion of a recursive graph, or, more generally, a recursive structure or data base. In this paper we summarize our recent work on recursive structures and data bases, including

(i) the high undecidability of many problems on recursive graphs,

(ii) somewhat surprising ways of deducing results on the classification of NP optimization problems from results on the degree of undecidability of their infinitary analogues, and

(iii) completeness results for query languages on recursive data bases

[1] D. Harel, *Towards a theory of recursive structure*, ***Proceedings of the Symposium on Theoretical Aspects of Computer Science***, Lecture Notes in Computer Science, Springer-Verlag, 1994.

► RICHARD KAYE, *Models of arithmetic: three perspectives.*

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Research in model theory of first-order Peano Arithmetic and its subtheories can be loosely divided into three main areas: 1. "algebraic" models; 2. models of "weak" theories and complexity theory; and 3. "strong" theories and their models. There are some connections between these areas and the dividing lines are not always so well defined, but each area has its own distinctive flavour.

Each of my three talks will be devoted to one of these areas. My intention is primarily to give the listeners a good idea of the type of work that is currently being carried out in each area (and necessarily some relevant older results), and to illustrate the flavour of the subjects by giving at least one sketch proof of an important result in each area. Time will not permit full proofs, nor a comprehensive survey of any of these topics, but I hope to whet the listeners' appetite for further investigation.

The three talks will each be mostly self-contained and will require the minimum of prior knowledge. They consist of

1. "Algebraic" models: Shepherdson's characterization of models of open induction as models consisting of integral parts of elements of their real closure. Wilkie's results. Van den Dries' results and more recent work on normal models of open induction.

2. Models of "weak" theories and complexity theory: The model-theoretic attack on complexity theory; the theories $I\Delta_0$ and $I\Delta_0 + \exp$; Parikh's theorem; Buss' hierarchy of theories, and the Krajíček–Pudlák–Takeuti result connecting this hierarchy with the polynomial-time hierarchy; relativizations of the hierarchy problems.

3. "Strong" theories: the theories $I\Sigma_n$, $B\Sigma_n$, and a rapid overview of "local" results in PA and its fragments; "nonlocal" results, including results by Gaifman and others; recursively saturated models and their structural properties.

► ALEXANDER KECHRIS, *The descriptive set theory of group actions and equivalence relations.*

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Descriptive set theory is the study of definable sets in Polish spaces. Although these include many of the spaces studied in mathematics, there are also spaces of fundamental importance which are obtained instead by starting with a Polish space X and a definable equivalence relation E on X and considering the quotient space X/E.

Examples include the space of orbits of a Borel automorphism or more generally of a Borel action of a Polish group, the space of countable models of a theory modulo elementary equivalence or isomorphism, the space of Turing degrees, etc.

In these talks, I will discuss some recent progress in the study and classification of definable equivalence relations on Polish spaces and their quotient spaces.

▶ DANIEL LEIVANT, *Transfer principles between logic-based approaches to computational complexity.*

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Machine independent approaches to computational complexity establish the importance of certain classes, provide insight into their nature, relate them to issues relevant to programming methodology and to program verification, suggest new separation tools, and offer concepts and methods for generalizing computational complexity to computing over arbitrary structures and to higher type functionals.

Machine independent characterizations fall, by and large, into three major groups: database queries (methods of finite model theory), applicative programs over free algebras (bounded recurrence, tiered recurrence), and proof-theoretic (restricted comprehension, bounded induction, tiered induction).

Towards a unified theory of computational complexity, we describe transfer principles that link various approaches, and in so doing we derive some new results.

▶ MACIEJ LIŚKIEWICZ AND RÜDIGER REISCHUK, *Lower bounds, hierarchies and closure properties, for sublogarithmic space classes.*

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We try to fully characterize the properties and relationships of space classes defined by Turing machines that use less than logarithmic space - may they be deterministic, nondeterministic or alternating. Because of the limited counting ability of such machines most of the standard simulation techniques do not work for sublogarithmic space classes. However, machines with such little space may still be quite powerful. Therefore, it was not obvious how to obtain analogs for inclusion and separation results known for classes above logspace.

We provide several examples of specific languages and show that machines using sublogarithmic space are unable to accept these languages. These lower bounds show that the complexity world below logarithmic space really behave differently. For example, certain closure properties do not hold. The restricted power of these machines makes it possible to prove explicit separations—even for alternating complexity classes—by combinatorial arguments and to obtain a hierarchy of non-relativized complexity classes without any unproven assumption.

Finally, these complexity classes are related to other classes within P, in particular to context-free languages.

► H. DUGALD MACPHERSON, *Variations of o-minimality.*

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One way of defining o-minimality is to say that a structure $\mathcal{M} = (M, <, \dots)$ is *o-minimal* if $(M, <)$ is totally ordered and, for every $\mathcal{N}$ elementarily equivalent to $\mathcal{M}$, every definable subset of N is quantifier-free definable by a formula just involving $<$ (and equality) (in fact, for o-minimality one doesn't need to mention $\mathcal{N}$). The talk concerns attempts to vary this definition by replacing $<$ with various other relation symbols (with corresponding axioms in place of those of a total order). I will describe joint work with Steinhorn on various such minimality notions, work with Haskell on notions of minimality (C-minimality and P-minimality) particularly appropriate for algebraically closed valued fields and for the p-adics, and joint work with Marker and Steinhorn on another related notion, weak o-minimality, appropriate for real closed convexly valued fields and related structures.

The theorems obtained take a similar form in each case. Any weakly o-minimal ordered field is real closed, any C-minimal field is an algebraically closed field with a definable valuation, and any P-minimal field is p-adically closed. A key ingredient in these results is topological dimension All these structures carry a topology with a uniformly definable basis. We define the dimension of any (parameter)-definable subset X of M^n to be the maximal r such that for some projection $\pi : M^n \longrightarrow M^r$, $\pi(X)$ has non-empty interior in M^r. In each case, it is shown that if $X_1, \dots, X_t$ are definable subsets of M^n then $\dim(X_1 \cup \ldots \cup X_t) = \mathrm{Max}\{\dim(X_1), \ldots, \dim(X_t)\}$ Simultaneously, continuity results are proved for definable functions $F^n \longrightarrow F$, and in some cases there are cell decomposition theorems. There are also characterizations of when algebraic closure has the exchange property (if it does, it gives the same notion of dimension).

In the talk, I will survey the relevant background from o-minimality. I will then describe the various new notions of minimality and their motivating examples, and discuss the main theorems.

► LARISA MAKSIMOVA, *On some aspects of the history of modal logics.*

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(1) First formal systems of modal logics (Lewis 1918, Lewis & Langford 1932). Modalities and strict implication. Various axiomatizations of modal systems.

(2) Applying modalities for interpretation of the intuitionistic logic (Gödel 1933). Translation of the intuitionistic logic into the modal system S4 Algebraic and topological semantics of S4 constructed by McKinsey and Tarski.

(3) From study of individual logical calculi to the research of families of propositional logics. The logic as a set of its theorems. Intermediate logics (Umezawa 1955, 1959) and their translation into modal logics between S4 and S5 (Dummett & Lemmon 1959). Universality of the algebraic semantics. The cardinality of the families of intermediate and of modal logics (Yankov 1968).

(4) Relational semantics of modal logics (Kripke 1959, 1963, Hintikka 1961, Kanger 1957). Normal and non-normal modal systems. Interrelations of algebraic and relational semantics. Representation theorems for Boolean algebras with operators (Jónsson & Tarski 1951).

(5) Investigation of Kripke-completeness of propositional modal systems. Deleting the reflexivity axiom. Examples of non-complete modal logics (with respect to the relational semantics) (Fine 1974, Thomason 1974) Generalized Kripke semantics

(Makinson 1970). Neighborhood semantics (Montague 1970) and topological semantics. Completeness and non-completeness.

(6) Decidability and non-decidability in propositional modal logics. Finite model property. Filtration. Gentzen formulations of some modal systems. Applying Rubin's theorem Examples of non-decidable propositional modal calculi (Thomason 1975)

(7) Quantifiers in modal logics (Barcan 1946). Barcan Formula.

▶ BRUNO POIZAT, *Théorie des modèles et géométrie algébrique*

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Cette conférence est consacrée à la convergence de la Théorie des modèles vers la Géométrie algébrique qu'on observe depuis quelques années. Bien des mathématiciens, surtout dans ce pays, trouveront contre-nature ce rapprochement entre la Logique et une manifestation si prestigieuse de l'activité scientifique ; cependant, si on le voit avec quelque recul, il semble inévitable : l'essentiel de l'étude des géomètres a pour sujet des êtres définissables dans un corps algébriquement clos, si bien qu'on considérerait la Géométrie algébrique comme la partie de la Théorie des modèles consacrée aux corps si cette étude n'était pas plus fine que celle qu'en font les logiciens. Ces derniers, par nécessité, ont dû chercher à comprendre les travaux des géomètres quand ils ont vu apparaître, dans le contexte abstrait de la classification des structures, des objets de même comportement que ceux de la Géométrie. En plusieurs occasions, ils se sont même demandé s'ils ne leur étaient pas identiques.

Un exemple d'une telle situation est la récente caractérisation abstraite, par Hrushovski et Zil'ber, d'une courbe lisse à partir de la topologie de Zariski de ses puissances cartésiennes.

Cette caractérisation est lourdement chargée d'histoire, étant l'aboutissement d'un long processus modèle-théorique. Mais c'est aussi la promesse d'un avenir où les géomètres manifesteront un intérêt réciproque pour la Théorie des modèles. En effet Hrushovski a déduit de cette caractérisation la première preuve complète, en toute caractéristique, de la conjecture dite de Mordell-Lang à propos des sous-groupes de type fini d'une variété abélienne. C'est à mon avis la première application significative de la Logique aux Mathématiques classiques.

Cette preuve a la propriété particulièrement intéressante d'utiliser une structure auxilliaire - un corps différentiellement clos en caractéristique nulle, un corps séparablement clos en caractéristique p - pour aboutir à un énoncé qui ne fait intervenir que la notion de corps. Elle vient de la maîtrise des propriétés modèle-théoriques de cette structure auxilliaire, dont on doit en particulier montrer que des fragments substantiels ressemblent aux courbes lisses dont il a été question plus haut. Vu sous cet angle, Mordell-Lang apparaît comme la partie émergeante d'un iceberg, comme une espèce de traduction superficielle de résultats structurels profondément engloutis.

Je dirai quelques mots plus précis sur les raisons qui ont mené les théoriciens des modèles à la Géométrie, j'essayerai d'expliquer en quoi consiste l'énoncé de Mordell-Lang - ce n'est pas un exercice si facile devant un auditoire peu familier du sujet - et je décrirai les grandes lignes de la démonstration.

▶ JEAN PIERRE RESSAYRE, *Finitistic applications of transfinite power series*.

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In the forties, Malcev and B. Neumann proved:

(0) every real closed field K is isomorphic to some field K' of transfinite power series (with the operations induced by the structure of these series).

In the nineties, three extensions and complements were obtained (see [1, 3, 2]):

(1) ensuring that K' is **truncation closed** (for every series $S \in K'$ of ordinal length α and for every $\beta < \alpha$, the sum of the first β terms of S also belongs to K'). This is important in the proof of (2), (3) below.
(2) extending (0) to the case where K, K' are models of the complete theory of ($\mathbb{R}, e^x$ restricted to $[0,1]$). (or of other expansions of the field $\mathbb{R}$ by restricted analytic functions).
(3) constructing in every real closed field K an **"Integral Part"**, that is a subring A providing every element x of K with a unique "integer part" $[x] \in A$ such that $[x] \leq x < [x]+1$. Moreover, if K is equipped with a function x^y satisfying the basic properties of this function over $\mathbb{R}$, then this Integral Part A can be chosen so that A^+ is closed under x^y.

The author applied (1) (2) (3) to questions of an effective nature: his results concern the polynomial time computability of certain functions over **N** or **R**, and eliminate complicated schemes from the recursive axiomatization of $(\mathbf{R}, e^x)$ provided by Macintyre-Wilkie (who prove its completeness under Shanuel's Conjecture). For most of these effective results, only the proof based on (1) (2) (3) exists, and the problem of finding a less exotic, more constructive proof seems to be a challenge.

[1] M. H. Mourgues and J. P. Ressayre, *Every real closed field has an integer part*, ***The Journal of Symbolic Logic***, vol. 52 (1993), pp. 641–647.

[2] J. P. Ressayre, *Integer parts of real closed exponential fields*, ***Proof theory, arithmetic and computational, complexity. Proceedings of the Prague 1991 workshop*** (P. Clote and J. Krajicek, editors).

[3] L. van den Dries, A. Macintyre, and D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, ***Annals of Mathematics***, vol. 140 (1994), pp. 183–205.

▶ SIMONA RONCHI DELLA ROCCA AND LUCA ROVERSI, *Lambda calculus and intuitionistic linear logic*.

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The *Curry-Howard isomorphism*, or "formulas-as-types" principle, can be seen as the basis of typed functional programming. In fact, through this isomorphism, the intuitionistic proof of a formula can be seen as a functional program, whose type is the formula itself. Then the computation process has its logic correspondence in the proof normalization procedure, or "cut elimination". Well known examples of such an isomorphism are the implicational fragment of the intuitionistic propositional logic and the simply typed λ-calculus, or the second order propositional logic and the second order λ-calculus of Girard and Reynolds.

The *linear logic*, introduced by Girard [3], seems to be particularly interesting from the computational point of view. In fact, it is more refined than the classical logic, in the sense that the use of the structural rules is explicitly controlled through a modal connective: more precisely, weakening and contraction rules can be applied only to modal formulas. Since weakening and contraction rules are naturally related to the operations of erasing and copying information, respectively, the linear logic can be seen as a model for a computational environment with an explicit control over the resources management. This features can be effectively studied using a language corresponding to the intuitionistic fragment of the linear logic, through the Curry-Howard isomorphism.

Until now, some languages inspired by this isomorphism have been designed, by Lafont [4], Abramsky [1], Mitchell and Lincoln [5] and Benton, Bierman, de Paiva and Hyland [2].

We propose a new language Λ_l, representing the intuitionistic linear proofs and having both a simpler syntax and an easier operational semantics than the previous proposed languages. It is well known that the λ-calculus is interesting and useful, as paradigmatic language for the intuitionistic calculus, because of its simplicity. It has just two term formation rules, abstraction and application, and one evaluation rule, the β-rule, which is essentially the substitution. In the same line, a language can be used as a paradigm for the linear intuitionistic computation, only if it preserves both the syntactical and the computational simplicity of the λ-calculus.

The language Λ_l is a fully typed language. It is obtained as a "decoration" of the intuitionistic linear proofs, given in natural deduction style Syntactically, the most peculiar features of Λ_l are that terms are built starting from two different sorts of variables, and that not only variables, but also *patterns*, can be bound. A pattern is an expression formed by a logic connective applied to variables (for example, $X \otimes Y$). The two different sorts of variables correspond respectively to modal and non modal premises. The fact that structural rules can be applied only to modal premises reflects that terms must obey to linearity constraints only on one sort of variables. Thanks to the use of patterns, the reduction rules (induced by the cuts of the logics) have a uniform definition: they are all extensions of the classical β-rule, based on pattern-matching. Consequently the computational behavior of Λ_l is very simple, and properties of the language, like the strong normalization, for example, can be easily proved by using standard techniques.

Moreover, we define a denotational semantics of Λ_l in a categorical setting. This allows the development of general tools for studying operational properties of the language. For example, we prove that the operational equivalence is extensional.

It is possible to define an untyped version of Λ_l, and a type assignment system giving it linear formulas as types. This is done by means of a function which erases the type information on terms of Λ_l. The type assignment inherits from Λ_l all the good properties, like subject reduction and strong normalization. Further, the principal type property holds for the type assignment, and it is possible to extend it to universal quantified types or intersection types, expanding the class of typable terms.

[1] S. Abramsky, *Computational interpretation of linear logic*, Technical Report 90/92, Department of Computing,Imperial College, London, 1990.

[2] N. Benton, G. Bierman, V. de Paiva, and M. Hyland, *Term assignment for intuitionistic linear logic*, Technical Report 262, Computer Laboratory, University of Cambridge, August 1990.

[3] J. Y. Girard, *Linear logic*, ***Theoretical Computer Science***, vol. 50 (1987), pp. 1–102.

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► JERZY TIURYN, *Simply typed lambda calculus with subtyping – the problem of type reconstruction.*

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Types are used in programming languages in order to decrease the possibility of making some kind of run-time errors, such as calling a procedure with incorrect actual parameters. So, from this point of view, types are decorations attached to pieces of programs. Types are very useful in programming but they often tend to be very large, making the program difficult to read. Thus the problem of type reconstruction comes naturally as the problem of deciding for a given program without types whenever it can be correctly typed. This problem is of a certain practical importance since it can be encountered in some programming languages, such as ML—in such languages the type reconstruction is being solved automatically by a compiler.

During the lecture we will discuss the issue of type reconstruction for the simply typed lambda calculus with the relation of subtyping. Sybtyping can be introduced as a partial order on the set of atomic types which is further extended to all simple types. The problem of type reconstruction for the simply typed lambda calculus with subtyping has been the subject of an intensive research lately. We will discuss the main results obtained in this area so far. In particular we will discuss the following topics:

- General upper and lower bounds on the complexity of this problem.
- Partial orders for which the problem is easy.
- Satisfiability of sybtype inequalities in term algebra — this is an algebraic problem (a generalization of the unification problem) to which the type reconstruction problem is reducible in polynomial time.
- The retraction problem for finite partial orders — this is another algebraic problem. The previous problem is exponential time reducible to it.

► JOHAN VAN BENTHEM, *Modal logics and process theories.*

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Dynamic logics in a broad sense are still gaining ground as a way of studying various kinds of process in computer science and recently also cognitive science and linguistics. These logics employ a two-level modal language combining propositions and programs, which is naturally interpreted over "labeled transition systems" (i.e., polymodal Kripke models). A key semantic invariance between such process models has turned out to be "bisimulation", a notion which has emerged independently in modal logic and computer science.

In this setting, there is a variety of operators which may be considered as "logical constants": propositional connectives, program operations, but also many operators moving back and forth between programs and propositions. Our main concern in this lecture will be to get a more systematic view of the natural candidates here, working toward functional completeness theorems that describe natural repertoires of logical constants in dynamic logic. For this purpose, we introduce two notions of "invariance" and "safety" for bisimulation, which allow us to classify all possibilities in the above categories (always modulo this central process equivalence).

Dynamic logic is essentially a marriage between two standard partners: propositional modal logic and a relational algebra for program expressions. Subsequently, we shall look at two ways of extending this framework. The first extension involves so-called "arrow logics" that generalize relational algebra, essentially, making the entire system a two-sorted modal logic. The second generalization involves current "process algebra".

We generalize the earlier notions of invariance and safety to these broader settings, and show how our functional completeness theorems must change. As a side effect of our analysis, we find a novel modal formulation for studying process algebra.

[1] VAN BENTHEM, ***Language in action. categories, lambdas and dynamic logic***, Studies in Logic, no. 130, North-Holland, Amsterdam, 1991.

[2] ———, *Programming operations that are safe for bisimulation*, Report 93-179, Center for the Study of Language and Information, Stanford University, 1993.

[3] ———, *Logic and the flow of information*, ***Proceedings 9th International Congress of Logic, Methodology and Philosophy of Science. Uppsala 1991*** (D. Prawitz, B. Skyrms, and D. Westerstahl, editors), Elsevier Science Publishers, 1994, pp. 693–724.

▶ MOSHE Y. VARDI, *Infinitary logic for computer science.*

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Infinitary logic $L_{\infty\omega}$ extends first-order logic by allowing infinitary conjunctions and disjunctions (i.e., conjunctions with an infinite number of conjuncts and disjunctions with an infinite number of disjuncts). One usually thinks of infinitary logic as a fairly esoteric logic, which is not of much interest in computer science. Surprisingly, a certain fragment $L^{\omega}_{\infty\omega}$ of $L_{\infty\omega}$ turns out to be of great interest in computer science. This fragment is obtained by restricting formulas to contain a finite number of distinct variables, though the formulas can be of infinite length; The advantage of dealing with $L^{\omega}_{\infty\omega}$ is that its the expressive power can be completely characterized in game-theoretic terms.

We will describe applications of this logic to the study of 0-1 laws and the expressive power of database query languages.

[1] PH. G. KOLAITIS and M. Y. VARDI, *On the expressive power of Datalog: tools and a case study*, ***Proceedings of the 9th ACM symposium on principles of database systems***, 1990, pp. 61–71. Full version to appear in ***Journal of Computer and System Sciences***.

[2] ———, *Fixpoint logic vs. infinitary logic in finite-model theory*, ***Proceedings of the 7th IEEE symposium on logic in computer science***, 1992, pp. 46–57.

[3] ———, *Infinitary logics and 0-1 laws*, ***Information and Computation***, vol. 98 (1992), pp. 258–294.

▶ BOBAN VELICKOVIC, *Forcing and* $0^{\#}$ *complexity of the reals in inner models of set theory.*

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We present results from a joint paper with Hugh Woodin.

The set of constructible reals is a Σ^1_2 set of reals. This set can have a simpler definition if either it is countable or if every real is constructible. Martin and Solovay showed that it is relatively consistent with ZFC that this set is uncountable and co-analytic.

H. Friedman asked if this set can be analytic or even Borel in a nontrivial way, i.e. both uncountable and not equal to the set of all reals. There is a related problem due to K. Prikry who asked if there can exist a nonconstructible perfect set consisting of constructible reals.

We give a strong negative answer to Friedman's question by showing that if M is an inner model of set theory such that the reals of M form an analytic set then either all reals are in M or $\aleph_1^M$ is countable. The proof relies on a new of regularity property of analytic sets and can be extended to projective sets under the appropriate large cardinal assumptions. However, we show that it is consistent to have an inner model of set theory M such that the reals in M form an uncountable F_σ set. Of course, in this case CH must fail in M.

Concerning Prikry's question we show that it is consistent that there exists an inner model M of set theory which computes correctly $\aleph_1$, there is a perfect set consisting of reals from M, and yet not every real is in M.

We also present some related results in the context of the axiom of determinacy and the failure of the axiom of choice.

▶ VICTOR VIANU, *Relational complexity.*

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One of the exciting developments in complexity theory is the discovery of a very intimate connection between computational complexity and logic. This intimate connection was first discovered by Fagin, who showed that the complexity class NP coincides with the class of properties expressible in existential 2nd-order logic. Although the relationship between descriptive and computational complexity is intimate, it is not without its problems. Indeed, there is a fundamental mismatch between computational devices and complexity, already noted by mathematicians like Tarski and Harvey Friedman. while computational devices work on *encodings* of problems, logic is applied directly to the underlying mathematical structures. As a result, machines are able to enumerate objects that are logically unordered. For example, while we typically think of the set of nodes in a graph as unordered, it does become ordered when it is encoded on the tape of a Turing machine. This "impedance mismatch" does not pose any difficulty in the identification of NP with existential 2nd-order logic since the logic can simply assert the existence of the desired order. The mismatch with logic becomes apparent, however, at complexity classes below NP. For example, the relationship between the class P and fixpoint logic is more complicated as a result of the mismatch. It is only when we assume built-in order that we get that P coincides with the class of properties expressible in fixpoint logic.

We present a device operating directly on structures, without encoding, called *relational machine*. There is a close connection between relational machine and infinitary logic with finitely many variables. Indeed, relational machines can be viewed as the "effective fragment" of this logic. The relational machine yields a new notion of complexity, quite different from classical computational complexity, which we call *relational complexity*. We show natural correspondences between relational complexity classes and logics, without any order assumption. For example, relational polynomial time corresponds *precisely* to the fixpoint queries, and relational polynomial space yields the while queries. Moreover, relational machines help overcome the "impedance mismatch" between logic and *classical* complexity, by acting as a mediator between logic and classical devices A typical result of this type is that *fixpoint* = *while* iff P = PSPACE. Similarly, we present logical analogs of the inclusions between complexity classes from P to EXPTIME, in terms of various fixpoint logics.

This work is largely motivated by the study of database query languages and their complexity. Database queries are posed against an abstract interface, so they lead to computations more accurately modeled by relational machines than classical devices. In particular, using the relational machine, we obtain results which point to new possibilities for optimizing queries, and evaluate this using probabilistic techniques related to 0-1 laws. This yields some surprising results on the expected complexity of many queries, which is shown to be lower than the overly pessimistic worst-case complexity

This talk is based on joint work with Serge Abiteboul, and partly with Kevin Compton and Moshe Vardi.

► FRANK WAGNER, *On the structure of stable groups*

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In this talk, we shall present results about the group-theoretic properties of stable groups. These can be classified into three main categories, according to the strength of the assumptions needed: chain conditions, generic types, and some form of rank.

Each category has its typical application: chain conditions often allow to deduce global properties from local ones, generic properties are used to get definable groups from undefinable ones, and rank (in some form or other) is necessary to interpret fields in certain group actions.

While originally the main input came from algebraic group theory, founded on the similarities between (finite) rank and dimension, increasingly methods borrowed from finite group theory have come to play their role.

We shall thus look at characteristic subgroups such as the Fitting subgroup, the soluble radical, and the Frattini subgroup, and at characteristic families of subgroups such as p-Sylow subgroups or Carter subgroups.

Finally, we shall propose a conjecture regarding the conjugacy of the Borel subgroups of a group of finite Morley rank.

Abstracts of contributed papers

▶ A. NAIT ABDALLAH, *The dynamics of logic systems and its applications*.
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Logic is usually conceived as a geometry of static truths. Reasoning about actions in the context of partial information naturally leads to viewing logic as a "dynamics of processes." This view yields a simple solution to a wide range of instances of the *frame problem*: Yale shooting problem, Robot problem, Projection problem, etc.

We use partial information ionic logic, a monotonic framework specifically designed for reasoning with partial or tentative information [1, 2]. Given a set of ionic formulae Ψ, the justification ordering on ionic interpretations $i, i' \in \Delta_\infty$ is defined by: $i \leq_{\star p}^{\Psi} i'$ if and only if i accepts more justifications $\in \Psi$ and rejects less justifications $\in \Psi$ than i'. Let I be some linearly ordered set. Let $\mathcal{F}$ the set of partial information ionic formulae. A syntactic path is a monotone mapping $\Phi : I \to \mathcal{P}(\mathcal{F}), n \mapsto \Phi_n$ such that $n \leq n'$ implies $\Phi_n \subseteq \Phi_{n'}$. Given a syntactic path Φ, the characteristic surface $\Sigma(\Phi)$ is defined as being the set

$$\Sigma(\Phi) = \prod_{n \in I} (\Omega_{min}(\models)(\Phi_n) / \equiv_\star^{G(\cup_n \Phi_n)})$$

where (i) for any set of formulae Ψ, $\Omega_{min}(\models)(\Psi) = \{m \in \Delta_\infty : m \text{ is a } \sqsubseteq\text{-minimal model of } \Psi\}$, where $\sqsubseteq$ is the information ordering ; (ii) $G(\cup_n \Phi_n)$ is the set of formulae having a justification occurrence in some formula of $\cup_n \Phi_n$; and (iii) $\equiv_\star^{G(\cup_n \Phi_n)}$ is the equivalence relation generated by the ordering relation $\leq_\star^{G(\cup_n \Phi_n)}$, where $(i_0, J, i_1) = i \leq_\star^{G(\cup_n \Phi_n)} i' = (i_0', J', i_1')$ if and only if $i \leq_{\star p}^{G(\cup_n \Phi_n)} i'$ and $i_0 \sqsubseteq i_0'$ and $i_0' \sqsubseteq i_1$.

A semantic path for Φ is a $\sqsubseteq$-chain $(m_n)_{n \in I}$ of interpretations such that $\forall n \in I$, m_n is a model of Φ_n, and $\sqcup_n m_n$ is a model of $\cup_n \Phi_n$. A semantic path (m_n) is extremum if and only if for any other semantic path (m_n') $\forall n \in$, whenever m_n and m_n' cand be compared for the justification ordering , then $m_n \leq_{\star p}^{G(\cup_n \Phi_n)} m'n$. If $\star$ is a universal ionic operator, then any semantic path determines an element of $\Sigma(\Phi)$ and conversely [2].

The following variational principle (least action principle) is introduced: "*Of all the possible paths in $\Sigma(\Phi)$ that it might take to get from one state to another, the point representing a (physical) logic system takes a path (m_n) which is extremum, i.e., the smoothest possible on the characteristic surface representing the system.* This principle provides a solution to the frame problem.

Example: (Yale shooting problem). *After performing an action, things normally remain as they were After loading a gun, the gun is loaded. After a man is shot with a loaded gun, he dies. After loading a gun, waiting and shooting Fred, is it the case that Fred will die?* The characteristic surface of this problem has 10 semantic paths, one of which is minimal among the other paths. When the evolution of the system follows tis path, Fred is dead after being shot.

Example: (Robot problem). *After performing an action, things normally remain the same. After locking the robot's forward gear, the gear is locked if it can be locked. After trying to move, the robot will move forward if the reverse gear is locked and the forward gear is unlocked.*

Aftyer trying to move, the robot will move backward if the forward gear is locked and the reverse gear is unlocked. If the robot is moving backward, then the reverse gear is unlocked. If the robot is moving forward, then the forward gear is unlocked. Initially the reverse gear is locked, the forward gear is unlocked, the robot is not moving at all, and the forward gear can be locked. After waiting, locking the forward gear, and trying to move the robot, is it the case that the robot stand still? The characteristic surface of this problem has one extremum path. When the evolution of the system follows this path, locking the forward gear was successful, and the robot is not moving forward and is not moving backward.

[1] M. A NAIT ABDALLAH, *Commonsense reasoning = logic + partial information (abstract)*, ***Proceedings of the 1992 European Summer Meeting of the ASL***, 1992

[2] ———, ***The logic of partial information***, EATCS Research Monographs in Theoretical Computer Science, Springer Verlag, 1994.

▶ ZOFIA ADAMOVICZ, *Consistency of* $\neg exp$ *and* $\exists^*_m$ *definability of all small* Σ_1 *definable numbers.*

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Let $\exists^*_m$ denote the collection of Σ_1 formulas of restricted complexity – these are translations of $\exists\Pi^b_m$ (where Π^b_m is Buss's class) formulas to a language without terms.

We prove that for a sufficiently large m there is a model of the theory $I\Delta_0 + \Omega_2 + \neg exp$ in which for every Σ_1 definition $\phi(x)$ there is an $\exists^*_m$ definition $\varphi(x)$ such that

$$\phi(x) \Leftrightarrow \varphi(x)$$

for sufficiently small x.

▶ DAVID ALBRECHT, FRANK A. BÁUERLE, JOHN N. CROSSLEY AND JOHN JEAVONS, *Curry-Howard terms for linear logic.*

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In [4] J. Y. Girard presented a Gentzen-style system for linear logic together with a strong normalization theorem based on proof-nets. In [3] Crossley and Shepherdson presented an extension of the Curry–Howard correspondence to full predicate logic (with possibly non-logical axioms). The Curry–Howard notation permits mechanization of the normalization process.

Now we present a natural deduction system of introduction and elimination rules for first order full linear logic, Curry–Howard terms for deductions in this logic and a strong normalization theorem which uses a modified version of Girard's candidates for reducibility.

The logical system is a further development of those of Troelstra [5] and Benton *et al* [2] and the rules follow fairly traditional patterns.

On the other hand the new Curry–Howard terms involve substantial modifications to the lambda calculus, in particular the introduction of new operators ι, κ, τ and ω with their associated reduction rules. (The system of terms goes beyond that of Abramsky [1].) More importantly the definition of substitution (equivalent to replacing one sub–proof by another) has to be updated.

THEOREM 1. *There is a one-one correspondence between Curry–Howard terms and proofs in first order full linear logic.*

THEOREM 2. *Every Curry–Howard term for first order full linear logic strongly normalizes.*

[1] S. Abramsky, *Computational interpretations of linear logic*, Technical Report 90/20, Department of Computing, Imperial College, London, October 1990.

[2] N. Benton, G. Bierman, V. de Paiva, and M. Hyland, *A term calculus for intuitionistic linear logic*, ***Typed lambda calculi and applications. proceedings of the international conference TLCA '93 March 16–18, Utrecht, The Netherlands***, LNCS, no. 664, Springer-Verlag, 1993, pp. 75–90.

[3] J. N. Crossley and J. C. Shepherdson, ***Extracting programs from proofs by an extension of the Curry-Howard process***, Birkhäuser, 1993

[4] J.-Y. Girard, *Linear logic*, ***Theoretical Computer Science***, vol. 50 (1987), pp. 1–102.

[5] A. S. Troelstra, ***Lectures on linear logic***, no. 29, Stanford, CA: Center for the study of language and information, 1992.

▶ KLAUS AMBOS-SPIES, SEBASTIAAN A. TERWIJN AND XIZHONG ZHENG, *Resource bounded randomness and weakly complete problems.*

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We introduce and study resource bounded random sets based on Lutz's concept of resource bounded measure ([4,5]). We concentrate on n^c-randomness ($c \geq 2$) which corresponds to the polynomial time bounded (p-) measure and which is adequate for studying the internal and quantitative structure of E=DTIME(2^{lin}).

First we show that the class of n^c-random sets has p-measure 1. This provides a new, simplified approach to p-measure 1-results.

Next we compare randomness with genericity (in the sense of [1, 2]) and we show that n^{c+1} - random sets are n^c-generic whereas the converse fails. From the former we conclude that n^c-random sets are not p-btt-complete for E.

Our technical main results describe the distribution of the n^c-random sets under p-m-reducibility. We show that every n^c-random set in E has n^k-random predecessors in E for any k≥1 whereas the amount of randomness of the successors is bounded. We apply this result to answer a question raised by Lutz [3]: We show that the class of weakly complete sets has measure 1 in E and that there are weakly complete problems which are not p-btt-complete for E.

[1] K. Ambos-Spies, H. Fleischhack, and H. Huwig, *Diagonalizations over polynomial time computable sets*, ***Theoretical Computer Science***, vol. 51 (1987), pp. 177–204.

[2] ———, *Diagonalizing over deterministic polynomial time*, ***Proceedings of CSL 1987***, Lecture Notes in Computer Science, no 329, Springer Verlag, 1988, pp. 1–16.

[3] J. H. Lutz, *Weakly hard problems*, (preliminary version), April 1993, submitted for publication; extended abstract in ***Proceedings of the 9th Structure in Complexity Theory Conference***, IEEE Computer Science Press, 1994. pp. 146–161.

[4] ———, *Category and measure in complexity classes*, ***SIAM Journal of Computing***, vol. 19 (1990), pp. 1100–1131

[5] ———, *Almost everywhere high nonuniform complexity*, ***Journal of Computer and System Science***, vol. 44 (1992), pp 220–258.

▶ MATTHIAS BAAZ, CHRISTIAN G. FERMÜLLER AND RICHARD ZACH, *Proof theory of finite-valued logics.*

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Proof theoretic calculi for finite-valued logics (sequent calculi, tableaux, natural deduction) have been proposed several times since the 50s, but results about them have usually not gone further than soundness and completeness theorems.

The approach used is based on many-sided sequents, say, $\Gamma = \Gamma_1 \mid \dots \mid \Gamma_m$, if the logic under consideration is m-valued. Here, Γ_i is a finite sequence of formulas. Such a sequent is true in a given structure, if some Γ_i contains a formula A which takes the truth value i (in the structure). Based on this generalized notion of Gentzen's sequents, a calculus can be constructed for any finite-valued first-order logic with usual connectives and so-called distribution quantifiers [3]. The ordinary universal and existential quantifiers belong to this very general class of quantifiers. The construction is effective: for any finite-valued matrix a calculus can be computed (see [1]).

If Γ and Δ are two sequents, the Γ, Δ denotes the sequent obtained by concatenating the corresponding Γ_i and Δ_i. [i: A] is the sequent which is empty everywhere, except that it has the formula A at position i.

The calculus contains the following cut rule:

$$\frac{\Gamma, [i : A] \quad \Gamma', [j : A]}{\Gamma, \Gamma'}$$

where i and j are distinct. We show how many-valued resolution [2] can be used to obtain the cut-elimination theorem for such calculi, in a similar uniform way as the construction of the calculi itself. The proof also (obviously) works for the classical case and points the way to obtaining resolution-based interactive cut-elimination procedures.

[1] M. Baaz and C. G. Fermueller, *Resolution for many-valued logics*, ***Proceedings LPAR 1992***, LNAI, no. 624, Springer, pp. 107–118.

[2] M. Baaz, C. G Fermueller, A. Ovtrucki, and R. Zach, *MULTLOG. A system for axiomatizing many-valued logics*, ***Proceedings LPAR '93,***, LNAI, no. 698, Springer, Berlin, 1993, pp. 345–347.

[3] W. A Carnielli, *Systematization of finite many-valued logics through the method of tableaux*, ***Journal of Symbolic Logic,***, vol. 52 (1987), pp. 473–493.

► S. A. BADAEV, *Some problems on computable enumerations.*

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Let $\mathcal{A}$ denote a class of r.e. sets A surjective mapping $\alpha : \omega \to \mathcal{A}$ is called enumeration of $\mathcal{A}$. If the set $\{\langle n, x\rangle : x \in \alpha n\}$ is r.e., then enumeration α is called computable. Enumeration α is said to be reducible to enumeration $\beta (\alpha \leq \beta)$, if $\beta = \alpha f$ for some recursive function f If for every enumeration β of $\mathcal{A} \beta \leq \alpha$ implies $\alpha \leq \beta$, then α is called minimal enumeration of $\mathcal{A}$.

Theorem 1. *Enumeration α of $\mathcal{A}$ is minimal iff for every r.e. set W if $\alpha(W) = \mathcal{A}$ then there exists r.e. equivalence η such that $\forall x \forall y \exists z \in W((\langle x, y\rangle \in \eta \to \alpha x = \alpha y) \& \langle x, z\rangle \in \eta)$*

This criteria published in [1] was used widely for solving some problems of the theory of enumerations [3]. In particular, it had been obtained natural classification of minimal enumeration and had been round an elementary proof of an analogy of speed-up theorems [2]. One of the recent results in this direction is the following.

Theorem 2. *Let computable class $\mathcal{A}$ contain any pair of embedded sets $A \subset B$ such that the set $N \setminus B$ can be separated from the set A by some r.e. set. Then $\mathcal{A}$ has infinitely many pairwise* *non-reducible computable enumerations*

[1] S. A. BADAEV, *About one of Goncharov's problems*, ***Siberian Mathematics Journal***, vol. 32 (1991), no. 3, pp. 532–534.

[2] ———, *On minimal enumerations*, ***Siberian Advances in Mathematics***, vol. 2 (1992), no. 1, pp. 1–30.

[3] YU. L. ERSHOV, ***Theory of enumerations***, Nauka, Moscow, 1977.

► JOAN BAGARIA AND ROGER BOSCH, *Definable fragments of Martin's Axiom.*

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By a definable poset we mean a poset belonging to one of the following classes: the class of projective posets, the class of constructive (from a given real) poset or the class of ordinal-definable (with a real as a parameter) poset.

We study the axioms that we obtain by restricting Martin's Axiom to one of these of definable posets. We construct models for the axioms by iterating only definable posets and we show that many of the consequences of Martin's Axiom do not hold in them. As a corollary, we obtain a much stronger form of a theorem of H. Judah and S. Shelah in [1].

[1] H. JUDAH and S. SHELAH, *Souslin forcing*, this JOURNAL, vol. 53 (1988), no. 4, pp. 1188–1207.

► STEFANO BARATELLA AND RUGGERO FERRO, *Non standard regular finite set theory.*

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Our goal is to introduce a set theory negating the cantorian axiom of infinity, but, at the same time, still able to support modern mathematics.

Lying underneath there is a notion of infinity, different from the cantorian one. Indeed we will admit the existence of natural numbers of which we do not recall the entire process of construction. These numbers may be regarded as infinite. So we get another notion of infinity that takes into account the role of the observer.

In the theory **NRFST** (*Non standard Regular Finite Set Theory*) that we propose, in addition to the usual membership predicate, there is an undefined unary predicate, *standard*, whose intended interpretation is that it holds on the elements that can be grasped by the observer.

All the elements of the observed world form the *internal universe*.

We will assume that the observer has a vague notion of the non standard elements of the observed world. To be able to consider these vague notions we need a further undefined unary predicate, *internal*, that will distinguish between the objects of the observed world and the extensions of vague notions associated to them.

In our presentation we would like to introduce the axioms of **NRFST**, their heuristic motivations, and to report on the basic features and relative consistency results of theory.

► CATALIN BARBACIORU, *The topology of implicative BCK-algebras.*

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A BCK-algebras is an algebra of type $\langle 2, 0\rangle$ which satisfies the following axioms:

1) $((x^*y)^*(x^*z)^*(z^*y)) = 0$

2) $(x^*(x^*y)^*y) = 0$

3) $x^*x = 0$
4) $0^*x = 0$
5) $x^*y = y^*x = 0$ imply $x = y$.

For an implicative BCK-algebra $(X,^*, 0)$ any prime ideal is maximal and is characterized by the property that for any x, y in I either x^*y or y^*x is in I. The set of prime ideals is denoted by $B(X)$

The family $(S_X(\mathcal{X}))$ is a topological base of compact open sets of $B(X)$, where $S_X(\mathcal{X}) = (I \in B(X) : \mathcal{X} \notin I)$.

Denoting by s the topology defined by this base then the topological space $(B(X), s)$ verifies the following properties :

i) admits a topological base of compact open sets;

ii) for any compact open sets U, V of $B(X)$ the set $U \cap *(B(X) - V)$ is a compact open set;

iii) for any distinct I, J in $B(X)$, there exists a compact open set U such that $I \in U$ and $J \notin U$.

Let $f : X \to Y$ be a morphism of implicative BCK-algebras. We denote by $B(f)$ the map $B(f) : B(Y) \to B(X)$ defined by $B(f)(J) = f^{-1}(J)$ Then:

a) if V is a compact open set of $B(X)$ then $B(f)^{-1}(V)$ is a compact open set of $B(Y)$ (so $B(f)$ is a continuous map);

b) for any compact set U of $B(X)$ we have $B(f)^{-1}(*(B(X) - U)) = *(B(f)^{-1}(B(X) - U))$.

Let C be the category whose objects are the topological spaces with properties i), ii), iii) and whose morphisms are the maps which satisfy a), b). Then B becomes a contravariant functor from the category of implicative BCK-algebras into the category C. For an object A of C let $T(A) = (U \subseteq A$. is a compact open set).

We define $U^*V = U \cap (A - V)$ for any U, V in $T(A)$. Then $(T(A), *, 0)$ is an implicative BCK-algebra. For A, B in ObC and f in $Hom_c(A, B)$ we define $T(f) : T(B) \to T(A)$ by $T(f)(V) = f^{-1}(V)$. Then T becomes a contravariant functor. Furthermore the maps $(S_X)_{X \in BCK_i}$ define a functorial morphism S between the covariant functors 1_{BCK_i} and $T \circ B$.

We remark that the one-to-one morphism S_X is an isomorphism iff X is a superior semilattice.

We may conclude that the dual category BCK_i^v, of superior semilattices positive implicative BCK-algebras, is the subcategory of C, image of BCK_i^v, through the functor B.

► ANATOLIJ P. BELTIUKOV, *An intuitionistic formal theory with realizability in polynomial time.*

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A formal intuitionistic theory is proposed (equivalent to the Peano arithmetic by expressive power) with realizability of proved formulas in polynomial time. It can be expressed and proved in the theory, that the polynomially bounded Turing machines (with polynomial time counter) really complete their work and that the results of this work really exist. A POLYNOMIAL total algorithm f with the property

$$\forall x\, A(x, f(x))$$

can be algorithmically extracted from any proof of any formula

$$\forall x\, A(x, f(x))$$

in the theory. Here x and y can be vectors.

The theory is intended to use for deductive synthesis of algorithms with bounded computational complexity. The matter is that algorithms with the same property can be extracted from ordinary intuitionistic proofs but these algorithms may be so hardly computable that their usage in practical programming is impossible.

The theory operates with words over some finite alphabet, particularly, with numbers in the binary notation. It uses operations of concatenation and word multiplication

$$(w^\frown v = w\ .\ \ .\ w \ (|v| \text{ times}), \text{ where } |v| \text{ is the length of } v),$$

word equality, "subword" relation, lexicographical word ordering. The theory has a sufficiently large axiom system to describe main properties of these operations and relations. The main difference between the theory and Peano arithmetic is induction axiom scheme restriction. Only the WORD INDUCTION of the following form is permitted:

$$A() \wedge \forall x\ (A(x) \longrightarrow A(xs_1) \wedge\ \ldots \wedge A(xs_n)) \longrightarrow (x)A(x)$$

where $\{s_1, \ldots, s_n\}$ = the alphabet with which words the theory works. All the quantifiers of formulas, by which induction can be fulfilled, must be bounded. All values of existentially quantified variables situated outside negated formulas and premises of implications must be less (in lexicographical order) than some terms in the language

$$(\exists x\ (x < t \wedge A(x))).$$

The values of other quantified variables must be bounded by subword relation with some terms

$$((Qx)(x \text{ subword } t \wedge A(x))).$$

These boundedly quantified formulas describe precisely NP.

► VLADIMIR Y. BELYAEV, *HF-logic, computability over reals and other structures.*

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I consider the formalization of algorithm notion over any model A of finite language L which use BSS-machines [3] and notion of the list superstructure $HL(A)$ [1]. BSS-machines include as base operations the computabilities of terms of language L and relations of L in branch nodes. It is not possible to use elements of A except input elements, constants from L and values of terms from its. $HL(A)$ is elements of $|A|$, finite lists of elements from $|A|$, finite lists of elements and lists and so on. The necessary operations are added to L. $HL(A)$ has obvious similarity with $HF(A)$ – the minimal admissible set with urelements from A. This analogy leads to natural interdependence of HF-logic and computability over A. In particular we have the characterization of relations on A definable by formulas of HF-logic in recursion theory terms.

Let $\Sigma_0^s = \Sigma_0^r$ – class of recursion relations on $HL(A)$, Σ_{n+1}^s (Σ_{n+1}^r) is class of domains (range) of BSS-computable functions with oracle from Σ_n^s (Σ_n^r). This hierarchies was defined by author and A. Myasnykov in [2].

THEOREM 1. *Let $X \subseteq |A|^k$. Then X is definable by Σ_n-formula of HF-logic iff X is Σ_n^r-relation*

Let A be the reals $(L = \{+, \cdot, -, ^{-1}, 0, 1, \leq\})$.

THEOREM 2. *In A the classes Σ_n^s and Σ_n^r are hierarchies of Borel and projective sets respectively in **effective** descriptive set theory*

[1] I. V. ASHAEV, V. Y. BELYAEV, and A. G. MYASNIKOV, *Approaches to the theory of generalized computability*, ***Algebra i Logika***, vol. 32 (1993), no. 4, pp. 349–386.

[2] V. BELYAEV and A. MYASNIKOV, *Theorems on hierarchy in generalized recursion theory*, to appear.

[3] L. Blum, M. Shub, and S. Smale, *On a theory of computation and complexity over the real numbers: NP-completeness*, ***Bulletin of the American Mathematical Society***, vol. 21 (1989), no. 1, pp. 1–47.

► A. BERARDUCCI AND M. OTERO, *A recursive nonstandard model of normal open induction.*

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Models of (normal) open induction are those (resp. normal) ordered domains which are an integral part of the real closure of their fraction field.

In 1964 Shepherdson gave a recursive nostandard model of open induction. His model is not normal and does not have infinite prime elements.

Here we present a recursive nonstandard model of normal open induction with an unbounded set of infinite prime elements. Answering questions posed by Macintyre and Marker in [2]and Kaye in [1]. The construction is done using techniques of Shepherdson in [3] and Wilkie in [4].

[1] R. Kaye, *Open induction, Tennenbaum phenomena, and complexity theory*, ***Arithmetic, proof theory and computational complexity*** (Clote and Krajicek, editors), Oxford Logic Guides, no. 23, Oxford, 1993, pp. 222–237.

[2] A. Macintyre and D. Marker, *Primes and their residue rings*, ***Annals of Pure and Applied Logic***, vol. 43 (1989), pp. 57–77.

[3] J. Shepherdson, *A nonstandard model for a free variable fragment of number theory*, ***Bulletin de L'Academie Polonaise Science***, vol 12 (1964), pp. 79–86.

[4] A. Wilkie, *Some results and problems on weak systems of arithmetic*, ***Logic colloquium*** *'77* (Mcintyre et al , editors), North-Holland, 1978, pp. 285–296.

► G. M. BIERMAN AND VALERIA C. V. DE PAIVA, *Yet another intuitionistic modal logic.*

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In this work we investigate the categorical proof theory of a intuitionistic version of the modal logic S4. The formulation of intuitionistic versions of well-known classical modal logics is far from settled in the literature. The most popular approach is to adapt your favourite intuitionistic model to suggest the theorems which the modal logic should satisfy. The problem with this approach is that different adaptations and different choices of models results in different, *non-equivalent*, intuitionistic modal logics.

In this work we propose an intuitionistic modal logic, $\mathbf{S4}_p$, for which we provide equivalent axiomatic, sequent calculus and natural deduction formulations. It turns out that our proposal is equivalent to that proposed by Prawitz in his monograph (hence the 'p' subscript). However our natural deduction formulation is significantly more elegant than that proposed by Prawitz.

Another contribution of this work is the development of a notion of a categorical model of $\mathbf{S4}_p$. It turns out that some interesting variants of well-known categorical concepts need to be used. It should be possible to use these new categorical concepts, à la Kripke-Joyal, to suggest new Kripke-like structures.

► TERRY BOSWELL, *Historical reflections on the logical analysis of quasi-definite descriptions.*

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It is characteristic of Frege's logic that such phrases as "some animal", which are sometimes termed quasi-definite descriptions, are analyzed in the literal sense of the word. They resolve and disappear upon analysis. This is one aspect of his rejection of the subject-predicate analysis of statements, and constitutes one of his major achievements in logic. Whereas in Frege's analysis of such phrases no logical structure remains which corresponds directly to them, logicians since his time have often attempted to reconstruct forms within a Fregean framework which do so. This has partly been done in the interest of making logic more "natural", as for example when a natural deduction system allows the choice of single letters which stand for such phrases. But it is also for example characteristic of Hilbert's ε-calculus and of the generalized theory of quantifiers. Certain pre-Fregean uses of symbolism in logic, for example in Aristotle and Leibniz, have also retained such a correspondence.

► JÖRG BRENDLE, *σ-ideals related to classical forcing notions.*

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With each of the usual tree forcings $\mathcal{I}$ (e.g., $\mathcal{I} = $ Sacks forcing $\mathcal{S}$, Laver forcing $\mathcal{L}$, Miller forcing $\mathcal{M}$, Mathias forcing $\mathcal{R}$, etc.) we associate a σ-ideal i^0 on the reals as follows: $A \in i^0$ iff for all $T \in \mathcal{I}$ there is $S \leq T$ (i.e. S is stronger than T or, equivalently, S is a subtree of T) so that $A \cap [S] = \emptyset$, where $[S]$ denotes the set of branches through S. (so, s^0 is the ideal of Marczewski null sets, r^0 is the ideal of Ramsey null sets (nowhere Ramsey sets) etc.)

We show (in ZFC) that whenever i^0, j^0 are two such ideals, then $i^0 \setminus j^0 \neq \emptyset$. E.g., for $\mathcal{I} = \mathcal{S}$ and $\mathcal{J} = \mathcal{R}$ this gives a Marczewski null set which is not Ramsey, extending earlier partial results by Aniszczyk, Frankiewicz, Plewik, Brown and Corazza and answering a question of the latter.

For these ideals i^0, we will also discuss the covering number $\mathrm{cov}(i^0)$, the smallest size of a family of sets from i^0 covering the reals.

► DUMITRU BUSNEAG, *Bounded Hilbert and Hertz algebras of fractions.*

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A *Hilbert algebra* (Ha) is a triple $(A, \rightarrow, 1)$, where A is a non-empty set, $\rightarrow$ is a binary operation on A, $1 \in A$ is an element such that the next axioms is verified for every $x, y, z \in A$:

(1) $x \rightarrow (y \rightarrow x) = 1$,
(2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,
(3) If $x \rightarrow y = y \rightarrow x = 1$ then $x = y$.

If A is a Ha , then the relation $x \leq y$ iff $x \rightarrow y = 1$ is a partial order on A (called *natural ordering on* A); with respect to this ordering, 1 is the largest element of A. A *bounded Ha* (bHa) is a *Ha* with a smallest element 0 .

A *Hertz algebra* (Hza) is a Ha H with the property that for every $x, y \in H$ the infimum $x \wedge y$ (relative to the natural ordering) exists in H.

Examples:

(1) If $(B, \vee, \wedge, \neg, 0, 1)$ is a Boolean lattice, then $(B, \rightarrow, \wedge, 0, 1)$ is a bounded Hertz algebra, where $\rightarrow$ is defined by $x \rightarrow y = (\neg x) \vee y$, for $x, y \in B$.

(2) If $(X, \mathcal{T})$ in a topological space, then $(\mathcal{T}, \rightarrow, \cap, X, 0)$ is a bounded Hertz algebra, where for $D, D' \in \mathcal{T}, D \rightarrow D' = \mathrm{int}\,((X - D) \cup D'), 1 = X$ and $0 = \emptyset$

The origin of Hilbert algebras is in mathematical logic. In 1923, Hilbert was the first who remarked the possibility of study a very interesting part of the classical propositional calculus, taking as axioms only the Hilbert's ones concerning only for logical implication (this field is known as "*positive implicative propositional calculus*").

In the rest of this resume, by A we denote a bHa. For $x, y, \in A$, we denote by

$x \,|\!\sqcup\, y = x^* \to y$, where $x^* = x \to 0$ (we have $x, y, \leq x \,|\!\sqcup\, y$).

A Ha A' is called *Hilbert algebra of fractions of A* if :

(1) A is a Hilbert subalgebra of A' and

(2) For every $a', b', c' \in A', a' \neq b'$, there exists $a \in A$ such that $a \,|\!\sqcup\, a' \neq a \,|\!\sqcup\, b'$ and $a \,|\!\sqcup\, c' \in A$ (as a notational convenience, we write $A \leq A'$ to indicate that A' is a Ha of fractions for A). $Q(A)$ is the *maximal Ha of quotients of A* if $A \leq Q(A)$ and for every A' with $A \leq A'$, then there exists an injective morphism of Hilbert algebras $\mathrm{i} : A' \to Q(A)$.The notions of *Hertz algebra of fractions* and *maximal Hertz algebra of quotients* is defined in an analogous way as for Hilbert algebras. In [D. Busneag · Hilbert algebra of fractions and maximal Hilbert algebras of quotients, Kobe Journal of Math., 5(1988), pp. 161–172] and [D. Busneag : Hertz algebras of fractions and maximal Hertz algebra of quotients, Mathematica Japonica, 3(1993), pp 1–8] we have proved the existence of $Q(A)$ and $Q(H)$ (H a bHza).

The aim of this communication is to prove the next result: *For every bHa A (resp. bHza H) there exists a non-empty set I (resp. I') and a filter F (resp. F') on I (resp. I') such that $Q(A)$ (resp. $Q(H)$) is a subalgebra of a reduced power A^I/F (resp. $H^{I'}/F'$)*

► ALEXANDER CHAGROV, *The undecidability of the tabularity problem for modal logics.*
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A modal logic is called *tabular* if it is characterized by a finite model (or frame). Let $L + \varphi$ ($L \oplus \varphi$) be the closure of $L \cup \{\varphi\}$ under substitution and modus ponens (respectively, substitution, modus ponens and necessitation) and K_n the minimal normal modal logic with n necessity operators $\Box_1, \ldots, \Box_n$.

THEOREM 1. (i) *For no $n \geq 1$, there is an algorithm which can determine, given a formula φ, whether $K_n + \varphi$ is tabular.*

(ii) *For no $n \geq 1$, there is an algorithm which can determine, given a formula φ, whether $K_n \oplus \varphi$ is tabular.*

THEOREM 2. (i) *Let L be a tabular extension of K_n, for $n \geq 1$. There is no algorithm which can determine, given a formula φ, whether $K_n + \varphi = L$. In particular, the consistency problem for extensions of K_n is undecidable.*

(ii) *Let L be a normal tabular extension of K_n, for $n \geq 2$. There is no algorithm which can determine, given a formula φ, whether $K_n \oplus \varphi = L$ In particular, the consistency problem for normal extensions of K_n is undecidable.*

THEOREM 3. *Suppose L is a normal tabular extension of K_1 such that $L \nvdash \Diamond\top$ There is no algorithm which can recognize, given a formula φ, whether $K_1 \oplus \varphi = L$*

It is well known (D. Makinson) that the consistency problem for normal extensions of K_1 is decidable. The second (and the last) decidable property of K_1's extensions known to me is the property of coincidence with K_1.

It may be of interest to note that in parallel with Theorem 3 one can prove

THEOREM 4 *Let L be a consistent normal finitely axiomatizable extension of the provability logic G (e.g., G, $G3$, $G \oplus \Box\bot$). There is no algorithm which can recognize, given a formula φ, whether $K_1 \oplus \varphi = L$*

► LILIA A. CHAGROVA, *Undecidability of the first-order definability of intuitionistic formulas on finite frames.*
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It is well-known that there are difficulties when we transfer negative results about modal formulas to intuitionistic ones. For examples, remind Shehtman's result: there are normal extensions of $S4Grz$, which are incomplete with respect to the topological semantics, but analogous examples for superintuitionistic logics aren't known.

A purpose of the present report is to transfer the main result from [1] to intuitionistic case

THEOREM 1. *The problem of recognizing, given any intuitionistic formula, whether it is first-order definable on the class of all finite intuitionistic frames, is algorithmically undecidable.*

[1] A. V. CHAGROV and L. A. CHAGROVA, *Algorithmic problems concerning first-order definability of modal formulas on the class of all finite frames*, Preprint ML-93-07. University of Amsterdam, 1993.

► WITOLD CHARATONIK, *Solving set constraints with projections.*

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Set constraints give a natural formalism for many problems in program analysis and type inference. The first general results concerning the decidability of set constraints were obtained by Heintze and Jaffar , who studied the so-called definite set constraints. The decidability of systems of set constraints without projections was established by A Aiken and E.L. Wimmers. Later other proofs have been obtained R. Gilleron, S. Tison, and M. Tommasi gave a proof based on automata theoretic techniques and L. Bachmair, H. Ganzinger, and U. Waldmann [1] gave a simple and very elegant proof using the decision procedure for the first order theory of monadic predicates. One more algorithm was presented by A Aiken, D. Kozen, M. Vardi, and E.L. Wimmers.

The problem of decidability of systems of set constraints with projections is much more difficult. There were three different approaches to solve this problem. In each of them the first step toward a general solution was solving the problem of decidability of negative set constraints – a special case of set constraints with projections Three independent proofs were given by Aiken, Kozen and Wimmers, Gilleron, Tison and Tommasi, and Charatonik and Pacholski [2]. So far only our approach gave the solution of the general problem.

Our method is an extension of the results presented in [1] and [2] In [1] the case of negative occurrences of projection symbols was solved by introducing for each projection expression a predicate symbol encoding this expression. In case of unrestricted occurrences of projection symbols the same predicate gives only some approximation of the desired set. Extending the methods of [2] we are able to check if this approximation is correct. The first observation is that each solution of a system of set constraints gives a finite set called a *history* which is sufficient to reconstruct a (perhaps different) regular solution Then, the proof proceeds by distinguishing a small subset, *the skeleton* of a history, which is small, is not always a history, but can be collapsed to obtain a small history. Now, what is left, is to make an exhaustive search through all *small histories.*

The results appeared in W. Charatonik and L. Pacholski, *Set constraints with projections are in NEXPTIME*, 35th annual IEEE symposium on foundations of computer science, 1994, pp. 642–653.

[1] L. BACHMAIR, H. GANZINGER, and U. WALDMANN, *Set constraints are the monadic class*, ***Eighth annual IEEE Symposium on Logic in Computer Science***, 1993, pp 75–83.

[2] W. CHARATONIK and L. PACHOLSKI, *Negative set constraints with equality*, ***Ninth annual IEEE symposium on logic in computer science***, 1994, pp. 128–136.

► MARCELLO D'AGOSTINO AND MARCO MONDADORI, *Classical logic as the limit of a sequence of cut-bounded feasible logics.*

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In this paper we address the problem of coping with the exponential worst-case complexity of proof-search in analytic systems of deduction developed in the Gentzen tradition. Elsewhere [*The taming of the cut*, in Journal of Logic and Computation vil. 4, 1994, pp. 285–319] we have proposed a new notion of classical deduction based on a system called KE and proved that even its *analytic* restriction is essentially more efficient than the tableaux method. KE is a refutation system, like the tableaux method, but all its "operational" rules are linear while the *only* branching rule is a "structural" rule expressing a form of cut (which we call PB). A natural strategy to cope with the problem of exponential worst-case complexity consists in imposing an upper bound on the number of applications of this rule. Call $KE(n)$ the system obtained from KE by allowing at most n analytic applications of PB. We show that :

- $KE(1)$ has a decision procedure which runs in time $O(n^2)$.
- For every fixed n, $KE(n)$ has a polynomial time decision procedure.

Obviously, the set of classical tautologies is obtained as the limit of the sets of tautologies of each $KE(n)$. The interesting point is that low-depth systems are powerful enough for a wide range of applications. In the final part of the paper, we investigate alternative characterizations of the sets of tautologies of each $KE(n)$.

► G. D'AGOSTINO, A. MONTANARI AND A. POLICRITI, *Translating modal formulae as set-theoretic terms.*

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Translation methods for modal logics into classical first-order logic allow to use mechanical theorem provers for predicate calculus as modal theorem provers. A common feature of all the existing translation methods is that in order to be applied directly, the underlying modal logic must have a first-order semantics [2]. The main motivation of the present work is to find a translation method applicable to all normal complete finitely-axiomatizable modal logics, regardless of whether they are first-order complete or not, and whether a semantic characterization is given or not for them. In [1] we propose a set-theoretic translation method that represents any Kripke frame as a set, with the accessibility relation modeled using the membership relation $\in$. The whole translation is carried out in the following finite set theory Ω:

$$x \in y \cup z \leftrightarrow x \in y \vee x \in z, \qquad x \in y \setminus z \leftrightarrow x \in y \wedge x \notin z,$$
$$x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y), \qquad x \in Pow(y) \leftrightarrow x \subseteq y.$$

Ω does not contain the extensionality axioms as well as the axiom of foundation. We will make an essential use of the latter fact since we will be forced to work in universes containing non-well-founded sets and it will be convenient to use universes satisfying AFA.

A modal formula $\phi(P_1, \ldots, P_n)$ is mapped in a set-theoretic term $\phi^*(x, x_1, \ldots, x_n)$ inductively defined as follows:

$$P_i^* \equiv x_i, \qquad (\phi \vee \psi)^* \equiv \phi^* \cup \psi^*, \qquad (\neg\phi)^* \equiv x \setminus \phi^*, \qquad (\Box\phi)^* \equiv Pow(\phi^*)$$

We prove that for $H = K + \psi(P_{j_1}, \ldots, P_{j_m})$, where $\psi(P_{j1}, \ldots, P_{jm})$ is an axioms' schema and K is the minimal normal modal logic, the following holds:

$\vdash_H \phi(P_1, \ldots, P_n)$ iff

$$\Omega \vdash \forall x(\mathit{Trans}(x) \wedge \mathit{Axiom}_H(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots x_n))),$$

where $\mathit{Trans}(x)$ is the formula $\forall y(y \in x \rightarrow y \subseteq x)$ and $\mathit{Axiom}_H(x)$ is the formula

$$\forall x_{j_1}, \ldots, \forall x_{j_m}(x \subseteq \psi^*(x, x_{j1}, \ldots, x_{jm})).$$

The finite axiomatizability of Ω guarantees the possibility of implementing mechanical proof search procedures via the deduction theorem or more specialized techniques. In particular, we are currently exploring the possibility of applying theory resolution techniques incorporating decidability results we obtained about Ω [1]. Moreover, the translation into set-theoretic terms, built using U, $\setminus$, *Pow*, guarantees that the translation of any formula is carried out in the same set-theoretic class of formulae, which does not contain equality (as well as Ω). The set-theoretic class of formulae corresponding to a given modal logic H is therefore completely defined by the translation of the axioms of H. The above points mark the difference between our method and more naive techniques that could be proposed to map modal formulae in a set-theoretic setting, passing (for example) through second order logic.

[1] G. D'Agostino, A. Montanari, and A. Policriti, *Translating modal formulae as set-theoretic terms*, Research Report, Dipartimento di Matematica e Informatica, Università di Udine, 1994.

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- ANUJ DAWAR AND ERICH GRÄDEL, *Generalized quantifiers and 0-1 laws.*

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In this paper we study 0-1 laws for extensions of first-order logic by Lindström quantifiers. We state necessary and sufficient conditions on a quantifier Q expressing a graph property, for the logic FO[Q] – the extension of first-order logic by means of the quantifier Q – to have a 0-1 law.

We use these conditions to show, in particular, that

FO[Ham], where Ham is the quantifier expressing Hamiltonicity, does not have a 0-1 law, and that FO[Rig], where Rig is the quantifier expressing rigidity, does have a 0-1 law. Blass and Harary [1] pose the question whether there is a logic which is powerful enough to express Hamiltonicity or rigidity and which has a 0-1 law (see also [2]). It is a consequence of our results that there is no such regular logic (in the sense of Ebbinghaus [3]) in the case of Hamiltonicity, but there is one in the case of rigidity. This is because, FO[Q] is the minimal regular logic that can define the property expressed by Q.

We also consider sequences of vectorized quantifiers, and show that the extensions of first-order logic obtained by adding such sequences generated by the quantifiers for planarity and 3-colourability both have 0-1 laws.

[1] A. Blass and F Harary, *Properties of almost all graphs and complexes*, ***Journal of Graph Theory***, vol. 3 (1979), pp. 225–240.

[2] K. J. Compton, *0-1 laws in logic and combinatorics*, ***NATO advanced study institute on algorithms and order*** (I. Rival, editor), Kluwer, 1989, pp. 353–383.

[3] H.-D. Ebbinghaus, *Extended logics: The general framework*, ***Model-theoretic logics*** (J. Barwise and S. Feferman, editors), Springer-Verlag, New York, 1985, pp. 25–76.

- JAMSHID DERAKHSHAN, *The structure of topologically CZ-groups satisfying local periodicity conditions.*

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Recall that a group is called (topologically) CZ (as a generalization of linear groups) if it carries a Noetherian T_1-topology such that $x \longrightarrow x^{-1}$, $x \longrightarrow ax$, xa, and $x \longrightarrow a^x$ (for any a) are continuous. Since their introduction by Kaplansky in 1951 [5], a central question has been to determine their structure under additional conditions and to discover the discrepancies between their structure and that of (linear) algebraic groups. Note that there are non-algebraic groups which are CZ in the verbal topology (see for example [1]).

We present a solution of the above problem when the groups satisfy local finiteness conditions. Our results do not use the classification of finite simple groups. We first give a direct proof of the nilpotency of connected locally finite CZ-groups of bounded exponent. This generalizes the results of Bryant-Wilson [2], Bryant-Yen [3], Higgins [4] and Wehrfritz [7]. Otto Kegel [6] provides a proof in the case of the chain condition on centralizers using the classification of finite simple groups It turns out that under certain conditions the simple CZ-groups are just $PSL_2(K)$ for K an infinite locally finite field of odd characteristic. The linearity follows from much fewer assumptions. We prove that $\aleph_0$-categorical CZ-groups with *max–cc* and definable multiplication are abelian by finite. We conjecture that groups of finite Moley rank (or stable) are CZ in some (definable) CZ-topology.

The Burnside conjecture that periodic groups are locally finite fails for CZ-groups, although they are quasi-compact. An application of the Zelmanov Lie algebraic theory of compact groups [8] to study the structure of periodic stable CZ-groups and periodic CZ-skew linear groups appears to be a promising direction for further research.

[1] Roger Bryant, *The verbal topology of a group*, ***Journal of Algebra***, vol. 48 (1977), pp. 340–346.

[2] Roger Bryant and John Wilson, *On locally finite CZ-groups*, ***Quarterly Journal of Mathematics at Oxford***, vol. 30 (1979), pp. 265–270.

[3] Roger Bryant and Ted Yen, *Sylow subgroups of locally finite groups*, ***Arch. Math.***, vol. 31 (1978), pp. 117–119

[4] Carl Higgins, *Solubility in locally soluble cz-groups*, ***Journal of the London Mathematical Society***, vol 15 (1977), pp. 81–87.

[5] Ian Kaplansky, ***An introduction to differential algebra***, Paris, 1957.

[6] Otto Kegel, *Four lectures on Sylow theory*, ***Group theory*** (Kai Nah Cheng and Yu Kaiang Lerong, editors), Walter de Gruyter, 1989.

[7] Bertram Wehrfritz, *A note on periodic locally soluble groups*, ***Arch. Math.***, vol. 18 (1967), pp 577–579.

[8] Efim Zelmanov, *On periodic compact groups*, ***Israel Journal of Mathematics***, vol. 77 (1992), pp. 83–95.

▶ CARLOS A DI PRISCO, *About perfect sets of real numbers*.

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A partition relation of pairs of reals is used to obtain a consistency result about perfect sets of real numbers. Sacks forcing and arguments of A. Miller are used to show the consistency of the partition relation of pairs, and another forcing argument is used to prove the following:

Theorem 1. *Con(ZFC + There is an inaccessible cardinal) implies Con(ZF + DC + There is an ultrafilter on ω + Every set of real numbers contains or is disjoint from a perfect set)*

This theorem also follows from a result of J. Henle which makes use of the dual Ramsey property of Carlson and Simpson.

We will comment on ways in which the theorem can be extended.

[1] A. Miller, *Infinite combinatorics and definability*, ***Annals of Pure and Applied Logic***, vol. 41 (1989), pp. 179–203.

[2] T. J. Carlson and S. G. Simpson, *A dual form of Ramsey's Theorem*, ***Advances in Mathematics***, vol 5 (1984), pp. 265–290.

► V. P. DOBRITSA, *Efficiency and set of computable indexations*

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Introducing of an efficiency into abstract structures such as groups, algebras and models is one of the basic subjects of the theory of constructive models. Various and deep results had been obtained in this direction by efforts of many authors from different countries Consideration of classes of constructive models with respect to their uniform efficiency leads to the notion of computable of constructive models Now the theory of computable classes of constructive models received a large development effort.

The following stage of such kind of considerations is to study the set of computable indexations of computable class of constructive models. This set under the direct sum of computable indexations forms upper semilattice. Connections between efficiency and the set of computable indexations are given in the following statement.

Theorem. a) *There exists computable class of constructive models with non-computable set of computable indexations.*

b) *There exists computable class constructive models with computable set of computable indexations, but whose upper semilattice of computable indexations is not constructivizable.*

c) *There exists computable class of constructive models, whose upper semilattice of computable indexations is constructivizable.*

► DANIEL DZIERZGOWSKI, *Kripke models of intuitionistic TT and NF.*

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Let us define the *intuitionistic part* of a classical theory T as the intuitionistic theory whose proper axioms are identical to the proper axioms of T. For example, Heyting arithmetic HA is the intuitionistic part of classical Peano arithmetic PA.

It's a well-known fact, proved by Heyting and Myhill, that the excluded middle can be derived in the intuitionistic part of ZF. So ZF is equal to its intuitionistic part.

In this paper, we mainly prove that TT, Russell's Simple Theory of Types, and NF, Quine's "New Foundations", are not equal to their intuitionistic part. So, an intuitionistic version of TT or NF, is more naturally obtained than an intuitionistic version of ZF.

In the first section, we present a simple technique to build Kripke models of the intuitionistic part of TT This technique is a simplification of Fitting's forcing-like method (see [1]). We illustrate it with short examples bringing to light bad properties of finite sets if we keep Russell's definition (i.e., the set of finite sets is the smallest set F such that $\emptyset \in F$ and $x \in \{z\} \in F$ for all $x \in F$ and for all z).

The remaining sections are devoted to NF and one of its simple consistent fragment, NF_2. For each of these theories, we exhibit two suitable classical models, and show how to combine them in order to obtain non trivial Kripke models of the intuitionistic restrictions of NF_2 and NF. More precisely, in the case of NF, we use a result of Körner ([2]) to obtain two models $\mathcal{M}$ ond $\mathcal{M}'$ of classical NF, such that, on the one hand, $\mathcal{M} \models \varphi[a_1, \ldots, a_n]$ iff $\mathcal{M} \models \varphi[a_1, \ldots, a_n]$, for all *stratified* formulæ φ and all $a_1, \ldots, a_n \in M$, and, on other hand, $\mathcal{M} \in \sigma$ and $\mathcal{M}' \not\models \sigma$ for some well-chosen *non-stratified* sentence σ. Then by putting $\mathcal{M}$ "on top of" $\mathcal{M}$, we obtain a Kripke model of the intuitionistic part of NF, where the excluded middle is not satisfied for some non-stratified sentence.

[1] MELVIN CHRIS FITTING, ***Intuitionistic logic model theory and forcing***, Studies in Logic, North-Holland, Amsterdam - London, 1969.

[2] FRIEDERIKE KÖRNER, *Cofinal indiscernibles and their application to New Foundations*, ***Mathematical Logic Quarterly***, vol. 40 (1994), pp. 347–356.

► RYSZARD FRANKIEWICZ AND PAWEL ZBIERSKI, *Fat P-sets in the space ω^**.

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We prove that—consistently—in the space $\omega^* = \beta\omega \setminus \omega$ there are no P-sets with the $\mathfrak{c}$-*cc* and any two fat P-sets with the $\mathfrak{c}^+$-*cc* are coabsolute. Moreover, each fat P-set with the $\mathfrak{c}^+$-*cc* has a π-base tree of height ω.

► MARCELO F. FRIAS, ARMANDO M. HAEBERER, PAULO A. S. VELOSO AND GABRIEL A. BAUM, *Representability of fork algebras*.

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We consider *proper relation algebras* (*PRA*)

$$\mathcal{P} = \langle P, \cup, \cap, |, \sim, {}', \emptyset, \Delta, V\rangle$$

of binary relations over a set $\mathcal{U}$, with *composition* $|$, *transposition* $\sim$, *diagonal relation* Δ and *universal relation* V as usual. Their abstract versions are *relational algebras* (*RA*'s)

$$\mathcal{R} = \langle R, +, \bullet, ;, -, \breve{}, 0, 1, \infty\rangle$$

characterized by the usual set of axioms [2].

Consider a set $\mathcal{U}$ with an injective operation $\star : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ (constructing pairs $y \star z$). This induces an operation on subsets of $\mathcal{U} \times \mathcal{U}$:

$$r \lhd s = \{\langle x, y \star z\rangle \in \mathcal{U} \times \mathcal{U} : \langle x, y\rangle \in r \wedge \langle x, z\rangle \in s \wedge \langle y, z\rangle \in V\}.$$

A *proper fork algebra* (*PFA*) is a structure

$$\mathcal{F} = \langle F, \cup, \cap, |, \lhd, \sim, {}', \emptyset, \Delta, V\rangle$$

such that F is closed under $\lhd$ and the reduct

$$\mathcal{F} = \langle F, \cup, \cap, |, \sim, {}', \emptyset, \Delta, V\rangle$$

is a *PRA*.

The abstract version of a *PFA* is an *abstract fork algebra* (*AFA*): a structure

$$\mathcal{A} = \langle A, +, \bullet, ;, \nabla, -, \breve{}, 0, 1, \infty\rangle$$

such that

$$\langle A, +, \bullet, ;, -, \breve{}, 0, 1, \infty\rangle$$

is an *RA* and the following axioms hold:

(1) $r \nabla s = (r; \pi) \bullet (s; \rho)$,
(2) $(r \nabla s); (p \nabla q)\breve{} = (r; p\breve{}) \bullet (s; q\breve{})$,
(3) $\pi \nabla \rho \preceq 1$, and
(4) $0 \neq t \preceq \infty \otimes \infty \to (\exists v)(\exists w)(0 \neq v \otimes w \preceq t)$;

where *projections* and *cross* product are defined by

$$\pi = (1\nabla\infty)^{\smile} \quad \rho = (\infty\nabla 1)^{\smile} \text{ and } r \otimes s = (\pi; r)\nabla(\rho; s).$$

These projections satisfy the properties of quasi-projections [3], but are rather special. In fact, every atom of an atomic *AFA* is functional.

Other properties of atomic *AFA*'s are: $r \neq 0 \to \pi; r \neq 0 \wedge \rho; r \neq 0$, $r \otimes s = 0 \leftrightarrow r; \infty; s = 0$, $r \otimes s \neq 0 \to \breve{\pi}; (r \otimes s); \pi = r \wedge \breve{\rho}, (r \otimes s); \rho = s$, $(r \otimes s = t \otimes q) \wedge r \neq 0 \wedge s \neq 0 \wedge t \neq 0 \wedge q \neq 0 \to (r = t \wedge s = q)$,

$r \otimes s$ is an atom iff both r and s are atoms with $r, \infty; s \neq 0$, and if r is an atom with $r \preceq \infty\nabla\infty$ then $r = (r; \pi)\nabla(r; \rho)$.

THEOREM 1 (Representability). *Every (complete) atomic* AFA *is isomorphic to a* PFA [1]. (Thus we have a finitely axiomatizable algebra for Classical First-Order Logic with Equality [4]).

An isomorphism $\mathcal{H}$ is defined as in [2] based on certain pairs of atoms of 1, namely those $\langle\alpha, \beta\rangle$ with $\alpha; \infty; \beta \neq 0$ On them, $\otimes$ is an injective operation, as required. The crucial steps now are $\mathcal{H}(\infty) = V$ and $\mathcal{H}(r\nabla s) = \mathcal{H}(r) \lhd \mathcal{H}(s)$.

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[2] B. JÓNSSON and A. TARSKI, *Boolean algebras with operators, part II*, ***American Journal of Mathematics***, vol. 74 (1952), pp. 127–162.

[3] A. TARSKI and S. GIVANT, *A formalization of set theory without variables*, ***American Mathematical Society Colloquium Publications***, vol. 41 (1987).

[4] P. A. S. VELOSO and A. M. HAEBERER, *A finitary relational algebra for classical first-order logic*, ***Bulletin, Section of Logic, Polish Academy of Sciences***, vol. 20 (1991), pp. 52–62.

► SY D. FRIEDMAN AND DORSHKA C. WYLIE, *Core models and fine structure*

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We present a modification of the existing constructions of core models for large cardinals which allows for a smoother fine structure theory.

Our hierarchy is based on embeddings rather than extenders and is designed to allow truncation and condensation properties close to those present in the L-hierarchy.

It is hoped that with such a hierarchy the the coding method for L can be lifted without difficulty to core models.

► A. J. GIL, A. TORRENS AND V. VERDÚ, *An algebraizable Gentzen system for the three-valued Lukasiewicz propositional logic.*

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Let $\mathcal{L} = \{\neg, \to, \wedge, \vee\}$, be a propositional language of type (1,2,2,2) and let $A_3 = (\{0, 1/2, 1\}, \neg, \to, \wedge, \vee)$ be the three-element Lukasiewicz algebra. The three-valued Lukasiewicz deductive system is the pair $S_3 = (\mathcal{L}, \vdash_3)$, where the consequence relation $\vdash_3$ is defined in the following way: for all $\Sigma \cup \{\varphi\} \subseteq \mathcal{F}m_{\mathcal{L}}$, $\Sigma \vdash_3 \varphi$ iff for all $h \in hom(\mathcal{F}m_{\mathcal{L}}, A_3)$, $h(\Sigma) \subseteq \{1\}$ implies $h\{\varphi\} = 1$.

By using [5] we obtain an axiom system for a three-dimensional Gentzen system $\mathcal{G}_3 = (\mathcal{L}, \mathrel{|\!\sim}_{\mathcal{G}_3})$ (for the definition of n-dimensional Gentzen systems, see the abstract *Algebraizable Gentzen systems* in this volume) such that:

(1) This axiom system has the cut-elimination property.

(2) $\mathcal{G}_3$ is algebraizable in the sense of J. Rebagliato and V Verdú ([3] and [4]), and the variety generated by A_3 is the equivalent variety semantics for $\mathcal{G}_3$.
(3) $\mathcal{G}_3$ has the deduction Theorem (in the sense of [3], and [4]).
(4) $\mathcal{G}_3$ allows to obtain $\mathcal{S}_3$ in the following way:
for all $\{\varphi_1, \dots, \varphi_n, \varphi\} \subset \mathcal{F}m_{\mathcal{L}} \varphi_1, \dots, \varphi_n \vdash_3 \varphi$ iff $\emptyset \mid\!\sim_3 \varphi_1, \dots, \varphi_n \mid \varphi_1, \dots, \varphi_n \mid \varphi$

[1] W. J. BLOK and D. PIGOZZI, *Algebraizable logics*, ***Memoirs of the American Mathematical Society***, vol. 396 (1989).

[2] A. J GIL, A TORRENS, and V. VERDÚ, *Algebraizable Gentzen systems for the finite-valued Lukasiewicz logics*, manuscript, 1994.

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[4] ———, *Algebrizable Gentzen systems and the deduction theorem*, manuscript, 1994.

[5] R. ZACH, *Proof theory of finite-valued logics*, Diplomarbeit, Technische Universität Wien, 1993.

► VALENTIN GORANKO, *Hierarchies of strongly expressive modal and temporal logics*.

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We introduce and investigate extensions of the propositional modal and temporal languages with simple syntactic gadgets called *local names* and *reference pointers*. These provide an efficient mechanism for implicit designation of a semantic entity (possible world, time instant etc.) within a formula by inserting a "local name", and then making references to that entity from elsewhere in that expression by using "reference pointers".

A simple scheme within a formula φ looks like this : $\varphi = \dots \downarrow (\dots \uparrow \dots \uparrow \dots) \dots$, possibly iterated. Semantically (in Kripke semantics) it means the following when the evaluation process of the formula φ reaches the subformula $\downarrow (\dots \uparrow \, . \, . \uparrow \dots)$ the actual point (at which the evaluation is being done at that moment) of the model receives the local name $\uparrow$ and thus becomes the denotation for every "free" (not in the scope of another $\downarrow$) occurrence of the reference pointers $\uparrow$ within the subformula.

The formal languages thus extended come considerably closer to natural languages since the reference pointers play the role of adverbs ("there", "then"), so much missed in the modal and temporal languages.

Adding several reference sets "local name – pointers" we obtain hierarchies of increasingly expressive, very powerful modal and temporal languages. Eventually they converge to a common limit which covers the Π^1_1-fragment of the monadic second-order language for relational structures.

We discuss the expressive power of the languages in these hierarchies, axiomatize their logics and prove undecidability of the satisfiability problem in them.

Finally we consider extensions of these languages by means of *locally definable operators* and show a general completeness result for these extensions.

► L. GORDEEV, *Cut free formalization of predicate logic with finitely many variables*.

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Let $\mathcal{L}_n$ be $\{\top, \bot, \vee, \wedge, \neg, \forall, \exists\}$-language of the 1-order predicate logic without function symbols and equality ($\neg$ acting on atomic level), whose variables are v_i for $0 \leq i \leq n \leq \omega$. Let $\mathbf{RPC}_n$ be the term rewriting system in $\mathcal{L}_n$ with the following rules (*elementary reductions*).

$$
\begin{array}{lrcl}
(\tau) & \Delta \vee L \vee \Pi \vee \neg L & \longrightarrow & \top \\
(\iota) & A \vee \top & \longrightarrow & \top \quad \text{and} \quad \top \vee A \longrightarrow \top \\
(\hat{\iota}) & A \wedge \top & \longrightarrow & A \quad \text{and} \quad \top \wedge A \longrightarrow A \\
(\alpha) & A \vee (B \vee C) & \longrightarrow & A \vee B \vee C \\
(\delta) & (A \wedge B) \vee C & \longrightarrow & (A \vee C) \wedge (B \vee C) \\
 & & & \quad \text{and} \quad C \vee (A \wedge B) \longrightarrow (C \vee A) \wedge (C \vee B) \\
(\pi) & \Delta \vee \forall x A \vee \Pi & \longrightarrow & \Delta \vee \forall x A \vee A[\div x] \vee \forall y (\Delta[\div y] \\
 & & & \quad \vee A[x \div y][x/y] \vee \Pi[\div y]) \vee \Pi \\
(\sigma) & \Delta \vee \exists x A \vee \Pi & \longrightarrow & \Delta \vee \exists x A \vee A[x/t] \vee \exists y (\Delta[\div y] \\
 & & & \quad \vee A[x \div y][x/y] \vee \Pi[\div y]) \vee \Pi \quad .
\end{array}
$$

($L :=$ literals; $\Delta, \Pi :=$ "long" disjunctions or $\emptyset$; $A, B, C :=$ formulas; $x, y :=$ variables; $t :=$ terms; $[\div x]$, $[x \div y]$, $[x/t] :=$ "deletion", "subtraction", "substitution" which do not increase the length.) A is called *derivable* in $\mathbf{RPC}_n$ ($\mathbf{RPC}_n \vdash A$) iff there is a reduction chain starting with A and ending by $\top$. (*Reductions* arise by applying elementary reductions to arbitrary subformulas.)

THEOREM 1. $\mathbf{RPC}_n \vdash A$ *iff A is derivable in the standard modus ponens predicate calculus in the presence of at most n distinct variables*

THEOREM 2 *There is a polynomial embedding $\mu^* : \mathcal{L}_\infty \longrightarrow \mathcal{L}_3$ such that for any formula A of $\mathcal{L}_\infty$, $\mu^*(A)$ is a formula of $\mathcal{L}_3$, and for every $n > 3$*

$$\mathbf{RPC}_n \vdash \mu^*(A) \;\; \textit{iff} \;\; \mathbf{RPC}_\infty \vdash A \;\; \textit{iff} \;\; \models A$$

In particular, A is valid iff $\mu^*(A)$ is derivable in $\mathbf{RPC}_4$.

Theorem 2 is also true of the particular "minimal" language $\mathcal{L}_3$ containing only one binary predicate and no individual constant.

THEOREM 3. *For each $n > 2$ there is a sentence of the minimal language $\mathcal{L}_3$ (as above) which is derivable in* $\mathbf{RPC}_{n+1}$, *but not in* $\mathbf{RPC}_n$

If $\mathcal{L}_n$ also includes the equality "$\equiv$", the correlated calculus with equality, $\mathbf{RPCE}_n$, extends $\mathbf{RPC}_n$ (conservatively) by the following elementary reductions.

$$
\begin{array}{lrcl}
(\varepsilon) & x \equiv x & \longrightarrow & \top \\
(\gamma) & x \not\equiv y & \longrightarrow & x \not\equiv y \vee y \not\equiv x, \;\; if \;\; x \neq y \\
(\lambda) & \Delta \vee x \not\equiv y \vee \Pi & \longrightarrow & \Delta \vee \Delta[x/y] \vee x \not\equiv y \vee \Pi[x/y] \vee \Pi, \;\; if \;\; x \neq y
\end{array}
$$

THEOREM 4. $\mathbf{RPCE}_n \vdash A$ *iff A is derivable in the standard modus ponens predicate calculus with equality in the presence of at most n distinct variables.*

COROLLARY 5. *The theorems of Tarski's Relation Algebra are, modulo the canonical interpretation, the three-variable sentences derivable in* $\mathbf{RPCE}_4$.

► WILLIAM G. HANDLEY, *Nondeterministic summation mod $\mathcal{T}_3$, the semigroup of functions on* $\{0, 1, 2\}$.

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Deterministic summation *modulo* an arbitrary semigroup is an extension of the schemes for bounded sum and bounded product used to define the class of Csillag-Kalmar elementary functions. For an arbitrary class $\mathcal{R}$ of relations on natural numbers, one has the closure $\mathcal{T}_n - \Delta_0(\mathcal{R})$ under deterministic summation *mod* $\mathcal{T}_n$, the semigroup of functions with domain and codomain $\{0, 1, \ldots, n-1\}$.

Using P. Clote's definition of nondeterministic summation, which extends this even further, we consider the closure $\mathrm{N}\overline{\mathcal{T}_n} - \Delta_0(\mathcal{R})$ and show that

$$\mathrm{N}\overline{\mathcal{T}_3} - \Delta_0(\mathcal{R}) = \overline{\mathcal{T}_3} - \Delta_0(\mathcal{R})[= \overline{\mathcal{S}_3} - \Delta_0(\mathcal{R})].$$

This resolves a question in [1]. The case $n = 4$ remains open.

We also show that if $\mathcal{G}$ is any finite group then

$$\mathrm{N}\overline{\mathcal{G}} - \Delta_0(\mathcal{R}) = \overline{\mathcal{G}} - \Delta_0(\mathcal{R}).$$

[1] P. Clote, *Stack register machines: non-determinism, the linear time hierarchy, and uniform boolean circuit classes*, (Extended Abstract), 1993.

▶ KOSTAS HATZIKIRIAKOU, *WKL$_0$ and Stone's separation theorem for convex sets.*

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This paper is a contribution to the program of Reverse Mathematics; see [2]. We prove that Stone's Separation Theorem for convex sets (in countable vector spaces over a countable ordered field) is provable in WKL$_0$. Moreover, we prove that a version of it is equivalent to Weak König's Lemma As a corollary, we obtain a result due (independently) to I. Kalantari and R. Downey, i.e., there exist two r.e. convex subset of V_∞ which are disjoint but they cannot be separated by a recursive convex subset of V_∞ whose complement is also convex; see [1]. The main theorems of the paper are:

Theorem 1 (WKL$_0$). *Let V be a countable vector space over a countable ordered field C Let C be a Σ^0_1 convex subset of V such that $O \notin C$. Then, there exists a hypercone $K(o)$ of V such that $C \subseteq K(o)$.*

Theorem 2 (WKL$_0$). *Let V be a countable vector space over a countable ordered field F. Let C, D be disjoint convex sets Then, there exists a convex set H such that V/H is convex, $C \subseteq H$ and $D \subseteq V/H$.*

Theorem 3 (RCA$_0$). *The following are equivalent.*

(1) *Weak König's Lemma.*

(2) *Let V be a countable vector space over a countable ordered field F. Let C, D be disjoint Σ^0_1 convex sets. Then, there exists a convex set H such that V/H is convex, $C \subseteq H$ and $D \subseteq V/H$.*

[1] R. Downey, *Some remarks one a theorem of Iraj Kalantari concerning convexity and recursion theory*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 30 (1984), pp. 303–312.

[2] H. Friedman, S. G. Simpson, and R. Smith, *Countable algebra and set existence axioms*, ***Annals of Pure and Applied Logic***, vol. 25 (1983), pp. 141–181, Addendum in vol 27 (1985) pp 318–320

▶ LAURI HELLA AND JOUKO VÄÄNÄNEN AND DAG WESTERSTÅHL, *Definability of polyadic lifts of generalized quantifiers.*

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We study generalized quantifiers on finite structures. The best understood among such quantifiers are the monotone simple monadic ones. With every function $f : \omega \to \omega$ we can associate a monotone simple monadic quantifier Q_f by letting $Q_f x\phi$ say "there are at least $f(n)$ elements x satisfying ϕ, where n is the size of the universe". Our results hold also for the relativized version $Q'_f xy\phi\psi$ which says "of the n elements that satisfy ψ at least $f(n)$ satisfy ϕ". We say that f is *unbounded*, if $\forall m \exists n (m \leq f(n) \leq n - m)$, otherwise *bounded*. Canonical examples of unbounded functions are $\lfloor \frac{n}{2} \rfloor$, $\lfloor \sqrt{n} \rfloor$ and $\lfloor \log(n) \rfloor$, and of bounded ones $f(n) = 5$ for even n and $f(n) = n - 6$ otherwise.

In this paper we study *polyadic lifts* of such quantifiers Q_f. Unlike monadic quantifiers, that only talk about relative numbers of elements, polyadic quantifiers talk about properties of tuples. To say that a graph contains so and so many vertices of degree 3 one uses a monadic quantifier of the form Q_f. But to say that a graph contains a clique of such and such size, one needs a polyadic quantifier. *Lifting* is a way of transforming monadic quantifiers Q into polyadic ones $Lift(Q)$. We study the following two particular lifts.

The *k-ary Ramsey* quantifier defined by Q_f is the quantifier $Ram^k(Q_f)x_1 ... x_k\phi$ which holds in a structure of size n if and only if there is a subset X of the universe of the model such that $|X| \geq f(n)$ and every k-tuple of different elements from X satisfies ϕ. So with $Ram^2(Q_f)$ we can say a graph has a clique of size $\geq f(n)$.

The *k-ary Branching* of Q_f is the quantifier $Br^k(Q_f)x_1 ... x_k\phi$ which holds in a structure of size n if and only if there are sets $X_1, \ldots, X_k$ of size $\geq f(n)$ so that every k-tuple in $X_1 \times \ldots \times X_k$ satisfies ϕ. With $Br^2(Q_f)$ we can say "at least $f(n)$ boys in my class and $f(n)$ girls in your class have dated each other this year".

A natural question to ask about a lift is, whether $Lift(Q)$ is definable in $\mathcal{L}_{\infty\omega}(Q)$ or not, that is, whether the lift brings anything new at all. We have the following general answer to this question:

(1) $Ram^k(Q_f)$ is definable in $\mathcal{L}_{\infty\omega}(Q_f)$ if and only if f is bounded.

(2) $Br^k(Q_f)$ is definable in $\mathcal{L}_{\infty\omega}(Q_f)$ if and only if f is bounded.

Actually we prove more: if f is unbounded, then $Br^k(Q_f)$ is not definable in the extension of first order logic by any monadic quantifiers, and $Ram^{k+1}(Q_f)$ is not definable in the extension of $\mathcal{L}^{\omega}_{\infty\omega}$ by any finite number of k-ary quantifiers. We discuss also other lifts.

► SACHIO HIROKAWA, YUICHI KOMORI AND IZUMI TAKEUTI, *A reduction rule for Peirce's formula.*

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Implicational theorems in classical logic are characterized by

$$S : (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma, K : \alpha \to \beta \to \alpha$$

and *Pierce's formula* P : $((\alpha \to \beta) \to \alpha) \to \alpha$. The first two determine the intuitionistic logic and have the reduction rules $Sxyz \Rightarrow xz(yz)$ and $Kxy \Rightarrow x$. This two reduction rules correspond to the normalization of proofs in Natural Deduction System, hence to the β-reduction of typed λ-terms [1, 3]. What is a reduction rule for Pierce's formula? What kind of equivalence of proofs is derived from such a reduction rule? In other words, how do we extend the Curry-Howard isomorphism [2] to the classical logic ?

We introduce a reduction rule (P-reduction).

$$x^{\alpha\to\beta}(P^{((\alpha\to\beta)\to\alpha)\to\alpha}y^{(\alpha\to\beta)\to\alpha}) \xRightarrow{P} x(yx).$$

It represents the following transformation of proof figures in classical logic which eliminates superfluous occurrence of Pierce's axiom.

$$\dfrac{\begin{array}{c}\vdots D_1\\ \alpha\to\beta\end{array} \qquad \dfrac{((\alpha\to\beta)\to\alpha)\to\alpha \qquad \begin{array}{c}\vdots D_2\\ (\alpha\to\beta)\to\alpha\end{array}}{\alpha}}{\beta} \quad \xRightarrow{P} \quad \dfrac{\begin{array}{c}\vdots D_1\\ \alpha\to\beta\end{array} \qquad \dfrac{\begin{array}{c}\vdots D_2\\ (\alpha\to\beta)\to\alpha\end{array} \qquad \begin{array}{c}\vdots D_1\\ \alpha\to\beta\end{array}}{\alpha}}{\beta}$$

We prove the following theorem which says that all the proofs of the same formula are equivalent with respect to the βP–convertibility.

LEMMA $x^{\alpha\to\alpha}y^{\alpha} =_{\beta P} x(xy)$.

THEOREM $A^{\alpha} =_{\beta P} B^{\alpha}$

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[2] W. A. HOWARD, *The formulae-as-types notion of construction,* ***To H. B. Curry, essays on combinatory logic, lambda calculus and formalism*** (Hindley and Seldin, editors), Academic Press, 1980, pp. 479–490.

[3] D PRAWITZ, ***Natural deduction,*** Almqvist and Wiksell, Stockholm, 1965

▶ TAKAO INOUÉ, *A generalized Gödel's Theorem.*

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Let X be classical predicate (propositional) logic without equality **CQC** (**CPC**).

Let Y be intuitionistic predicate (propositional) logic without equality **IQC** (**IPC**).

For any atomic formula p of Y, we define two translations $id_{\delta\{p\}}$ and $id_{\iota\{p\}}$ from X to Y as follows: *for any formula A of X, $id_{\delta\{p\}}(A) = id_{\iota}(A) \vee p$ and*

$$id_{\iota\{p\}}(A) = \begin{cases} id(A) \supset p. \supset p & \text{if } p \neq \perp \text{ and } p \text{ does not occur in } id(A); \\ \neg\neg id(A) & \text{if } p = \perp; \\ id(A) & \text{otherwise} \end{cases}$$

where id is identity translation.

A pair $(\tilde{p}, \sigma)$ is said to be a *pre-G-sequence of Y with length n* if it satisfies the following conditions (1) and (2):

(1) $\tilde{p}$ is an ordered finite sequence $p_1, p_2, \ldots, p_n (n \geq 1)$ of atomic formulas of Y;

(2) σ is a mapping from $\{1, 2, \ldots, n\}$ to $\{\delta, \iota\}$.

A pre-G-sequence $(\tilde{p}, \sigma)$ of Y with length n, say $\tilde{p} = (p_1, p_2, \ldots, p_n)$ is said to be a *non-$\perp$-sequence of Y with non-$\perp$-index i*, if $\exists 1 \leq j \leq n(p_j \neq \perp)$ holds with $i = max\{j : p_j \neq \perp, 1 \leq j \leq n\}$, otherwise it is called a *$\perp$-sequence of Y*.

A pre-G-sequence $(\tilde{p}, \sigma)$ of Y with length n, say $\tilde{p} = (p_1, p_2, \ldots, p_n)$ is said to be a *G-sequence of Y*, if it satisfies one of the following conditions (a) and (b):

(a) it is a $\perp-$ sequence of Y;

(b) it is a non-$\perp$-sequence of Y with non-$\perp$-index i such that if $i \geq 2$, then $\forall 1 \leq j \leq i - 1[p_j = \perp \bigwedge \sigma(j) = \iota. \supset \neg\exists j + 1 \leq k \leq i(p_k \neq \perp \bigwedge \sigma(k) = \delta)]$.

Let $(\cdot)^{\circ}$ be Gödel translation from X to Y. For any pre-G-sequence $(\tilde{p}, \sigma)$ of Y with length n, say $\tilde{p} = (p_1, \ldots, p_n)$, we define a translation $\zeta_{\sigma}(\tilde{p})$ from X to Y as follows:

for any formula A of X,

$$\zeta_{\sigma}(\tilde{p})(A) = id_{\sigma(1)\{p_1\}}(id_{\sigma(2)\{p_2\}}(\cdots(id_{\sigma(n)\{p_n\}}(A^{\circ}))\cdots))$$

Let $\mathcal{G}$ be an equivalence class $\{\zeta_{\sigma}(\tilde{p}) : (\tilde{p}, \sigma)$ is a G-sequence of $Y\}/_{\equiv_c}$, which is called a *generalized Gödel translation from* X *to* Y, where $\equiv_c$ is C-equivalence (for (non-)C-equivalence, see [2])

Main results: For any pre-G-sequence $(\tilde{p},\sigma)$ of Y, $\zeta_\sigma(\tilde{p})$ is an embedding of X in Y iff $(\tilde{p},\sigma)$ is a G-sequence of Y (In particular, for any $\perp$-sequence $(\tilde{p},\sigma)$, $(\cdot)^\circ$ and $\zeta_\sigma(\tilde{p})$ are C-equivalent.) The cardinality of $\mathcal{G}$ is $\aleph_0$. We give a necessary and sufficient condition of the non-C-equivalence of elements of $\mathcal{G}$.

The full description of the results and their full proofs are given in [1].

[1] T. Inoué, *Corrections and additions to my paper "A note on unprovability-preserving sound translations", part I; part II; part III*, Forthcoming. It is possible that the titles of these papers will be changed.

[2] ———, *A note on unprovability-preserving sound translations*, ***Logique et Analyse***, vol. 33 (1990), pp. 243–257, (This paper was published in December, 1993).

► ALEXANDRE IVANOV, *Finite covers of ω-categorical structures and cohomology.*

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Let N be a countably categorical transitive structure and F be a finite set. A structure $M = N \cup (N \times F)$ with the projection $N \times F \to N$ and the N- relations is a *cover* of N if the natural homomorphism from $Aut(M)$ into $Aut(N)$ is surjective A cover M is split if it admits a covering expansion M' with trivial *kernel* (the kernel of $Aut(M') \to Aut(N)$) M is strongly split if there is a partition of $N \times F$ by some its transversals $P_1, \ldots, P_k (\forall x \in N \exists! f \in F : (x,f) \in P_i)$ such that the expanded structure $(M, P_1, \ldots, P_k)$ is a cover of N too.

We classify all finite covers of highly homogeneous structures. In particular, we prove that they can be non-split and can be split but not strongly split. We show that the Cameron's cohomology approach to two-graphs can be applied to construction of non-split covers of some wide class of structures including some highly homogeneous ones. We also apply these ideas to construction of *universal* covers. It is shown that some natural assumptions on the group of strong automorphism imply a complete characterization of finite covers.

► JIHAD JAAM, *Coloriages cycliques pour les hypergraphes complets associés aux nombres de Ramsey classiques ternaires.*

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Soit X un ensemble de n éléments, et soit $\mathcal{P}_r(X)$ l'ensemble des parties (une famille) à r éléments de X $\left(|\mathcal{P}_r(X)| = \binom{n}{r} = \frac{n!}{r!(n-r)!}\right)$. Le couple $H = \left(X, \mathcal{P}_r(X)\right)$ constitue un*hypergraphe complet de rang r et d'ordre n*. Les éléments de X et de $\mathcal{P}_r(X)$ désignent respectivement les sommets et les arêtes de H. Chaque arête de H est donc un ensemble de cardinal r Un tel hypergraphe est appelé une*n-clique de rang r* et se note K_n^r Nous allons introduire le théorème général de Ramsey sous forme de coloration des arêtes de H. Considérons m nombres entiers $k_1, k_2, \ldots, k_m > r$ et m couleurs différentes $\{1, 2, \ldots, m\}$. On définit une $(k_1, k_2, \ldots, k_m; r)$*-coloration* comme étant la fonction qui affecte à chaque arête de K_n^r une seule couleur parmi m couleurs possibles, de sorte qu'aucune clique de k_i sommets et de rang r n'ait toutes ses arêtes de couleur i. Cette fonction est appelée*une bonne coloration des arêtes de* K_n^r. Un tel hypergraphe est dit $(k_1, k_2, \ldots, k_m; r)$*-colorable.*

Le théorème de Ramsey (1930) affirme l'existence d'un plus petit entier fini $R(k_1, k_2, \ldots, k_m; r)$ tel que tout hypergraphe complet de rang r avec un nombre de sommets $n \geq R(k_1, k_2, \ldots, k_m; r)$ n'admette aucune bonne $(k_1, k_2, \ldots, k_m; r)$-coloration pour ses arêtes. Ce nombre s'appelle*le nombre de Ramsey classique de rang r en* $k_1, k_2, \ldots, k_m$. En d'autres

termes, tout coloriage des arêtes de K_n^r avec $n = R(k_1, k_2, \ldots, k_m; r)$, contient au moins une k_i-clique monochrome de rang r et de couleur i.

L'évaluation de chaque nombre de Ramsey est un problème NP-difficile. On se contente très souvent d'améliorer les bornes inférieures et supérieures de ces nombres. Ainsi, pour améliorer leur minorant, on cherche un hypergraphe K_n^r avec n le plus grand entier possible, pour lequel il existe une bonne $(k_1, k_2, \ldots, k_m, r)$-coloration. De même, pour améliorer leur majorant, on étudie le plus petit entier n $\left(n \geq R(k_1, k_2, \ldots, k_m; r)\right)$, pour lequel il n'existe aucune bonne $(k_1, k_2, \ldots, k_m; r)$-coloration. On construit très souvent les bonnes colorations on se servant principalement des propriétés des coloriages cycliques pour les graphes, introduites par J.G. Kalbfleisch en 1966. Nous introduisons dans cet article, les coloriages cycliques pour les hypergraphes.

Nous proposons quelques algorithmes d'optimisation stochastique pour évaluer (ou approximer) certains nombres de Ramsey classiques binaires et ternaires. Nous montrons que les coloriages cycliques existent non seulement pour les graphes complets associés aux nombres de Ramsey classiques binaires mais également pour les hypergraphes complets associés aux nombres de Ramsey classiques ternaires. Nous utilisons pour cela la notion de longueur des arêtes des hypergraphes. Nous démontrons quelques propriétés d'inexistence de certains coloriages cycliques pour ces graphes et hypergraphes.

Mots clés : coloriage cyclique, nombre de Ramsey, graphe, hypergraphe.

▶ FRED JOHNSON, *Contingent syllogisms.*

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Storrs McCall (***Aristotle's Modal Syllogisms***, North-Holland, Amsterdam, 1963) gives a deductive system, Q-L-X-M, that is intended to organize Aristotle's judgments about the validity of assertoric, apodeictic, and contingent syllogisms. Q-L-X-M is shown to generate theses that McCall overlooked. All syllogisms of the form $(QAbc \rightarrow (LAab \rightarrow x))$ and $(LAbc \rightarrow (QAab \rightarrow x))$ are theses, though Aristotle claims that Barbara-QLX and Barbara-LQX are invalid. That McCall overlooked no other theses is shown by means of a formal semantics for the system. Since there are 29 discrepancies with Aristotle's judgments about the validity of "basic syllogisms." An alternative to Q-L-X-M is proposed that generates fewer discrepancies. A natural and Aristotelian semantics is given for the latter system, using some of the semantic conditions given by Fred Johnson (*Models for Modal Syllogisms*, ***Notre Dame Journal of Formal Logic***, vol. 30 (1989) pp. 271–284 and *Modal Ecthesis*, ***History and Philosophy of Logic***, vol. 14 (1993), pp. 171–182) and S. K. Thomason (*Semantic Analysis of the Modal Syllogistic*, ***Journal of Philosophical Logic***, vol. 22 (1993), pp. 111–128).

▶ M'BOKA KIESE, *La logique chaotique.*

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Bien qu'elles se soient appuyées sur des techniques inouïes de l'intelligence artificielle, les mathématiques, en offrant une interprétation empirio-éclectique aux phénomènes du chaos et du désordre, restent tributaires de façon inductive, de la logique classique. Dès que l'on étudie de tels phénomènes, les principes aristotéliciens de l'identité, du tiers exclu générateurs des structures d'ordre, deviennent caducs. Cela nécessite, par conséquent de nouvelles méthodes, un nouveau langage et surtout une nouvelle logique. La pensée du chaos soulève une question plus radicale liée au renouvellement du paradigme de la rationalité mathématique dont les injonctions sont graves. L'adoption de nouveaux principes on ne peut plus stoïciens que voici:.

le principe d'altérité: A une proposition $e(c)$ (lire, "e contre c") s'urgit la proposition

alterne $c(e)$; on écrit $e(c)$ P $c(e)$ (lire, $\ll$ $e(c)$ *puis* $c(e)$ $\gg$) ;

le principe de densité: A deux propositions *alternes* $e(c)$ et $c(e)$, on peut trouver une tierce proposition h qui est l'*objet* de l' *altérité* telle que $[e,c]$ (h) $[c,e]$ (lire, "*e, c* contre h et *c, e* pour h") ;

le principe de mobilité: Deux propositions sont *en mouvement* $e(c)$ et $(e)c$ (lire"c pour e") ou forment *une controverse* quand elles s'opposent de telle manière que la proposition alterne de $e(c)$ dénie la propriété exprimée par $(e)c$; on écrit $e(c) \parallel (e)c$ (lire $\ll$ $e(c)$ *pendant que* $(e)c$ $\gg$*)*; grâce à ces principes, nous pouvons comprendre le déroulement des *situations chaotiques* au niveau logique.

Nous avons élaboré une fondation des *mathématiques chaotiques*. La *logique chaotique avec altérité* est la logique interne aux *mathématiques chaotiques*. La *théorie des modes* est la sémantique de la logique chaotique. *La théorie axiomatique des espèces*, avec ses axiomes propres est un modèle *concurrent* à théorie des ensembles selon Zermelo-Frankael. Les *Greegébres* sont une extension de la *théorie des espèces*. Pour ce qui concerne des *Espaces mbuuka*, ce sont des espaces non topologiques où l'idée de la forme ne se confond pas avec l'idée de l'ordre. Chaque théorie fondationnelle est une méthode conduisant vers la compréhension du chaos. Ces cinq théories s'interpénètrent comme les cinq doigts de la main, au point de former une quintessence pour les "mathématiques à venir."

► ANDRZEJ KISIELEWICZ, *Ideal calculus with two membership symbols.*

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Introducing double meaning for the membership relation makes possible to return to ideas of the original Frege proposal and base set theory on a single axiom-schema of comprehension.

The language of the presented system, denoted by IC2, is that of the first order predicate calculus with equality, supplemented by two binary predicate symbols $\in$ and $\mathcal{E}$. Any formula $\varphi(x)$ making no use of the symbol $\mathcal{E}$ is called *normal*. The *dual* of this is the formula $\overline{\varphi}(x)$ obtained from $\varphi(x)$ just by replacing each occurrence of $\in$ by that of $\mathcal{E}$. Let $R(y)$ abbreviate $\forall x(x \in y \leftrightarrow x \mathcal{E} y)$ (which means that for the set y both the membership relations coincide). A set y having this property or satisfying either $\forall x(x \in y \rightarrow R(x))$ or $\forall x(x \mathcal{E} y \rightarrow R(x))$ will be called *regular*. The nonlogical axioms of IC2 are the following.

I. AXIOM OF EXTENSIONALITY. *For all sets x,y,* $\forall z(z \in x \leftrightarrow z \mathcal{E} y) \rightarrow x = y$.

II. AXIOM OF COMPREHENSION. *For all regular sets* $x_1, \ldots, x_n$, $\exists y \forall x(x \mathcal{E} y \leftrightarrow \varphi(x) \wedge x \in y \leftrightarrow \overline{\varphi}(x))$, *where* $\varphi(x)$ *is any normal formula in which no other variable than* $x, x_1, \ldots, x_n$ *occurs free, and* $\overline{\varphi}(x)$ *is its dual.*

We show that the universe IC2 contains all (regular) sets whose existence is provided by the axioms of ZF, and therefore, IC2 allows as trouble-free development of mathematics as ZF does. No familiar paradoxes can be reproduced in IC2. The system contains a mechanism preventing reproducing antinomies. However, the following is still open:

Problem. *Is there any consistency proof of IC2 relative to a presently accepted system of set theory?*

► ULRICH KOHLENBACH, *Exploiting partial constructivity relatively to non-constructive lemmas in given proofs.*

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We show how a new monotone version of modified realizability can be used to extract bounds on $\bigvee w^2$ in sentences $G :\equiv \bigwedge u^1 \bigwedge l \leq_{\gamma} tu(\neg C \rightarrow \bigvee w^2 D)$ which are proved in (sub-)systems $\mathcal{T}_i^\omega$ of intuitionistic arithmetic E-HA$^\omega$ (in the language of all finite types) **plus arbitrary non-constructive analytical sentences** having the form $F :\equiv \bigwedge x^\delta(A \to \bigvee y \leq_\rho sx\neg B)$, where A, B, C, D are **arbitrary** formulas $\in \mathcal{L}(\mathcal{T}_i^\omega)$ (such that G, F are closed), γ, δ, ρ are arbitrary finite types, s, t are closed terms and $x_1 \leq_{\rho(\tau)} x_2 :\equiv \bigwedge y^\tau(x_1 y \leq_\rho x_2 y)$.

More precisely we prove results of the following type:

THEOREM. *If* $\mathcal{T}_i^\omega + F + IP_\neg(+AC) \vdash \bigwedge u^1 \bigwedge v \leq_\gamma tu(\neg C \to \bigvee w^2 Dw)$, *then there exists an effectively a closed term* Φ *of* $\mathcal{T}_i^\omega$, *such that*

$$\mathcal{T}^\omega + (b-AC) + F(+AC) \vdash \bigwedge u \bigwedge v \leq tu(\neg C \to \bigvee w \leq_2 \Phi u D w),$$

where AC denotes the schema of choice in the language of functionals of finite types, and

$(b-AC)$: $\bigwedge Z^{\tau\rho}(\bigwedge x^\rho \bigvee y \leq_\tau Zx\, A(x,y,Z) \to \bigvee Y \leq_{\tau\rho} Z \bigwedge x A(x, Yx, Z))$,
$IP_\neg$: $(\neg A \to \bigvee y^\sigma B) \to \bigvee y^\sigma(\neg A \to B)$ (y not free in A),
$\mathcal{T}^\omega$: $\mathcal{T}_i^\omega +$ classical logic.

EXAMPLES FOR LEMMAS F are:

(1) attainment of the maximum of $f \in C([0,1]^n, \mathbf{R})$ on $[0,1]^n$,
(2) mean value theorems,
(3) Cauchy-Peano existence theorem,
(4) Brouwer's fixed point theorem for $\mathbb{R}^n$,
(5) existence of best approximations,
(6) "lesser limited principle of omniscience",
(7) weak König's lemma WKL,
(8) comprehension for negated formulas : $\bigvee \Phi \leq_{0\rho} \lambda x^\rho.1 \bigvee y^\rho(\Phi y =_0 0 \leftrightarrow \neg A(y))$

▶ IVAN KOREC, *Theories of generalized Pascal triangles.*

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Generalized Pascal triangles /GPT/ are mappings of some cofinite subsets of $\mathbf{N} \times \mathbf{N}$ into finite sets associated to some finite algebras analogously as the classical Pascal triangle can be associated to the (infinite) algebra $\langle \mathbf{N}; +, 0 \rangle$. More formally, for every finite algebra $\mathcal{A} = \langle \mathbf{A}; *, o \rangle$ (with $\mathbf{A} \subseteq \mathbf{N}$ to avoid some technical problems) and every word $w = w_0 \dots w_{|w|-1} \in \mathbf{A}^+$ the function $G =$ GPT$(\mathbf{A}, w)$ will be defined by

$$G(x,y) = \begin{cases} \text{undefined} & \text{if } x+y < \mid w \mid -1, \\ w_x & \text{if } x+y = \mid w \mid -1, \\ \mathrm{o} * G(0, y-1) & \text{if } x = 0,\ y \geq \mid w \mid, \\ G(x-1, 0) * \mathrm{o} & \text{if } y = 0,\ y \geq \mid w \mid, \\ G(x-1, y) * G(x, y-1) & \text{if } x+y \geq \mid w \mid,\ x > 0,\ y > 0. \end{cases}$$

For example, if $\mathbf{Z}_n$ is the additive group modulo n then $B_n =$ GPT$(\mathbf{Z}_n, 1)$ is the Pascal triangle modulo n; we have $B_n(x,y) = \binom{x+y}{x}$ MOD n.

Theories of various structures related to GPT and classes of such structures will be investigated. Every considered structure will have the base set $\mathbf{N}$, it will contain a GPT and possibly some additional operations or relations, e g. the successor function $\mathbf{s}$ The elementary theories of such structures can be undecidable already for $\mid \mathbf{A} \mid = 3$; this bound is exact, and does not depend on $\mid w \mid$ and the presence of the additional operation $\mathbf{s}$ or $+$ It is diminished to 2 if the additional operation is $\times$ or the additional relation is the divisibility relation or the set of squares as a unary relation; the necessary GPT for that is B_2. Analogously it is diminished to 2 if theories of classes of structures are considered.

The addition $+$ is not definable in any structure $\langle \mathbf{N}; B_n \rangle$, n prime. However, it is definable in every such structure if $n > 0$ is composite. Moreover, if $n > 0$ has two distinct prime divisors, then $\times$ is definable, too, and hence the elementary theory of $\langle \mathbf{N}, B_n \rangle$ is undecidable.

The addition is definable in the structures $\langle \mathbf{N}; B_n, s \rangle$ n prime, and the defining formula can be chosen independently on n. The elementary theory of every structure $\langle \mathbf{N}; B_n, + \rangle$ is decidable for every prime n. In the proof a reduction to weak monadic second order (WMSO) theory of successor is used.

The operations $+$, $\times$ are not WMSO definable in $\langle \mathbf{N}; B_2 \rangle$. Nevertheless, the WMSO theory of the structure $\langle \mathbf{N}; B_2 \rangle$ is undecidable; the elementary theory of $\langle \mathbf{N}; +, \times \rangle$ can be interpreted in it. The same hold for every prime instead of 2.

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Getmanova's logic uses some rational numbers as logical values [1]. But it is convenient to use much more rational numbers for this aim. More exactly we can take any complete closed set of rational numbers which is symmetrical about zero. For such set we can extend Gotmanova's logic to the predicates Instead of "The exist X such that A" we can use "The supremum A under all X". Instead of "For all X we have A" we can use "Infimum A under all X". We will be interested only in such formulas which logical values are not less than zero (not less than $1/2$ in Getmanova's logic). The true values are greater than zero The false values are less than zero. And zero is a paradox as logical value. The operation of negation is the minus operation. The conjunction is the minimum of all arguments. Disjunction is maximum of all arguments Sequent axioms have the form $XyZ \to VyW$ or $W \to XeY$, where $e \geq 0$, or $XdY \to W$, where $d \leq 0$. The rules are defined as the usually in sequent calculus.

THEOREM 1. *In the proposed logical system the axioms are not false and the rules preserve not falsity*

THEOREM 2. *Every formula without quantifier which value is not less than zero can be proved in the proposed calculus.*

THEOREM 3. *Any propositional formula without logical constants is not false iff it is a tautology of classic propositional logic*

[1] A. GETMANOVA, ***Logic***, Progress Publishers, Moscow, 1989.

▶ T. G. KUCERA, *On the structure of indecomposable totally transcendental modules.*
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There are many interesting problems here, as a significant special case is that of indecomposable injective right modules over a (right) Noetherian ring, where we find some important longstanding problems of purely algebraic interest. In the last few years ideas originating in model theory have begun to provide some new insight into what were apparently strictly algebraic questions. In particular, both the elementary cycle series introduced by I. Herzog [1] and my regular hierarchy [2] are well defined and exhaustive in any indecomposable totally transcendental module, whereas the usual algebraic analysis of indecomposable injectives ("fundamental series") requires additional assumptions.

I will report on some recent work on the algebraic and model-theoretic properties of these two series. In particular, I will discuss possible applications to such algebraic problems as non-commutative localization theory and Jacobson's conjecture.

[1] I. Herzog, *Elementary duality of modules*, ***Transactions of the American Mathematical Society*** (1994).

[2] T. G. Kucera, *The regular hierarchy in modules*, ***Communications in Algebra***, vol. 12 (1993), pp. 4615–4628.

► KANAT KUDAIBERGENOV, *Atomic models of locally modular theories of finite rank.*

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Some conditions for LMFR (i.e., locally modular of finite rank) extra stable theories to have an atomic model over a set are indicated.

Also, some results on the existence of atomic models of different powers for such theories are obtained

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In [5], L. van den Dries, A. Macintyre and D. Marker describe a method for the construction of nonarchimedean exponential fields which are elementarily equivalent to $(\mathbb{R}, \exp)$. In this note, we will give an alternative method which demands less technical effort. We eliminate one of the two limit processes used in [5], by a direct construction of fields with (non-surjective) logarithms. This is based on the point of view as developed in [2] and [1].

Indeed, we shall construct a field K_ω as the union of a chain $K_\omega = \bigcup_{n \in \mathbb{N}} K_n$, where every K_n is a power series field with real coefficients and exponents in a divisible ordered abelian group G_n (which is the value group of the natural valuation v of K_n). Further, every K_n is endowed with a (non-surjective) logarithm, i.e. with an embedding

$$\ell_n : (K_n^{>0}, \cdot, 1, <) \longrightarrow (K_n, +, 0, <)$$

such that ℓ_n extends ℓ_{n-1} and $\ell_n(K_n^{>0}) = K_{n-1}$. Then clearly, $\ell_\omega = \bigcup_{n \in \mathbb{N}} \ell_n$ will be a surjective logarithm on K_ω.

More precisely, if $\mathcal{U}_n^{>0}$ denotes the group of positive v-units of K_n, then the logarithm ℓ_n on $\mathcal{U}_n^{>0}$ will just be given by the logarithmic Taylor series. This makes K_n into a model of the theory of $\mathbb{R}$ with restricted exponentiation, cf. [4].

It then suffices to define the logarithm on a group complement $\mathbf{B}_n$ of $\mathcal{U}_n^{>0}$ in $(K_n^{>0}, \cdot, 1, <)$. To do so, we take a group complement $\mathbf{A}_n$ to the valuation ring R_n of K_n such that there is an embedding

$$h_n : G_n \longrightarrow \mathbf{A}_n$$

satisfying

$$vh_n(g) > g \quad \text{for all } g \in G^{<0}. \tag{1}$$

Then $h_\omega = \bigcup_{n \in \mathbb{N}} h_n$ will also satisfy (1) We then obtain ℓ_ω on $\mathbf{B}_\omega = \bigcup_{n \in \mathbb{N}} \mathbf{B}_n$ by composing h_ω with the canonical isomorphism $\mathbf{B}_\omega \Rightarrow G_\omega = \bigcup_{n \in \mathbb{N}} G_n$, $b \mapsto -v(b)$. It is then shown that condition (1) will guarantee that $f_\omega = \ell_\omega^{-1}$ is an exponential satisfying the axiom scheme

$$x > n^2 \implies f_\omega(x) > x^n \quad (n \in \mathbf{N}).$$

Hence by a theorem of Ressayre (cf. [3]); (K_ω, f_ω) will be a model of the theory of $(\mathbf{R}, \exp)$.

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[2] S. KUHLMANN, *On the structure of nonarchimedean exponential fields I*, to appear in ***Archive of Mathematical Logic***.

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▶ ANDREI A. KUZICHEV, *Equality of terms in the theory of C-monoids*.

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The theory of C-monoids (Cartesian closed categories with one object) can be interpreted in the $\lambda\beta\eta$-calculus with surjective pairing [1] (the calculus Λ_S). The usual analysis of equational theories of this kind is based upon the method of Church and Rosser. Unfortunately, many attempts to find Church - Rosser reductions, generating equality of terms in the theory of C-monoids or in Λ_S, fail (see the corresponding conjecture in [1]). However, we can analyze equality of terms in Λ_S, using the following generalization of the Church-Rosser property.

Let R be a reduction and K be a non-empty set of terms of some equational theory. We say that R is Church - Rosser over $K (R \in CR(K))$, if, for all terms a, b and c such that aRb, aRc and $b \in K$, there exists $d \in K$ such that bRd and cRd.

Let K be the set of all the terms of the pure λ-calculus. Let the reduction Red be the transitive closure of $R \cup Pr$, where R is some special Church-Rosser reduction, generating equality of terms in the $\lambda\beta\eta$-calculus, and where Pr is the reduction, defined by axioms of the forms $\pi_i < a_1, a_2 > Pra_i (i = 1, 2)$ and $\langle \pi_1 a, \pi_2 a \rangle Pra$

THEOREM. $Red \in CR(K)$.

It means that Λ_S is a conservative extension of the $\lambda\beta\eta$-calculus. As an application of this result, we get an elementary consistency proof for Λ_S and for the theory of C-monoids.

[1] W. S. HATCHER and P. J. SCOTT, *Lambda-algebras and c-monoids*, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik***, vol. 32 (1986), pp. 415–430.

▶ GUNWON LEE, *Genesis (history of logic)*.

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We pursue the GENESIS of the formal systems as in REPUBLIC(509B). A formal system can be generated as a model which can be realized as a model; a model of becoming; genesis. A formal system which is consistent, hence can have a model, had been said to be DYNAMIS in Timaeus(31B). A formal system, as it is, is a myth in Timaeus (27–29) MYTHOS.

As in Woo-Suk Park, a formal system can be safely named as a fictional myth, MUTHOS. The consistency of the formal system can have a model or be dynamic. However only when the formal system has been realized as in Kleene, it can be said to be the system of genesis in Timaeus. Therefore the genesis of Timaeus can be formulated as follows:

(1) A myth can be dynamic if and only if the myth is consistent.

(2) A consistent myth can be genesis if and only if the consistent myth is physically realizable.

A formal system as a being OUSIA as in Cornford is a Platonic Form. And as it is consistent, it can be dynamic. Usually the formal system in Plato as a Form is received as the consistent, therefore there are no doubt that all the Platonic Forms are dynamic. However the becoming, genesis has been examined as a separated concept in Cornford. And this separated concept of genesis can be said to be the realizable consistent models in (1) and (2) of the above.

However there are a lot of formal systems which can not be consistent, hence the system can be said to be dynamic or the Platonic Forms. In that sense, the logical systems can be named as a myth as in Park, but all of them are not the formal systems. Hence myth can be the Platonic Forms or a mere NOEMA which can be in the other world or nowhere in Parmenides. Only the Platonic Forms can be dynamic enough to be able to generate. The generated Platonic Forms can be named as GENESIS.

[1] F. M. CORNFORD, ***Plato's cosmology***, The Bobbs-Merrill Company, 1975.

[2] WOO-SUK PARK, *The fictional metalogic of the fictionalist philosophy of mathematics*, ***Philosophy***, Korean Philosophical Association, Seoul, 1993, Autumn, pp. 357–398.

▶ KERKKO LUOSTO, *Hierarchies of monadic generalized quantifiers.*

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This study can be seen as a part of a larger research programme aimed at finding relationships between syntactic properties of a quantifier Q and the expressive power of the logics $\mathcal{L}_{\omega\omega}(Q)$, $\mathcal{L}_{\infty\omega}(Q)$ and $\mathcal{L}^{\omega}_{\infty\omega}(Q)$ generated by that quantifier. To pick up a few examples, consider the finite, relational vocabulary τ_Q of a Lindstrom quantifier Q. Denote by n_R the arity of $R \in \tau_Q$. Then the *arity* of the quantifier Q is

$$\operatorname{ar}(Q) = \max\{n_R \mid R \in \tau_Q\}.$$

Hella (1989) showed that for every $\alpha \geq 0$, the Magidor–Malitz quantifier Q^{n+1}_{α} is not definable in the logic $\mathcal{L}_{\infty\omega}(\mathbf{Q}_n)$ where $\mathbf{Q}_n$ is the collection of all quantifiers of arity n, whence the quantifiers Q^n_α form a strictly increasing hierarchy in expressive power. Oversimplifying, this means that the increase in arity $(n = \operatorname{ar}(Q^n_\alpha))$ accounts for the increase in the expressive power. This line of thought can be pursued even further. The *pattern* of a quantifier Q is

$$p_Q : \omega \to \omega,\ p_Q(n) = |\{R \in \tau_Q \mid n_R = n\}|.$$

Hence, two quantifiers Q and Q' have the same pattern iff there is a renaming $\varrho : \tau_Q \to \tau_{Q'}$. Hella, Luosto and Väananen (1994) give a linear order $\leq$ on patters such that if $p < p'$, then there exists a quantifier Q' with $p(Q') = p'$ which is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}^*_p)$ where $\mathbf{Q}^*_p = \{Q \mid p_Q = p\}$; this result holds especially in the realm of finite structures. What is lost when the hierarchy is refined is that whereas Hella's methods provide us with a back-and-forth characterization for the elementary equivalence of $\mathcal{L}_{\infty\omega}(\mathbf{Q}_n)$ (and $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{Q}_n)$), the latter result is purely existential in nature and is simply based on cardinality arguments.

Let us now restrict ourselves to monadic finite vocabularies and, for definiteness, to finite structures. In this case, the pattern may be identified with the *width* of Q, $\operatorname{wd}(Q) = |\tau_Q| = p_Q(1)$, and a combinatorial criterium can be given for non-definability of Q in the logic $\mathcal{L}_{\omega\omega}(\mathbf{Q}^*_p)$ for a monadic pattern p. What is crucial here is that monadic structures admit a simple classification: for such $\mathfrak{M}$, the unary relations $R^{\mathfrak{M}}$, $R \in \tau$, divide its universe in parts, the sizes of which determine $\mathfrak{M}$ up to isomorphism. Let $\bar{\kappa}_{\mathfrak{M}} \in \omega^l$ be the sequence of natural numbers obtained in this way, and let

$$\mathcal{R}(Q) = \{\bar{\kappa}_{\mathfrak{M}} \mid \mathfrak{M} \in K_Q,\ \mathfrak{M} \text{ is finite}\}$$

where K_Q is the defining class of the quantifier Q. Thus, the problems related to monadic quantifiers can be reduced to questions about relations on ω. This leads to the combinatorial concepts of the *rank* of the relation, $r(R)$, and its variant $r_T(R)$ which takes the sums of the components into account. The intuitive meaning of the equation $l = r(R)$ is that the relation

$R \subset \omega^n$, $n \geq l$, is essentially of the arity l

The main results are: 1° If a monadic quantifier Q is definable in terms of monadic quantifiers of width l, then $r_T(\mathcal{R}_Q) \leq 2^l$ 2° There are relations R on ω of arbitrary high ranks $r_T(R)$. As an application, I show that the non-trivial resumptions of the Härtig quantifier are not definable by monadic quantifiers.

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Let (w) be the set of all partitions of w. We order the set in the following way: for $X, Y \in (w)$, $X \leq Y$ iff each piece of X is a union of pieces of Y We consider finite modification of the ordering and write $X \leq_* Y$ iff all but finitely many pieces of X are unions of pieces of Y.

Having the lattice $((w), \leq_*)$ we reformulate the definitions from the classical van Douwen diagram and obtain two collections of dual coefficients. We 'draw' the diagrams and prove a number of facts concerning their cardinals. We describe implications of MA and its negations, and some consistency results

► PAOLO MANCOSU, *Kant and Bolzano on proofs by contradiction as a watershed between philosophy and mathematics.*

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In [Mancosu 1991] I showed how widespread was the concern about proofs by contradiction among philosophers and mathematicians in the seventeenth century Cavalieri, Descartes, Barrow, Guldin, Wallis and others argued for the superiority of ostensive proofs over proofs by contradiction. The latter types of proof were considered inferior epistemologically and heuristically Epistemologically, since they do not proceed from causes to effects and thus do not show why a result holds but only that it holds. Heuristically, because most proofs by contradiction, for example all the results proved by exhaustion by Archimedes, fail to show by which method the result was obtained, and thus again do not give insight into the real reasons for the validity of the results. This background provides the context for the central theme of the paper, i.e Kant's statements about proofs by contradiction in the section of the *Critique of Pure Reason* called "Transcendental Doctrine of the Method" and Bolzano's reaction to Kant's point of view. In "The discipline of pure reason in regard to its proofs" Kant compares mathematical and transcendental proofs, and sets down rules for the appropriate use of pure reason in its proofs. In particular the third rule reads : "The third rule peculiar to pure reason, in so far as it is to be subjected to a discipline in respect of transcendental proofs, is that its proofs must never be apagogical, but always ostensive. "As a result of the transcendental dialectic Kant bans the use of apagogical proofs from transcendental philosophy but allows it in mathematics. Indeed in mathematics it is not possible "mistakenly to substitute what is subjective in our representations for what is objective, that is, for the knowledge of that which is in the object". This is because mathematics proceeds by construction of concepts in intuition. That last part of the paper analyzes Bolzano's philosophy of mathematics with emphasis on his stand on proofs by contradiction. In particular, I will argue that Bolzano's proof, contained in the §530 of the *Theory of Science*, which shows that every indirect proof can be reduced to a direct one must be seen in the greater context of his opposition to Kant's philosophy of mathematics and to Kant's use of proofs by contradiction as a watershed between philosophy and mathematics.

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We prove that the problem: *given two Horn clauses* $\mathcal{H}1$ *and* $\mathcal{H}2$ *in the language of the first order logic without equality, decide if* $\mathcal{H}1$ *finitely implies* $\mathcal{H}2$ is undecidable.

1. Motivation and Previous Results: Questions concerning decidability of clause implication and similar problems are motivated by artificial intelligence and automated deduction and have been an area of some interest recently. That is because redundancy elimination is believed to be an important issue in optimization of automated theorem provers or logic programs and clausal knowledge bases. The problem if a (general) clause $\mathcal{H}$ implies a clause $\mathcal{G}$ has been proved to be undecidable by M. Schmidt-Schauss in 1988 ([10]). The decidability of a clause implication problem for Horn clauses (i.e. whether a Horn clause $\mathcal{H}$ implies a Horn clause $\mathcal{G}$) remained open until 1992 ([9]). In [8] a particular Horn clause was constructed, which implies an undecidable set of Horn clauses.

Meanwhile, some strong sufficient conditions for decidability of Horn clause implication where given ([11, 3, 4, 5, 10, 7]).

The world described by a knowledge base is always a finite set of objects. So, we can claim, that the example of a natural and practical question is not the decidability of the problem: *does there exist* ***any*** *structure in which a Horn clause* $\mathcal{H}$ *is satisfied and a Horn clause* $\mathcal{G}$ *is not* but rather of the one: *does there exist a* ***finite*** *structure in which a Horn clause* $\mathcal{H}$ *is satisfied and a clause* $\mathcal{G}$ *is not.* If the last is not true, we say that $\mathcal{H}$ finitely implies $\mathcal{G}$.

There are some similarities between decidability problems of clause implication and decidability problems of dependencies implication in relational database theory (see [6]).

Many types of dependencies have been considered and some results concerning undecidability of finite implication have been proved (e.g. [1], [2]). What is important for us is that in the development of the theory finite and classical implication problems were considered simultaneously. It is not the case in the area of clause implication problems, where, up to our knowledge, finite implication have not been considered before

2. The Results: Let $IMPL = \{<\mathcal{H},\mathcal{G}>: \mathcal{H}$ finitely implies $\mathcal{G}\}$ and let $RES = \{<\mathcal{H},\mathcal{G}>: \mathcal{G}$ can be derived from $\mathcal{H}$ by resolution$\}$. Since RES is a recursively enumerable set and, as it was proved in [9], not a recursive set, and since $IMPL$ is co-recursively enumerable set it is clear that resolution is not complete for finite implication in the sense, that it does not enumerate all elements of $IMPL$. We prove that $IMPL$ is undecidable. Moreover, we prove that RES and co-$IMPL$ are recursively inseparable.

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[2] A. K. Chandra and M. Y. Vardi, *The implication problem for functional and inclusion dependencies is undecidable*, ***SIAM Journal of Computing***, vol 14 (1985), no. 3, pp 672–677.

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logic 89, LNCS, no. 440, Springer, 1989, pp. 225–241.

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[9] J. Marcinkowski and L. Pacholski, *Undecidability of the Horn clause implication problem*, ***Proceedings of the 33rd annual symposium on the foundations of computer science***, 1992, pp. 354–362.

[10] M. Schmidt-Schauss, *Implication of clauses is undecidable*, ***Theoretical Computer Science***, vol. 59 (1988), pp. 287–296.

[11] R. B. Stillman, *The concept of weak substitution in theorem-proving*, ***Journal of the Association for Computing Machinery***, vol. 20 (1973), pp. 648–667.

► SERGEJ MARDAEV, *Smallest-fixed-point theorem for modal logic S4.*

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Smallest-fixed-point theorem for **Int**, **Grz** and **GL** were announced in [1, 2]. The theorem for **Int** follows from [3] also. For **S4** we prove that for a formula with parameters $\Box q_1, \dots, \Box q_n$ the smallest fixed point is constructed by iteration.

Modal propositional formulas are formed in the usual way from variables , the constant 1 and connectives $\&, \vee, \neg, \Box, \Diamond$.

Connective $\equiv$ is usual abbreviation.

We define

$$\begin{aligned} \varphi^0(q_1, \dots, q_n) &= 1, \\ \varphi^{n+1}(q_1, \dots, q_n) &= \varphi(\varphi^n(q_1, \dots, q_n), q_1, \dots, q_n). \end{aligned}$$

Theorem 1. *Let $\varphi(p, \Box q_1, \dots, \Box q_n)$ be a modal formula with parameters $\Box q_1, \dots, \Box q_n$ such that all occurrences of p are positive. There is a number m such that*

$$\vdash_{S4} \varphi^m(\Box q_1, \dots, \Box q_n) = \varphi^{m+1}(\Box q_1, \dots, \Box q_n).$$

The theorem has the obvious corollary for **Int**, which also follows from [3].

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► TAMÁS MIHÁLYDEÁK, *Partiality and dynamics in first order logic and in type theoretical logic.*

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As it is well known there are many possibilities to treat partial feature of information in logic. Different types of partial logic try to model some types of (first of all) semantic partiality on object level (and not on meta level). In the background a new informal concept of meaning appears (for example a characteristic difference can be made between sentence and proposition). It has many consequences not only for the philosophy of logic in general but for the meanings of central notions of logic

In dynamic semantics the general starting point is (as Groenendijk and Stokhof mentioned) “that the meaning of sentences resides in its information change potential.”

My paper deals with the following questions:

1. (1) Is there any room in a logical system for the notion of meaning of partial semantics and for "information change potential" ?
2. (2) What type of language and what type of semantics do we need ?
3. (3) Is there any real connection between two notions of meaning ? Is there any possibility to represent these notions in a correct semantic system ? Are there two different notions of meaning or can we define only one concept to represent these aspects of meaning.
4. (4) What does a partial dynamic semantic means for the central notions of logic ?
5. (5) Is there any real difference from theoretical point of view between first order logic of partial and dynamic semantics and type theoretical one ?
6. (6) Do we need some kind of intensional logic to connect partiality and dynamics ?

► ALEXANDER MITIN, *Undecidability of elementary theories of certain groups.*

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Let G_1 be the group of invertible and Lebesgue's measure preserving transformations of segment $[0,1]$. We identify two transformations if they differ on a set having zero measure.

Let G_2 be the group of measure preserving transformations of the set of reals.

Let G_3 be the group of invertible transformations of the set of integers.

The groups G_1 and G_2 are studied in ergodic theory [2] V. V. Ryzhikov stated a hypothesis that elementary theories of all G_1 and G_2 and G_3 groups are undecidable. We confirm the hypothesis For any group G of the groups G_1, G_2 and G_3, we prove the following

THEOREM *There is a translation (effective procedure) producing, for any given closed formula of arithmetic, a closed formula of the group language, which is true in G if and only if the given formula is true*

Other examples of groups having undecidable elementary theories are presented in [1].

The author is sincerely grateful to V. V. Ryzhikov and N K. Vereshchagin for fruitful discussions.

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[2] P. R HALMOS, ***Lectures on ergodic theory***, Tokyo, 1956.

► MALIKA MORE, *Rudimentary representations of spectra.*

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The spectrum of a first-order sentence is the set of cardinalities of its finite models. Let $\mathcal{S}$ be the class of all spectra and let $\mathcal{F}_k$ (resp BIN) be the class of all spectra represented by a sentence of a language containing only predicates of arity k (resp. one binary predicate) (see Fagin [1]). Let RUD be the set of unary rudimentary relations. We consider rudimentary relations as defined by (polynomially) bounded formulas of the relational language of arithmetics $L_{PA} \equiv \{\leq, +, .\}$ (see Harrow [2]). The following inclusions hold $\mathcal{F}_1 \subsetneqq \text{RUD} \subseteq \text{BIN} \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_k \subseteq \cdots \subseteq \mathcal{S}$, but it is unknown whether they are strict. Besides, RUD $=$ BIN is equivalent to RUD $= \mathcal{S}$ and implies the collapsing of Stockmeyer's hierarchy at a level strictly greater than 1 (see Woods [4]).

Here, we construct rudimentary representations of the above classes of spectra (see More [3]): *for any spectrum S of $\mathcal{F}_k$, there is an integer $q \geq 1$ such that the range of S under the function $f_{q,k}(x) = 2^{q \cdot x^k}$ is a rudimentary set.*

Let S be a spectrum represented by a sentence ϕ of the language $L \equiv \{R_0, \ldots, R_{q-1}\}$ consisting of q predicates of arity k. Let $G = \langle \mathcal{G}, \mathcal{L} \rangle$ be a finite L-structure of cardinality n.

In order to encode G into an integer, we consider in turn each k-tuple of elements of $\mathcal{G}$, and we describe its structure concerning the q relations of L. So, the encoding of G is represented by a binary word c of length $q.n^k$ i.e. by an integer strictly smaller than $f_{q,k}(n)$

Then we change the formulas f of L into bounded formulas of L_{PA} (denoted by $U(\phi)$) containing two new free variables n and c as follows: each atomic formula is changed into a bounded L_{PA} formula and each quantifier is bounded by n Then the following property implies our result: $x \in S(\phi)$ if and only if $f_{q,k}(x) = 2^{q x^k}$ satisfies $\exists n \leq m \exists c \leq m(m = f_{q,k}(n) \wedge U(\phi)(n,c))$.

Any function g, rudimentary as a relation and asymptotically greater than the function $f_{q,k}$ asymptotically changes any spectrum represented by a sentence of L into a rudimentary set. For instance, the function defined by $f(x) = 2^{x^x}$ (resp. $f_k(x) = 2^{(x^{2k})}$), which is asymptotically greater than all functions $f_{q,k}$ (resp. with fixed k) allows us to represent all spectra (resp. all spectra of $\mathcal{F}_k$) as rudimentary sets, provided that some cases are dealt with by hand.

[1] R. FAGIN, *A spectrum hierarchy*, ***Zeitschrift für Mathematiche Logik und Grundlagen der Mathematik***, vol. 21 (1975), pp. 123–134.

[2] K. HARROW, *Sub-elementary classes of functions and relations*, ***Ph.D. thesis***, New York University, 1973.

[3] M. MORE, *Définissabilité dans les graphes finis et en arithmétique bornée*, ***Ph.D. thesis***, Université d'Auvergne, Clermont, 1994.

[4] A. WOODS, *Some problems in logic and number theory, and their connections*, ***Ph.D. thesis***, University of Manchester, 1981.

► A. S. MOROZOV, *Automorphisms of decidable models*.

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We study the interplay between decidability for models and their automorphism groups. The results obtained imply that there is no specific relations between decidability of models and isomorphism types of automorphism groups of these, but if we consider such groups as sets of transformations then the situation is changed.

All necessary definitions may be found in [1].

A model M is said to be *recursive (decidable)* provided its universe is a recursive subset of ω and one can effectively recognize quantifier–free (all first–order) formulas $\varphi(\bar{x})$ and n–tuples $\bar{m}$ such that $M \models \varphi(\bar{m})$. $\mathrm{Aut}_r M$ stands for the recursive automorphism group of M.

THEOREM 1. *There exists a recursive model M such that for all decidable models N of arbitrary effective language $Aut\ M \neq Aut\ M$ and $Aut_r M \neq Aut_r N$ hold.*

THEOREM 2. *For each recursive model M there exists a decidable model N such that*

$$Aut\ M \cong Aut\ N.$$

THEOREM 3. *There exists a decidable ω–categorical model M such that for each recursive model N*

$$Aut_r N \cong Aut_r M^*$$

holds for some decidable model $M^* \cong M$

Considering constructive models, we can reformulate this result as follows·

COROLLARY. *There exists a ω–categorical countable model M such that the class of recursive* *automorphism groups coincides with*

$$\{\, Aut_r(M, \nu) \mid \nu \text{ is a strong constructivization of } M\}$$

up to the isomorphism.

[1] YU. L. ERSHOV, ***Decidability problems and constructive models***, Nauka, Moscow, 1980.

► LUDOMIR NEWELSKI, *Trace calculations and local $\mathcal{M}$-ranks.*

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Assume T is superstable and small. For a type $s(x)$ and a set A let the trace of s on A be

$$Tr_A(s(x)) = \{stp(a/A) \cdot s(a) \text{ holds}\}$$

. We show some basic properties of this notion, similar e.g. to the open mapping theorem of Lascar-Poizat.

I defined a multiplicity rank $\mathcal{M}$ on complete types over finite sets by the following condition:

(a) $\mathcal{M}(a/A) \geq \alpha + 1$ iff $\exists$ finite $B\big[B \supseteq A,\ a \mathrel{|\!\!\smile} B(A),\ \mathcal{M}(a/B) \geq \alpha$
and $Tr_A(a/B)$ is nowhere dense in $Tr_A(a/A)\big]$.

This rank yields meager (hence locally modular, regular) types (many of them if $I(T, \aleph_0) < 2^{\aleph_0}$). This is similar to U-rank yielding regular types. Also, $I(T, \aleph_0) < 2^{\aleph_0}$ implies $(\mathcal{M}(p) \geq \omega^\beta \Longrightarrow R_\infty(p) \geq 1 + 2\beta)$.

It is open whether $I(T, \aleph_0) < 2^{\aleph_0}$ implies $\mathcal{M} \leq U$.

Now fix a finite set C. We define local ranks $\mathcal{M}_C$ and $\mathcal{M}^e_C$ on types $p \in S(A)$, $A \supseteq C$ finite, by the following conditions:

(b) $\mathcal{M}_C(a/A) \geq \alpha + 1$ iff $\exists$ finite $B\big[B \supseteq A, a \mathrel{|\!\!\smile} B(A), \mathcal{M}_C(a/B) \geq \alpha$
and $Tr_C(a/B)$ is nowhere dense in $Tr_C(a/A)\big]$.

(c) $\mathcal{M}^e_C(a/A) \geq \alpha + 1$ iff $\exists$ finite $B, D\big[A \subseteq B \subseteq D, a \mathrel{|\!\!\smile} D(A), \mathcal{M}^e_C(a/D) \geq \alpha$
and $Tr_C(aB/D)$ is nowhere dense in $Tr_C(aB/B)\big]$.

$\mathcal{M}^e_C$ is an eventual version of $\mathcal{M}_C$ and has the additivity properties of $\mathcal{M}$ (while $\mathcal{M}_C$ does not). $\mathcal{M}^e_C$ also yields meager types, related more closely to C.

► A. I. OMAROV, *The syntactical description of filtered formulas.*

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A formula $\varphi(x)$ is *a filtered formula*, if for every filter $\langle J, E\rangle$, every collection of similar algebraic structures $\mathfrak{M}_i$, $i \in J$, and every $\bar{f} \in \prod \mathfrak{M}_{i/E}$ the following is satisfied:

$$\prod \mathfrak{M}_{i/E} \models \varphi(\bar{f}) \Longleftrightarrow \{i \mid \mathfrak{M}_i \models \varphi(\bar{f}(i))\} \in E.$$

We give a syntactical description of filtered formulas. E. A. Palyutin introduced a new class of formulas, called by P-formulas. The P-formulas are built by induction:

(1) atomic formulas are P-formulas;
(2) the set of P-formulas is closed under conjunction;
(3) if φ and ψ are P-formulas, then $(\exists x)\varphi\&(\forall x)(\varphi \longrightarrow \psi)$ is a P-formula.

THEOREM 1. *Let the Boolean algebra $2^J/E$ contains a nontrivial atomless element. Then a formula $\varphi(X)$ is filtered by the filter $\langle J, E\rangle$ iff it is equivalent to some P-formula.*

There are three different classes of filtered formulas:

(1) filtered by ultrafilter; by Łoś' Theorem this is the class of all formulas;
(2) filtered by filter $\langle J, E\rangle$ such that $2^J/E$ is an atomic Boolean algebra.
(3) filtered by filter $\langle J, E\rangle$ such that $2^J/E$ has an atomless element ; it is exactly the class of P-formulas

The class (b) of formulas is called *multiplicative.*

We give some theorems about multiplicative formulas as analogies to the corresponding theorems about P-formulas.

[1] A.I. Omarov, *On the multiplicative formulas*, ***Algebra i Logika***, vol. 32 (1993), no. 2, pp. 139–147, (in Russian).

► DMITRY E. PAL'CHUNOV, *Relatively finitely axiomatizable boolean algebras with distinguished ideals.*

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Boolean algebras with $\lambda \in \omega$ distinguished ideals (called below as I-algebras) are studied in the signature $\langle \cup, \cap, , 0, 1, I_1, \dots, I_\lambda\rangle$. All definitions one can find in [1]

Denote by $\mathfrak{L}^\lambda$ the Lindenbaum-Tarski algebra of the I-algebra class, and by F_n the n-th Frechet ideal of $\mathfrak{L}^\lambda$. Denote $T_n \rightleftharpoons \{\varphi \in \mathfrak{L}^\lambda \mid \neg\varphi \in F_n\}$. An I-algebra $\mathfrak{A}$ is called non-vanishing if for any decomposition $\mathfrak{A} = \mathfrak{B} \times \mathfrak{C}$, we have $\mathfrak{A} \equiv \mathfrak{B}$ or $\mathfrak{A} \equiv \mathfrak{C}$. A finitely axiomatizable non-vanishing I-algebra is called basic one. If $\mathfrak{A} \equiv \mathfrak{B} \times \mathfrak{C}$, then we denote $\mathfrak{B} \leq \mathfrak{A}$; $\mathfrak{B} < \mathfrak{A}$ means that $\mathfrak{B} \leq \mathfrak{A}$ and $\mathfrak{B} \not\equiv \mathfrak{A}$. A non-local I-algebra $\mathfrak{A}$ is called minimal if $\mathfrak{B} \equiv \mathfrak{A}$ for any non-local $\mathfrak{B} \leq \mathfrak{A}$.

Definition. An I-algebra $\mathfrak{A}$ is relatively finitely axiomatizable if for some $n \in \omega$ the I-algebra $\mathfrak{A}$ is finitely axiomatizable in the theory T_n and $\mathfrak{A} \models T_n$.

Theorem 1. *An I-algebra $\mathfrak{A}$ is called relatively finitely axiomatizable iff the number of non-local non-vanishing direct summands of $\mathfrak{A}$ is finite and for any minimal non local $\mathfrak{B} \leq \mathfrak{A}$ the following statements hold.*

a) *Almost for all basic $\mathfrak{C} \leq \mathfrak{B}$ there exists only (modulo the elementary equivalence) basic $\mathfrak{D} \leq \mathfrak{B}$ such that $\mathfrak{D} \not\leq \mathfrak{C}$ and $\mathfrak{C} \not\leq \mathfrak{D}$.*

b) *If $\mathfrak{B} < \mathfrak{C} \leq \mathfrak{A}$ and $\mathfrak{C}$ is non-vanishing, then $\mathfrak{C}^n \not\leq \mathfrak{A}$ for some $n \in \omega$.*

c) *There exists local $\mathfrak{C} \leq \mathfrak{A}$ such that for any basic $\mathfrak{D} \leq \mathfrak{A}$ if $\mathfrak{D} \not\leq \mathfrak{C}$, then $\mathfrak{D} \leq \mathfrak{M}$ for some minimal non-local $\mathfrak{M} \leq \mathfrak{A}$.*

Theorem 2. *For any relatively finitely axiomatizable I-algebra $\mathfrak{A}$, the elementary theory $Th\,\mathfrak{A}$ is decidable.*

Theorem 3. *For any $n \in \omega$, the sentence set F_n and the theory T_n are decidable.*

[1] D. E. Pal'chunov, *Direct summands of boolean algebra with distinguished ideals*, ***Algebra and logic***, vol. 31 (1992), no 5, pp. 499–537

► LORENZO PEÑA, *Strengthening entailment logic into a logic of fuzziness.*

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We propose a chain of strengthenings of system **E**. The **E** functor '→', duly strengthened, receives a construal as meaning 'to the extent [at least] that'. The resulting systems are closer to **CL** (Classical Logic). They are not relevant, in so far as they fail to comply with the relevantist constraint about variable-sharing; yet, they comply with the constraint of use in proof, with some adaptations The resulting systems of fuzzy or gradualistic logic are intermediary between classical and entailmental logic. The approximation to **CL** goes further than that. In fact, several of the systems contain the whole **CL**, and are indeed conservative extensions thereof. Only, the **CL** negation '$\neg$' is given a different reading from what is customary. It is now read 'not...at all'. Thus viewed, classical negation is complete negation - the functor which maps anything true, to whatever extent, into complete falsehood, and the utterly false into complete truth In some important sense, those systems are to **CL** as relevant logic was meant to be to intuitionistic logic. All those systems have the laws of non-contradiction and excluded middle. They are paraconsistent copulative systems.

- P0 = E P1 = P0 + Factor
- P2 = P1 + Linearization ($\ulcorner p \to q \vee q \to p\urcorner$)
- P3 = P2 + the Self-Self Principle ($\ulcorner N(p \to p \to N(p \to p)) \to .p \to p\urcorner$)
- P4 = P3 + IF [Implicational Funnel] ($\ulcorner p \to q \vee .p \to q \to r\urcorner$)
- P4.5 = P4 + Mingle = **CL**. Mingle is $\ulcorner p \to\ p \to p\urcorner$
- P5 = P4 + one of these: Aristotle ($\ulcorner N(p \to Np)\urcorner$), Boethius ($\ulcorner p \to q \to N(p \to Nq)\urcorner$) or Contradiction (namely, for some particular sentential constant j: $\ulcorner j \wedge Nj\urcorner$)
- P6 = P4 + both these principles: $\ulcorner p \to q \to .NHNHp \to Hq\urcorner$ and $\ulcorner Hp \to q \vee Np \to r\urcorner$ ('H' is read: 'It is altogether true that')
- P3.5 = P3 + those two principles involving 'H' = P6 minus IF
- P7 = P6 + $\ulcorner Hp \vee Np\urcorner$
- P8 = P3.5 + these two: $\ulcorner \alpha\urcorner$ and $\ulcorner \alpha \to p \vee .p \to q\urcorner$ (' α ' is a sentential constant meaning the conjunction of all truths)
- P8.5 = P8 + $\ulcorner H\alpha\urcorner$ = **CL** P9 = P8 + $\ulcorner NH(\alpha \to \alpha)\urcorner$ P10 = P8 + $\ulcorner HN(N\alpha \to \alpha)\urcorner$

P8 contains P7; P10 contains P9; P9 contains P5; P8 is a conservative extension of CL (if classical negation, '$\neg$', is defined as 'HN'); P5 and such systems as they contain P5 are not just paraconsistent but contradictorial (they have the theorem of Heracleitus: $\ulcorner N(p \to p)\urcorner$). (As is well-known, $\ulcorner N(p \to p) \to (p \to p) \to (p \to p) \to N(p \to p) \to N(N(p \to p) \to (p \to p) \to (p \to p) \to N(p \to p))\urcorner$ is a theorem of **E**; the apodosis is the negation of the protasis; when we add Aristotle or Boethius, the negation of the theorem becomes a theorem, too. But in P1 we prove $\ulcorner p \wedge Np \to N(q \to q)\urcorner$). System P8 and those above it allow to implement classical conditional (which can be defined in the usual way with strong negation, '$\neg$', and disjunction, '$\vee$') in the relevantly recommended way: $p \supset q$ iff there is some truth, namely α, such that $p \wedge \alpha \to q$.

► M. C. PERETYAT'KIN, *On semantic classes of sentences*

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Theories T and S are said to be *semantic similar over the list L of model-theoretic properties*, if there is a recursive isomorphism $\mu : L(T) \longrightarrow L(S)$ between their Lindenbaum algebras such that any complete extension $T*$ of the theory T and the corresponding complete extension $S*$ of theory S, $S*$ and $\mu(T*)$ have identical description over properties of L.

Let σ be a finite signature, and $SL(\sigma$ be the set of all sentences of σ considered as finitely axiomatizable theories. The class $E \subseteq SL(\sigma)$ is a *semantic class over L* if E is closed under semantic similarity over the list L. Relations of semantic similarity appeared in connection with expressiveness problem of finitely axiomatizable theories [1]. Semantic classes are natural analogues to index set in algorithm theory. Some analogues of Rice's theorem (levels up to 3) are proved for this classes. We investigate some other properties of semantic classes.

A semantic class E is called *weakly decidable* if all theories F of E are decidable, and is called *uniformly decidable* if there is an algorithm representing the uniform procedure to the decision problem for all theories of E.

Theorem 1. *Let E be a semantic class over the list L only consisting of complete theories. Then E is uniformly decidable.*

Theorem 2. *Let E be a semantic class over the list $L = 0$ which is weakly decidable. Then E is uniformly decidable if and only if E only consists of complete theories*

[1] M. G. Peretyat'kin, *Expressive power of finitely axiomatizable theories, parts I, II, III*, ***Siberian Advances in Mathematics***, vol. 3 (1993), no. 2, pp. 153–197, no. 3, pp. 123–145, no. 4, pp. 131–201.

▶ REGIMANTAS PLIUŠKEVIČIUS, *Saturated disjunction property for a linear temporal logic.*

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We consider a propositional linear temporal logic with ○ ("next") and □ ("always") as temporal operators. The sequent S is called primary if

$$S = \Sigma_1, \bigcirc^n\Pi_1, \bigcirc^n\Box\Omega \to \Sigma_2, \bigcirc^n\Pi_2, \bigcirc^n\Box\Delta\ (n > 0),\ \text{where}\ \Sigma_1, \Sigma_2, \Pi_1, \Pi_2 = \varnothing$$

consists of logical formulas without ○ and □; $\Omega, \Delta = \varnothing$ or consists of logical formulas without ○.

Lemma 1. $\vDash$ *$(S \iff \bigwedge_{i=1}^{n} S_i)$, where $S_i (1 \leqslant i \leqslant n)$ is a primary sequent; the set $\{S_1, \dots, S_n\}$ is called a primary expansion of S and denoted by $P(S)$.*

Lemma 2. *Let $S \in P(S)$, then $\vDash S \Rightarrow \vDash \Sigma_1 \to \Sigma_2$ or $\vDash S \Rightarrow S^* = \Pi_1, \Box\Omega \to \Pi_2, \Box\Delta$; the sequent S^* is called the alternating one.*

Let us define the notion of the resolvent of a sequent S. Let us apply Lemma 2 to S_j from $P(S)$ and let $S^*_{j_2}$ be the alternating sequent of this application

Let $R^*e(S) = \{S^*_{1_2}, \dots, S^*_{n_2}\} = \{S_1, \dots, S_k\} (k \leqslant n)$. Let us drop from $R^*e(S)$ all members which can be obtained from other by the structural rule of weakening. Then the obtained set will be called a resolvent of S and denoted by $Re(S)$.

Let us define the notion of the k-th resolvent of a sequent S (denoted by $Re^k(S)$) and the notion of the saturated sequent. $Re^0(S) = S$; $Re^1(S) = Re(S)$.

Let $S_i \in Re^k(S)$ and $R^*e^{k+1}(S) = \bigcup_i Re(S_i)$ Then $Re^{k+1}(S)$ is obtained from $R^*e^{k+1}(S)$ by dropping all the sequents that can be obtained from some sequent $S^* \in R^*e^{k+1}(S)$ or by some sequent $S' \in Re^l(S) (0 \leqslant l < k + 1)$ by the structural rule of weakening. Let for some n $Re^n(S) = \varnothing$, then the set $\bigcup_0^n Re^{n-1}(S)$ will be called the saturated one of S and denoted by Sat$\{S\}$.

Lemma 3. *Let $S^* \in$ Sat $\{S\}$, then $Re(S^*) \in$ Sat $\{S\}$ or can be obtained from some $S' \in$ Sat $\{S\}$ by the structural rule of weakening*

Theorem 1. *Let $\vDash S = \Pi_1, \Box\Omega \to \Pi_2, \Box A_1, \dots, \Box A_n$ and S be saturated, then there exists $\Box A_j$ such that $\vDash \Pi_1, \Box\Omega \to \Pi_2, \Box A_j$ or $\vDash \Box\Omega \to \Box A_1, \dots, \Box A_n$ $(1 \leqslant j \leqslant n)$.*

Remark. Let $S = P, \Box(P \supset (\bigcirc P \wedge \Box(C \vee D) \wedge \Box(C \supset \bigcirc C) \wedge \Box(D \supset \bigcirc D))) \to \Box C, \Box D$, then $\vDash S$, however S is not saturated and Theorem 1 fails for the sequent S.

Theorem 1 is the main point for the generalization of [1], [2] to the case of non-Horn-like sequents.

[1] R. Pliuškevičius, *On the saturation principle for a linear temporal logic*, Lecture Notes in Computer Science, no. 713, Springer, pp. 289–300.

[2] ———, *On saturated calculi for a linear temporal logic*, Lecture Notes in Computer Science, no. 711, Springer, 1993, pp. 640–649.

▶ JORDI REBAGLIATO AND VENTURA VERDÚ, *The deduction theorem for Gentzen systems.*

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Let $\mathcal{L}$ be a propositional language and let $\mathcal{G}$ be an n-dimensional Gentzen system (see the abstract *Algebraizable Gentzen systems*

We define the notion of the deduction-detachment theorem for an n-dimensional Gentzen systems $\mathcal{G}$. When $\mathcal{G}$ is an n-dimensional deductive system we obtain the deduction detachment theorem stated in [3] and [2].

Let $\mathcal{G}$ be an algebraizable Gentzen system and let K be its equivalent quasivariety semantics. We prove that $\mathcal{G}$ has the deduction-detachment theorem iff the quasivariety K has equationally definable principal relative congruences. The proof of this theorem is similar to the proof of the analogous theorem for deductive systems given in [2].

Several examples are given.

[1] W. J. Blok and D. Pigozzi, *Algebraizable logics*, ***Memoirs of the American Mathematical Society***, vol. 396 (1989).

[2] ———, *The deduction theorem in algebraic logic*, Preprint, 1989.

[3] ———, *Local deduction theorems in algebraic logic*, ***Algebraic logic, colloquium of the mathematical society of János Bolyai*** (H. Andréka, J. D. Monk, and I. Nemeti, editors), North Holland, 1991, pp. 75–109.

[4] J. Rebagliato and V. Verdú, *El teorema de la deducción para sistemas de Gentzen*, ***Actas del I Congreso de la Sociedad de Lógica, Metodología y Filosofía de la Ciencia*** (Madrid) (E. Bustos, J. Etcheverría, E. Pérez Sede no, and M. I. Sánchez Balmaseda, editors), 1993, pp. 118–120.

[5] ———, *On the algebraization of some Gentzen systems*, ***Fundamenta Informaticae***, vol. 18 (1993), pp. 319–338.

[6] ———, *Algebraizable Gentzen systems and the deduction theorem*, Manuscript, 1994.

[7] G. Takeuti, ***Proof theory***, Studies in Logic, no. 81, North-Holland, 1975.

▶ VLADIMIR V. RYBAKOV, *Tabular modal and intermediate logics without independent basis for admissible inference rules.*

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Modal logic has nowadays a lot of applications to Computer Science. Tabular modal logics (logics with the set of truth values forming a finite algebra) have some special interest because finite many-valued logics reflect behavior of programs carrying out computations. Inference rules, generally speaking, simulate programming instructions. The study of inference rules for tabular (finite many-valued) logics is rather developed popular field. In [2] W. Rautenberg proved that every 2-element matrix has finite base for valid inference rules. P. Wojtylak [4] constructed 5-element matrix which has no finite basics for valid inference rules, 4-element matrix with this property was found later by W. Dziobiak [1]. Similar results are known for tabular modal logics. In [3] an example of tabular modal logic without finite basis for valid inference rules was found and it is show that any finite subdirectly irreducible modal K4-algebra has a finite base for valid inference rules. A considerably old open problem is :

whether any tabular modal logic has finite base for admissible inference rules (all those with respect to which the logic is closed) ? Note that admissible rules contain among them all valid inference rules but not conversely. We are going to answer this question.

There exists a finite generated (rooted) partially ordered set (frame) **F** of depth 3 such that

THEOREM 1. *The tabular modal logic* **LM(F)** *of the frame* **F** *has no basis for admissible rules in finitely many variables (in particular, has no finite basis) The superintuitionistic tabular intermediate logic* **LI(F)** *generated by* **F** *has no basis in finitely many variables for admissible inference rules.*

THEOREM 2. *The quasi-varieties generated by free algebras of countable rank from varieties of modal and pseudoboolean algebras corresponding to* **LM(F)** *and* **LI(F)** *have in these varieties only finite number of covering quasi-varieties.*

From this theorems we immediately extract:

THEOREM 3. *Tabular logics* **LM(F)** *and* **LI(F)** *do not have even independent basis for admissible inferences rules.*

In the same time, logics **LM(F)**, **LI(F)** have some finite basis for valid inference rules by [3].

[1] W. DZIOBIAK, *A finite matrix whose set of tautologies is not finitely axiomatizable*, ***Reports on Mathematical Logic***, vol. 25 (1991), pp. 105–112.

[2] W. RAUTENBERG, *2-element matrices*, ***Studia Logica***, vol. 40 (1981), pp. 315–353.

[3] V. V RYBAKOV, *Bases for quasi-identities of finite modal algebras*, ***Algebra and Logic***, vol. 21 (1982), no. 2, pp. 219–227.

[4] P. WOJTYLAK, *An example of a finite through finitely non-axiomatizable matrix*, ***Reports on Mathematical Logic***, vol. 17 (1984), pp. 39–46.

► ANDREAS SCHLÜTER, *A theory of rules for enumerated classes of functions.*

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We define an applicative theory $\mathbf{CL}_2$ similar to combinatory logic which can be interpreted in classes of functions possessing an enumerating function in the same way as classical combinatory logic is interpreted in the recursion-theoretic (partial) model. In contrast to the classical case, it is not necessarily assumed that the enumerating function itself belongs to that function class. Thereby we get a variety of possible models including e.g. the classes of primitive recursive, recursive, partial recursive, elementary, polynomial-time computable or ε_0-recursive functions.

We show that in $\mathbf{CL}_2$ a major part of the metatheory of enumerated classes of functions can be developed. Namely, a kind of λ-abstraction can be defined and abstract versions of the S^m_n- and (primitive) recursion theorems are proved.

A theory closely related to $\mathbf{CL}_2$ can be used to replace the applicative part of Feferman's theories for explicit mathematics. So this work can also be seen as an answer to a question by Feferman to formulate a theory for explicit mathematics in which operations can be interpreted as primitive recursive or even more feasible ones.

Finally, the proof-theoretical strength of theories with primitive recursive operations is compared to the classical systems of explicit mathematics formalizing general recursive operations.

[1] ANDREAS SCHLÜTER, *A theory of rules for enumerated classes of functions*, to appear in ***Archive for Mathematical Logic***.

► ANTON SETZER, *Proof theoretical strength of Martin-Löf type theory.*

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THEOREM. *The proof theoretical strength of extensional and intensional Martin-Löf Type Theory with W-type and one universe in the formalization* à la Russel *is* $\psi_{\Omega_1}(\Omega_{I+\omega})$.

The proof of this theorem has two directions, here we will concentrate on the proof of an upper bound by embedding Martin-Löf Type Theory in a set theory other versions as well. KPi^+ is an extension of Kripke-Platek set theory by adding axioms demanding the existence of a recursive inaccessible I and of finitely many admissibles above it.

We will essentially interpret closed types as the the set of terms, introduced by some introductory rule, and close this set under reduction. For the interpretation of the W-type we have an inductive definition, which we carry through by iterating an operator up to the next admissible. In the case of the universe we need a fixed point of an operator, which leads to the next admissible (since the universe is closed under W-type), therefore we iterate an operator up to an admissible, closed under admissibles, namely I. In order to interpret W-types, which are built using the universe, we need finitely many admissibles above I.

We conclude, that all Π^1_1-sentences provable in Martin-Löf Type Theory are provable in KPi^+ (we only have to restrict quantification over sets of natural numbers to elements of the least admissible containing ω). Therefore Tarski and Russel version of the Type Theory have as upperbound the strength $\psi_{\Omega_1}(\Omega_{I+\omega})$, which is "slightly" bigger than the strength of $(\Delta^1_2 - CA) + BI$, KPi and Feferman's theory T_0.

For the proof of the lower bound, which can be found in [2], we can only give a flavour. We use essentially, that we can achieve W-types with branching degrees, which are not elements of the universe, by using the sum of products of types, where the first component is possibly empty depending on the boolean value of a primitive recursive relation. Using this and well ordering techniques from proof theory we can show, that Martin-Löf Type Theory proves transfinite induction up to $\psi_{\Omega_1}(\Omega_{I+n})$.

[1] JAEGER, ***Theories for admissible sets: A unifying approach to proof theory***, Bibliopolis, 1986.

[2] A SETZER, *Proof theoretical strength of Martin-Löf type theory with W-type and one universe*, ***Ph.D. Thesis***, Universitat München, 1993.

► ALTYNBEK SHARIPBAY, *Construction of proof theory for digital circuits.*

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The proof theory on language of unary predicates of the second order is constructed for Digital Circuits (DC), which process some input signals, coming in discrete moments of time, into output signals.

In this language there are one constant "1" initial moment of time and one unary functional symbol "↑" immediate successor, two kinds of variables: $t_1, t_2, \dots, t_s$—the moments of time and $X_1, X_2, \dots, X_k$—the string of signals in given set of their states. Terms are defined as usually. Atomic formulas is as like as $X(t) = \alpha$, which means that at th t-th moment of time the state of signal X is α, where "=" is graphical equality. Formulas are produced from atomic by the application of logic operations and introducing quantifiers to the both kinds of variables. There are the rules of deduction for admissible connections of DC's.

Behavioural and structural specifications for the hierarchy of gradually complicated DC's are constructed by those formulas and rules: the names of behavioural specifications of basic DC considered as axioms, provide the first level, then applying rules of deduction to these axioms we can construct structural specifications for the second level of hierarchy and we can use the behavioural specifications of the second level as axioms to construct structural specifications for third level etc.

There is the theorem about completeness of this theory. The problem of verification for DC is algorithmically decidable

► V. B. SHEHTMAN, *Some results in neighbourhood semantics for modal and intermediate propositional logics.*
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Neighbourhood (or topological) semantics for modal and intermediate logics dates back to Tarski and Tang (1938). It is well-known that every Kripke-complete logic is neighbourhood-complete. The converse holds neither for S4-logics, nor for intermediate logics ([2, 4]) On the other hand, there exist neighbourhood-incomplete S4-logics [3]; for intermediate logics, the corresponding problem remains open.

A related notion is "strong completeness" (or "compactness"): a logic L is called strongly Kripke-complete (S-K-complete) if every L- consistent theory has a model on some frame for L. The corresponding definition has sense also in the neighbourhood semantics (S-N-completeness).

DEFINITION. A class of generated Kripke S4-frames is called "steady" if it is closed under generated subframes and adding roots. A modal logic is called "steady" if it is complete w.r.t. a steady class.

THEOREM 1. *Every steady S4-logic is S-N-complete.*

THEOREM 2. *Every Kripke-complete intermediate logic is S-N-complete.*

Due to some results from [1] and [5], these theorems allow to construct many examples of S-N-complete logics which are not S-K-complete.

[1] K. FINE, ***The Journal of Symbolic Logic***, vol. 50 (1985), pp. 619–651.

[2] M. GERSON, ?, ***Studia Logica***, vol. 34 (1975), pp. 334–342.

[3] ———, ***The Journal of Symbolic Logic***, vol. 40 (1975), pp. 141–147.

[4] V. B. SHEHTMAN, *Topological models of propositional logics*, ***Semiotika i Informatika***, vol. 15 (1980), pp. 74–98, (In Russian).

[5] T. SHIMURA, *On strong Kripke completeness of intermediate logics*, Manuscript, 1992.

► ZELJKO SOKOLOVIĆ, *Minimal subsets of differentially closed fields.*
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Joint work with Ehud Hrushovski.

We will indicate how to obtain a strong structure theory for strongly minimal sets definable in a differentially closed fields of characteristic zero. It turns out that strongly minimal sets are locally modular, or else equivalent to the field of constants. Using some classical objects one explicitly describes the locally modular nontrivial strongly minimal sets up to nonorthogonality, and as a consequence shows that there are continuum many nonisomorphic countable differentially closed fields proving a conjecture of S. Shelah.

► KLAUS-HILMAR SPRENGER, *Grzegorczyk hierarchies on term algebras.*
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We want to generalize the Grzegorczyk hierarchy $\{E_n\}_{n \geq 0}$ on natural numbers and

Weihrauch's Grzegorczyk hierarchy $\{E_n^k\}_{n\geq 0}$ on words over an alphabet of size $k \geq 2$ to Grzegorczyk hierarchies on arbitrary term algebras. By a term algebra $\mathcal{T}^\sigma$ of signature σ we mean the set of terms generated by a set of finitely many function symbols with at least one of arity 0 and ≥ 2.

As there is a canonical notion of primitive recursion for functions on term algebras, we have the hierarchy of recursion number classes $\{R_n^\sigma\}_{n\geq 0}$ on term algebras.

Terms can be considered both as words and trees; the measures of length and height of a term induce quasi orderings on $\mathcal{T}^\sigma$, which are appropriate to define two notions of bounded primitive recursion. After introducing two different sequences of Ackermann functions, the corresponding Grzegorczyk hierarchies $\{L_n^\sigma\}_{n\geq 1}$ and $\{H_n^\sigma\}_{n\geq 0}$ are defined. Both hierarchies coincide for all (super)elementary classes (i.e. for $n \geq 3$), and even the well-known theorem "$R_n = E_{n+1}$ for all $n \geq 2$" can be transferred to $\{L_n^\sigma\}$ and $\{H_n^\sigma\}$. However they are incomparable in the subelementary cases. $\{E_n\}$ and $\{E_n^k\}$ can trivially be embedded in $\{L_n^\sigma\}$ and $\{H_n^\sigma\}$ for a suitable σ. But which of these hierarchies is the "best generalization" of $\{E_n\}$ and $\{E_n^k\}$?

On the one hand $\{H_n^\sigma\}$ is related to $\{R_n^\sigma\}$ like $\{E_n^{(k)}\}$ is to $\{R_n^{(k)}\}$, while $\{L_n^\sigma\}$ is not for $n = 1, 2$. But some important functions, e.g., *length*(t), *equal*(s, t), are not resp. not obviously in H_2^σ. For that reason $\{H_n^\sigma\}$ doesn't provide a good notion of feasibility.

On the other hand, for each $n \geq 2$ the class L_n^σ turns out to be computationally equivalent to a word class E_n^k, and hence to a Turing machine time computation class. Thereby several natural functions like *length*(t), *height*(t), *equal*(s, t), *treeunion*(s, t) can easily be shown to be efficiently computable. Especially L_2^σ is equivalent to polynomial time computability. So there exists a notion of feasible term algebra functions. Some properties of $\{E_n\}$ and $\{E_n^k\}$ carry over to $\{L_n^\sigma\}$; for example: L_{n+1}^σ is universal for L_n^σ for all $n \geq 2$.

In this sense $\{L_n^\sigma\}$ is the better generalization of $\{E_n\}$ and $\{E_n^k\}$.

► ZBIGNIEW STACHNIAK, *Axiomatic varieties for nonmonotonic theories.*

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Properties of monotonic inference systems and properties of their theories are strongly linked. These links, however, are much weaker in nonmonotonic inference systems. In the absence of monotonicity, the meaning of a theory, as well as the role of theories in the analysis of inferences is radically different from the role theories play in monotonic calculi. Theories alone are not sufficiently expressive for the analysis of nonmonotonic inferences.

In the context of nonmonotonicity another syntactic notion assumes the role a theory plays in monotonic inference systems. The notion of the axiomatic variety for a theory captures all possible ways a theory can be axiomatized. With this notion, we recover many syntactic and semantic links that exist between monotonic inference systems and their theories but which are absent in nonmonotonic context. In this paper we introduce axiomatic varieties and study their basic properties. We show how axiomatic varieties can be used to identify, define, and compare nonmonotonic inference systems. The semantic links between properties of axiomatic varieties and properties of inference systems are also discussed. We formulate syntactic and semantic criteria for selected properties of nonmonotonic calculi (cummulativity, loop-cummulativity, and cut).

► KATRIN TENT, *Classifying totally categorical groups (and others).*

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Since the discovery of the group configuration by Zilber and Hrushovski, we know that groups abound almost everywhere in 1-based theories.

It has been know for some time [2] that minimal groups are just vector spaces over their (division) ring of definable $*$-endomorphisms (modulo $acl(\emptyset)$) and, more generally, that algebraic dependence of independent generics is connected 1-based groups can be completely described in terms of $*$-endomorphisms.

We show that for certain 1-based groups (i.e. groups in which the only definable set are boolean combinations of cosets of definable subgroups, by [3], in particular totally categorical ones), all structure is determined by the ring of $*$-endomorphisms of some (possibly larger) definable group. While is was know from pp-elimination of quantifiers for modules [1] that all modules are 1-based, Theorem 1 gives a partial converse

We call an element *reducible* if it is interalgebraic with a finite set of elements of smaller rank. Otherwise it is irreducible. A group is said irreducible if a generic of it is or, equivalently, if all its elements are irreducible.

We show the following:

THEOREM 1., *Let T a unidimensional 1-based theory, and let H be a group whose generics are irreducible with all minimal types being locally finite. Then there is an irreducible group G in H^{eq} containing H as a subgroup such that any connected subgroup of G^n, for $n < \omega$ is the connected component of a subgroup linearly defined over the ring $End^*(G)$. Furthermore, G is generated by H under $End^*(G)$.*

G is essentially a pure QE-module over the ring of definable endomorphisms of G with no extra structure.

[1] W. BAUR, *Elimination of quantifiers for modules*, ***Israel Journal of Mathematics***, vol. 25 (1976), pp. 64–70.

[2] E. HRUSHOVSKI, *Locally modular regular types*, ***Classification theory*** (J. T. Baldwin, editor), Lectures notes in Mathematics, Springer Verlag, 1987, pp. 132–164.

[3] E. HRUSHOVSKI and A. PILLAY, *Weakly normal groups*, ***Logic colloquium '85***, North Holland, Amsterdam-New-York, 1987, pp. 233–244.

► TINKO TINCHEV, *Intersection and graded modalities*.

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As is well known the intersection of two modalities is not expressible. Nevertheless, one can prove that adding to minimal (normal) modal logic the axiom saying that intersection-modality is subset of each of modalities, we obtain a completeness with respect to Kripke models [2]. The detailed analysis in [3] shows that an algorithm exists which for any formula gives its formal proof or tree-like finite model of its negation.

Also, an algorithm exists with the same property, but for graded modalities [4]. Here we mention only that this algorithm is in a sense "orthogonal" to the constraint method of Donini *et al.* [1].

Analyzing these algorithms we find the cause for the difficulties described in [5] for the case of intersection of graded modalities (at least 3): it is needed to add new axioms reflecting the well known combinatorial principle for counting "inclusion and exclusion". For the obtained formal system about graded modalities with intersections there is an algorithm with the above mentioned property. This gives decidability and completeness with respect to finite tree-like models.

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[2] G. GARGOV, S. PASSY, and T TINCHEV, *Modal environment for boolean speculations*,

Plenum Press, New York, 1986.

[3] T. TINCHEV, *A note on intersection of modalities*, ***Annuaire de l'Universite de Sofia***, Faculté de Mathématiques et Informatique, vol. 87, livre 1 (1993), in print

[4] ———, *Semantic tableau method for graded modalities*, Manuscript, March 1993.

[5] W. VAN DER HOEK and M DE RIJKE, *Counting objects in generalized quantifier theory, modal logic, and knowledge representation*, ***Vrije Universiteit Amsterdam***, Rapportnr. IR-307, 1992

► SERGEI TUPAILO, *Normalization for the first order classical logic with Hilbert's epsilon-symbol.*

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We prove normalization theorems for three theories: $\mathbf{LK}\varepsilon$, the first order classical logic with Hilbert's epsilon-symbol, $\mathbf{LK}\varepsilon{=}$, $\mathbf{LK}\varepsilon$ + equality, and $\mathbf{LK}\varepsilon{=}\mathbf{E}$, $\mathbf{LK}\varepsilon{=}$ + extensionality.

There were a number of preceding results, proving normalization for different epsilon-calculi. Yasuhara (1982) proved *normal form theorems* (theorems about existence of a normal form) for theories similar to those considered in this paper; Mints (1989) established a normal form theorem for the second order classical logic. Both these proofs use model-theoretic approach. There is also a number of constructive proofs for partial cases. Leisenring (1969) proved normalization theorem for derivations of epsilon-free sequents in a calculus without extensionality Mints (1973, 1975) gave a proof for the first order intuitionistic systems. Tupailo (1993) established normalization for a weak subsystem of analysis with restricted occurrences of epsilon-symbol. Note that the transfer from intuitionistic systems to classical is not straightforward and requires additional ideas. Still there were no constructive proof of normalization for general case: derivations of arbitrary sequents in a calculus with equality and extensionality.

It turned out to be very useful to formulate rules of inference dealing with epsilon-terms in the form of "generalized cuts", i.e. with a property that each formula in the conclusion occurs in the premises. We take epsilon-rule in the form

$$\frac{\Gamma \longrightarrow \Theta, Ft \qquad F[\varepsilon x Fx], \Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta}\ \varepsilon$$

and extensionality rule in the form

$$\frac{\Gamma \longrightarrow \Theta, \forall x(Fx \leftrightarrow Gx) \qquad \varepsilon x Fx = \varepsilon x Gx, \Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta}\ ext$$

This choice of rules makes standard cut-elimination go through without any changes, and enables to define epsilon-elimination (elimination of non-necessary epsilon-terms) simply as a substitution of t for $\varepsilon x Fx$ in an appropriate epsilon-rule, which makes it into a cut In the case of $\mathbf{LK}\varepsilon{=}\mathbf{E}$, when extensionality is present, the scheme is simple as well: when we are eliminating $\varepsilon x Fx$ from a derivation, first ext-rules mentioning $\varepsilon x Fx$ are to be killed by means of additional cuts (ext is replaced by cuts); after that cuts are eliminated as usual. We hope that this device for eliminating extensionality will be useful in higher order systems.

Skolem's and *Herbrand's theorems* (proved constructively) are obtained as immediate corollaries of normalization.

► DIMITER VAKARELOV, *Modal rules for intersection.*

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It is a well known fact that irreflexivity is not modally definable. In [1] Gabbay shows how to axiomatize irreflexivity by a special rule of inference. In [2] van Benthem noticed that irreflexivity is expressible in a modal language with propositional quantifiers by the formula (Irr) $(\exists p)(\neg p \wedge \Box p)$. We generalize these results for intersection of modalities.

Let $\mathcal{L}$ be a modal language with modalities $\langle\alpha\rangle$, $\langle\beta\rangle$ and $\langle\alpha\cap\beta\rangle$ interpreted in frames of the form $(W, R(\alpha), R(\beta), R(\alpha\cap\beta))$, satisfying the condition $(\cap)$ $R(\alpha\cap\beta) = R(\alpha)\cap R(\beta)$. This condition is not modally definable in $\mathcal{L}$, but, in a language with propositional quantifiers, $(\cap)$ is definable by the following formula

$$\langle\alpha\cap\beta\rangle A \Leftrightarrow (\forall p)(\langle\alpha\rangle(A\wedge p) \vee \langle\beta\rangle(A\wedge\neg p)).$$

Dealing with the axiomatization of the intersection in a non quantified modal language we can simulate the quantifier rule and the Barcan axiom by a special derivation rule, which generalizes Gabbay's irreflexivity rule:

$$\frac{A_0 \Rightarrow [\gamma_1](A_1 \Rightarrow [\gamma_2](A_2 \Rightarrow \ldots [\gamma_n](A_n \Rightarrow \langle\alpha\rangle(A\wedge p) \vee \langle\beta\rangle(A\wedge\neg p))\ldots))}{A_0 \Rightarrow [\gamma_1](A_1 \Rightarrow [\gamma_2](A_2 \Rightarrow \ldots [\gamma_n]A_n \Rightarrow \langle\alpha\cap\beta\rangle A)\ldots))}, \quad n = 1, 2, \ldots$$

p is a variable, not in $A_1, \ldots, A_n, A$ and γ_i is any of α and β.

As an application we can find similar rules for some other constructs as $\mathrm{Loop}\alpha = \langle\alpha\cap \mathrm{Id}\rangle\mathrm{true}$ and complement $\langle-\alpha\rangle A$.

[1] D. M. GABBAY, *An irreflexivity lemma with applications to axiomatizations of conditions on linear frames*, ***Aspects of philosophical logic***, Reidel, Dordrecht, 1981, pp. 67–89.

[2] J. VAN BENTHEM, ***Modal logic and classical logic***, Bibliopolis, 1985.

PAULO A. S. VELOSO, ARMANDO M HAEBERER AND MARCELO F. FRIAS, *Fork algebras as algebras of logic.*

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Fork algebras are extensions of relational algebras by a *fork* operator, which enables the introduction of projections by definition (making them special *quasi-projective* algebras [2]). Their abstract calculus, motivated by programming needs, provides an algebra for Classical First-Order Logic with Equality (CFOLE), which is finitary [1], in contrast with *Cylindric* and *Polyadic* algebras.

The *Elementary Theory of Fork Relations* (ETFR) is a two-sorted theory, with an injective binary operation $\star$ over sort $\boldsymbol{\imath}$, extending Tarski's *Elementary Theory of Binary Relations* (ETBR) [3] formulated with a ternary predicate symbol -[-,-] over sorts $\boldsymbol{\ell}$, $\boldsymbol{\ell}$ and $\boldsymbol{\imath}$, by a new binary operation ∇ over sort $\boldsymbol{\ell}$, with defining axiom

$$(\forall x : \boldsymbol{\imath})(\forall y : \boldsymbol{\imath})(r\nabla s[x,y] \leftrightarrow (\exists v : \boldsymbol{\imath})(\exists w \cdot \boldsymbol{\imath})(y \approx v\star w \wedge r[x,v] \wedge s[x,w])).$$

THEOREM 1 (Expressiveness of ETFR). ETFR *is a conservative extension of* CFOLE, *where for every formula* $\varphi(x_1, \ldots, x_n)$ *of* CFOLE *there exists a closed term* r *of sort* $\boldsymbol{\ell}$ *such that.*

$$\mathsf{ETFR} \vdash (\forall x_1, \ldots, x_n : \boldsymbol{\imath})(\varphi(x_1, \ldots, x_n) \leftrightarrow r[\star(x_1, \ldots, x_n), \star(x_1, \ldots, x_n)]),$$

where $\star(x_1, \ldots, x_n) = x_1 \star (x_2 \star \ldots (x_{n-1} \star x_n) ..)$ [3]

The *Abstract Language of Fork Relations* (ALFR) is obtained by forgetting $\boldsymbol{\imath}$, $\star$ and -[-,-]. The *Abstract Calculus of Fork Relations* (ACFR) is obtained by adding to the usual axioms for abstract atomic relation algebras [2] the following axioms

(1) $r\nabla s = (r;\pi) \bullet (s;\rho)$
(2) $(r\nabla s);(p\nabla q)^{\smile} = (r;p^{\smile}) \bullet (s;q^{\smile})$
(3) $\pi\nabla\rho \preceq 1$
(4) $0 \neq t \preceq \infty \otimes \infty \rightarrow (\exists v)(\exists w)(0 \neq v \otimes w \preceq t)$

where *projections* and *cross* product are defined by

$$\pi = (1\nabla\infty)^{\smile} \quad \rho = (\infty\nabla 1)^{\smile} \text{ and } r \otimes s = (\pi; r)\nabla(\rho; s).$$

THEOREM 2 (Algebra of Logic). ETFR *is a conservative extension of* ACFR. *For every formula* φ *of* CFOLE *there exists a closed term* $\dagger(\varphi)$ *of* ALFR *such that* CFOLE $\vdash \neg\varphi$ *iff* ACFR $\vdash \dagger(\varphi) = 0$ *This mapping* $\dagger$ *induces a Boolean-algebra isomorphism between the Lindenbaum algebra of formulas of* CFOLE *and the algebra of closed terms of* ALFR *modulo provable equality under* $1 \otimes 1 = 1$

[1] M. F. Frias, G. A. Baum, A. M. Haeberer, and P. A. S. Veloso, *A representation theorem for fork-algebras*, *PUC-Rio Res Rept.*, MCC 29, 1993, To appear in: ***Bulletin of the Section of Logic, Polish Academy of Sciences***.

[2] A. Tarski and S Givant, *A formalization of set theory without variables*, ***American Mathematical Society Colloquium Publication***, vol. 41 (1987).

[3] P. A. S. Veloso and A. M Haeberer, *A finitary relational algebra for classical first-order logic*, ***Bulletin of the Section of Logic, Polish Academy of Sciences***, vol. 20 (1991), no. 2, pp. 52–62.

► ANDREAS WEIERMANN, *How to characterize provably total functions by local predicativity.*

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The aim of this paper is to give a straightforward characterization of the provably total functions of Peano-Arithmetic, of Peano-Arithmetic plus the scheme of transfinite induction over initial segments of certain primitive recursive well-orderings and of Kripke Platek set theory with the axiom of infinity. Of course, our results are standard results from classical proof theory.

However, not only experts in proof theory and subrecursive hierarchies but also a wider readership interested in general proof-theoretic techniques will find this paper interesting, since we avoid completely the use of heavy metamathematical machinery like e.g. Mints' continuous cut elimination. Up to now the most direct approach of characterizing provably total functions of Peano-Arithmetic in terms of the fast growing hierarchy below level ε_0 was proposed in [Buchholz and Wainer (1987)]. Compared with this reference this paper's approach has the following advantages. Based on a new approach to subrecursion which is developed in [Buchholz, Cichon and Weiermann (1994)] we develop a very smooth theory of deduction relations $<_n$ These have much nicer closure-properties (for example compatibility with the natural sum) than the relations $\prec_n$ used in [Buchholz and Wainer (1987)]. The verification of these properties can be done by simple one-line-proofs. Moreover this theory can be generalized straightforwardly to stronger ordinal notation systems without much complication. As an immediate corollary to the usual cut elimination procedure for infinitary derivations we obtain the desired result by local predicativity style methods [cf. Pohlers (1981)] in a very perspicuous way. Also, our approach may be applied with minor modifications uniformly to very strong theories. To indicate this we apply our methods to $KP\omega$. In the meantime the author's idea has been successfully applied to the theory KPM, the strongest theory for which an ordinal analysis has been published to-date (cf. [Rathjen (1991)]).

► A. D. YASHIN, *The Smetanich logic has the unique Novikov complete extension.*

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Let B the intuitionistic propositional logic, $L(\varphi)$ be the extension of the intuitionistic propositional language by a new propositional constant symbol φ and $H(\varphi)$ is a logic in the language $L(\varphi)$ such thay $H \subseteq H(\varphi)$.

Definition 1 (P. Novikov). $H(\varphi)$ *determines* a new intuitionistic logical constant φ iff

(1) $H(\varphi)$ is a conservative extension of H;
(2) for every formula B without occurrences of φ, the logic $H(\varphi) + \{\varphi \equiv B\}$ is not conservative over H.

DEFINITION 2 (P. Novikov). $H(\varphi)$ is said to be *complete* iff

(1) $H(\varphi)$ is a conservative extension of H;
(2) for every formula $B \notin H(\varphi)$, the logic $H(\varphi) + \{B\}$ is not conservative over H.

EXAMPLE (Smetanich [1]). The logic $Sm = H + \{\neg\neg\varphi, \varphi \supset (p \vee \neg\varphi)\}$ determines a new intuitionistic logical constant.

It is clear (by Zorn Lemma) that every conservative over H logic $H(\varphi)$ has a Novikov completion.

The *Novikov's problem* was formulated ([1]) as follows: to give an explicit example of a complete logic which determines a new intuitionistic logical constant.

THEOREM. *Sm has the unique Novikov complete extension. This completion is finitely axiomatizable, it is characterized by a simple class of finite tree frames, and it is decidable.*

[1] YA. S. SMETANICH, *On completeness of the propositional calculus with additional unary operation*, ***Proceedings of Moscow Mathematical Society***, vol. 9 (1960), pp. 357–371.

▶ MICHAEL ZAKHARYASCHEV AND ALEXANDER ALEKSEEV, *The decidability of finitely axiomatizable normal modal logics containing K4.3.*

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By the well known results of Bull and Fine, all modal logics above **S4.3** have the finite model property, finite axiomatizations and so are decidable. However, normal extensions of **K4.3**, also characterized by Kripke frames of width 1 without infinite ascending chains (i.e., by *Noetherian chains of clusters*), but possibly with irreflexive points, do not necessarily have the finite model property and finite axiomatizations and may be undecidable (see Kracht [1]). Yet, by using the canonical formulas of Zakharyaschev [2], we show that finitely axiomatizable normal logics above **K4.3** are decidable. All the frames below are assumed to be Noetherian chains of clusters; NExt**K4.3** denotes the set of all normal extensions of **K4.3**.

THEOREM 1. *For every frame* **G** *and every canonical formula* $\alpha(\mathbb{F}, \mathbf{D}, \bot)$, $\mathbb{G} \not\models \alpha(\mathbf{F}, \mathbf{D}, \bot)$ *iff there is an injective cofinal subreduction of* **G** *to* **F** *satisfying the closed domain condition* (CD) *for* **D**.

In other words, every refutation frame for $\alpha(\mathbb{F}, \mathbb{D}, \bot)$ can be obtained from **F** by inserting some Noetherian chains below clusters $C(x) \in \mathbf{F}$ generated by points x such that $\{x\} \notin \mathbb{D}$ and possibly by enlarging some non-degenerate clusters in $\mathbb{F}$.

Let $\alpha(\mathbb{F}, \mathbb{D}, \bot)$ be a canonical formula, $\mathbb{F} = \langle W, R\rangle$ and $C(x_0)R \ldots RC(x_n)$ all distinct clusters in $\mathbb{F}$. By a *type for* $\alpha(\mathbb{F}, \mathbf{D}, \bot)$ we mean any n–tuple $t = \langle \xi_1, \ldots, \xi_n\rangle$ such that, for $i \in \{1, \ldots, n\}$, either $\xi_i = m$ or $\xi_i = m+$, for some $m < \omega$, or $\xi_i = \omega$, with $\xi_i = 0$ if $\{x_i\} \in \mathbf{D}$. Given a type $t = \langle \xi_1, \ldots, \xi_n\rangle$ for $\alpha(\mathbf{F}, \mathbb{D}, \bot)$, we define a *t-extension* of $\mathbb{F}$ as the frame **G** that is obtained from **F** by inserting between each pair $C(x_{i-1})$, $C(x_i)$ either a descending chain of m irreflexive points, if $\xi_i = m < \omega$, or a descending chain of $m + 1$ points of which only the last (lowest) one is reflexive, if $\xi_i = m+$, or an infinite descending chain of irreflexive points, if $\xi_i = \omega$.

THEOREM 2 *If $L \in$ NExt**K4.3** and $\alpha(\mathbf{F}, \mathbb{D}, \bot) \notin L$, then there is a type t for $\alpha(\mathbf{F}, \mathbb{D}, \bot)$ such that the t-extension of $\mathbb{F}$ validates L and refutes $\alpha(\mathbb{F}, \mathbf{D}, \bot)$.*

It follows that every logic in NExt**K4.3** is characterized by a recursively enumerable set of (possibly infinite) frames.

Let $\mathbb{H}$ be a finite frame, k the number of irreflexive points in $\mathbb{H}$, $t = \langle \xi_1, \dots, \xi_n \rangle$ a type for $\alpha(\mathbf{F}, \mathbb{D}, \bot)$ and $\mathbb{G}$ the t-extension of $\mathbb{F}$. Construct a subframe $\mathbf{G}_k$ of $\mathbf{G}$ by "cutting off" the infinite descending chains inserted in $\mathbb{F}$ just below their $k + 1$th points and let X be the set of all these $k + 1$th points.

THEOREM 3. $\mathbf{G} \not\models \alpha(\mathbb{H}, \mathbf{E}, \bot)$ *iff there is an injective cofinal subreduction f of $\mathbb{G}_k$ to $\mathbb{H}$ satisfying (CD) for $\mathbf{E}$ and such that $X \cap \operatorname{dom} f = \emptyset$.*

Thus, given a type t for $\alpha(\mathbf{F}, \mathbf{D}, \bot)$ and a canonical formula $\alpha(\mathbf{H}, \mathbb{E}, \bot)$, only finitely many steps is required to verify whether the t-extension of $\mathbb{F}$ refutes $\alpha(\mathbb{H}, \mathbb{E}, \bot)$, and so we can prove the decidability of finitely axiomatizable logics in NExt**K4.3** by using Harrop's old good argument.

THEOREM 4. *All finitely axiomatizable normal extensions of **K4.3** are decidable*

[1] M. KRACHT, *Prefinitely axiomatizable modal and intermediate logics*, ***Mathematical Logic Quarterly***, vol. 39 (1993), pp. 301–322.

[2] M. ZAKHARYASCHEV, *Canonical formulas for **K4**, Part I: Basic results*, ***The Journal of Symbolic Logic***, vol. 57 (1992), pp. 1377–1402.

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NOTICES

- Election: In the 1994 ASL Election, George Boolos was elected President and Menachem Magidor was elected Vice President. Yuri Gurevich and Haim Gaifman were elected to the Executive Committee of the Council, and Elisabeth Bouscaren and Carlos Di Prisco were elected to the Council. All terms are for three years beginning January 1, 1995.
- The First Sacks Prize has been awarded to Dr. Gregory Hjorth of the California Institute of Technology, as author of the best dissertation in the field of logic during 1993 and 1994. Hjorth completed his Ph.D. degree in 1993 under the direction of Professor W. Hugh Woodin at the University of California, Berkeley. His thesis research in descriptive set theory was singled out by the selection committee for its surprising consequences concerning the relationship between projective sets and large cardinals. The Sacks Prize will be offered for the second time in 1995; eligible are students who complete their degrees between September 30, 1994, and September 30, 1995. Nominations from thesis advisors are due by September 1, 1995. The Sacks Prize Fund seeks additional donations to the fund, in hopes of being able to award the Prize annually. A donor has established a matching fund which will match 25% of all donations made during 1994–98 from individuals who made a first donation in 1993 (and there is a cap on the amount that will be matched). Another donor has set up a similar program which applies to those who did not contribute in 1993. Checks should be made payable to the Sacks Prize Fund and sent to: Ms. Sonia Franklin-Sparrock, Assistant Administrative Officer, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, Attention Sacks Prize Fund.
- The 1995 ASL European Summer Meeting (Logic Colloquium '95) will be held August 9–17, 1995, in Haifa, Israel. For further information write to the following address (preferably by email): Logic Colloquium '95, Yvonne Sagi, Department of Computer Science, Technion – Israel Institute of Technology, 32000 Haifa, Israel; email: `logic95@cs.technion.ac.il`.
- The 1995–96 ASL Winter Meeting will be held in conjunction with the Annual Meeting of the American Mathematical Society, during January 10–13, 1996, in Orlando, Florida. Abstracts of contributed papers from ASL members (300 words, one page limit) should be sent by the deadline of October 20, 1995, to the Program Chairman, William J. Mitchell, Department of Mathematics, University of Florida, Gainesville, Florida 32611; email: `mitchell@math.ufl.edu`.
- The 1995–96 ASL Annual Meeting will be held March 9–12, 1996, at the University of Wisconsin in Madison, Wisconsin. The Program Committee consists of S. Buss, P. Cholak, H. J. Keisler, Chair, D. A. Martin, and M. Resnik. Chair of the local organizing committee is Steffen Lempp. Abstracts of contributed papers from ASL members (300 words, one page limit) should be sent by the deadline of January 12, 1996, to: H. J. Keisler, Mathematics Department, University of Wisconsin, Madison, Wisconsin 53706; email: `keisler@math.wisc.edu`.
- The 1996 ASL European Summer Meeting (Logic Colloquium '96) will be held July 7–13, 1996, in San Sebastián, Spain. The organizing committee will be chaired by Jesus Larrazabal. The program committee will be chaired by Daniel Lascar.

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- THE 1996–97 ASL ANNUAL MEETING will be held March 22–25, 1997, at the Massachusetts Institute of Technology in Cambridge, Massachusetts. Chair of the local organizing committee is Sy Friedman.
- THE 1997 ASL EUROPEAN SUMMER MEETING (LOGIC COLLOQUIUM '97) will be held in early July, 1997, in Leeds, England. The local organizing committee consists of B. Cooper, J. Derrick, F. Drake, D. Macpherson, A. Slomson, J. Truss, and S. Wainer.
- THE 1995 ANNUAL CONFERENCE OF THE AUSTRALASIAN ASSOCIATION FOR LOGIC will be held June 30–July 2, 1995, at the University of New England Armidale, New South Wales, Australia. The Conference Secretary, from whom further information may be obtained, is Dr. Leslie Roberts, Department of Philosophy, University of Queensland; email: `lroberts@lingua.cltr.uq.oz.au`.
- THE TENTH LATIN AMERICAN SYMPOSIUM ON MATHEMATICAL LOGIC will be held July 24–29, 1995, in Bogota, Colombia. Further information may be obtained from: Secretary X-SLALM, Departamento de Matematicas, Universidad de los Andes, Apartado Aereo 4976, Bogota, Colombia; email: `xslalm@cdcnet.uniandes.edu.co`; telephone: (571) 2838885; Fax: (571) 2841890.
- THE DIMACS SUMMER SCHOOL ON APPLIED LOGIC AND ALGORITHMS will be held during August, 1995, as the first program in their 1995-96 Special Year. It will consist of one-week courses on Finite Model Theory, Proof Complexity, and Computer Aided Verification. Although registration at the door is permitted, those who preregister by June 30, 1995 will receive priority if the Summer School is oversubscribed. For forms and further information, use any of the following: WWW, Mosaic, Lynx, or NetScape to `http://dimacs.rutgers.edu`; telnet to `info.rutgers.edu 90`, email: `center@dimacs.rutgers.edu`; telephone: 908-445-5928; Fax: 908-445-5932; paper mail to DIMACS Center, Rutgers University, P.O. Box 1179, Piscataway, New Jersey 08855-1179.
- THE RESMOD SCHOOL ON MODEL THEORY AND REAL ALGEBRAIC GEOMETRY will be held in Segovia, Spain on June 28–July 1, 1995. Information may be obtained from: `resmod95@sunall.mat.ucm.es` or Secretaria Departamento de Algebra (RESMOD School), Facultad de Matematicas, Universidad Complutense, Ciudad Universitaria, Madrid, E-28040 Spain.
- THE SECOND BARCELONA LOGIC MEETING will be held June 29–July 1, 1995, in Bellatera (Barcelona), Spain. Further information may be obtained from: Centre de Recerca Matemàtica (CRM), Apartat 50, 08193 Bellatera (Barcelona), Spain; email: `crm@crm.es`.
- THE EUROPEAN CONFERENCE ON SYMBOLIC AND QUANTITATIVE APPROACHES TO REASONING AND UNCERTAINTY will be held July 3–5, 1995, at the University of Fribourg, Switzerland. For further information contact: Professor J. Kohlas, Institute of Informatics, Regina Mundi, CH-1700 Fribourg, Switzerland; telephone: +41-37-219-560; Fax: +41-37-219-670; email: `ECSQARU.iiuf@unifr.ch`.
- THE SEVENTH CONFERENCE ON COMPUTER-AIDED VERIFICATION (CAV'95) will be held July 3–6, 1995, in Liege, Belgium. Further information may be obtained from the Program Chair, Pierre Wolper, Université de Liege, Institut Montefiore, B28, Grande Traverse 10, B-4000 Liege, Belgium; email: `cav95@montefiore.ulg.ac.be`. Information may also be retrieved using anonymous ftp from `ftp.montefiore.ulg.ac.be` in the directory `pub/cav95`.
- THE 2ND WORKSHOP ON LOGIC, LANGUAGE, INFORMATION AND COMPUTATION (WoLLIC '95) will be held July 26–28, 1995, in Recife, Brazil. The chair of the program committee is P. Veloso. To obtain further information about the workshop, contact R. de Queiroz, Departamento de Informatica, Federal University of Pernambuco at Recife, P.O. Box 7851,

Recife, PE 50732-970, Brazil; email: `wollic95@di.ufpe.br`; telephone: +55 81 271 8430; Fax: +55 81 271 4925.

- A CONFERENCE ON CATEGORY THEORY AND COMPUTER SCIENCE (CTCS-6) will be held August 7–11, 1995, at the University of Cambridge, England, in conjunction with the third Cambridge Summer Meeting in Category Theory. Further information may be obtained from Dr. David Pitt, Department of Mathematical and Computing Sciences, University of Surry, Guildford, Surrey, GU2 5XH United Kingdom.

- A WORKSHOP ON ADVANCES IN TYPE SYSTEMS FOR COMPUTING (ATSC) will be held August 14–18, 1995, at the Isaac Newton Institute for Mathematical Sciences, Cambridge, UK. For further details contact: J. Mitchell, Department of Computer Science, Stanford University, Stanford, CA 94305-2140, USA; email: `jcm@cs.stanford.edu`. Details of this event are available via the Newton Institute WWW homepage, at URL `http://www.newton.ac.uk/`.

- THE SEVENTH EUROPEAN SUMMER SCHOOL IN LOGIC, LANGUAGE AND INFORMATION will be held August 14–25, 1995, in Barcelona, Spain. Further information may be obtained from: ESSLLI95, GILCUB, Avda. Vallvidrera 25, 08017 Barcelona, Spain; telephone: +43 3 2033597, Fax: +43 3 2054656; email: `esslli95@gilcub.es`.

- A WORKSHOP ON COMPUTABILITY AND COMPLEXITY IN ANALYSIS will be held August 19–20, 1995, at the FernUniversität, Hagen, Germany. The email address for the workshop is `cca@fernuni.hagen.de`.

- THE SECOND INTERNATIONAL CONFERENCE IN MATHEMATICAL PHILOSOPHY: THE NATURE OF TRUTH IN MATHEMATICS will be held September 13–21, 1995, in Mussomeli, Sicily, Italy. Invited speakers include D. S. Bridges, A. Conte, H. G. Dales, E. Effros, H. Field, D. Isaacson, V. Jones, G. Lolli, A. Macintyre, P. Maddy, Yu. Manin, D. A. Martin, P. Martin-Löf, Y. Moschovakis, G. Oliveri, D. Prawitz, T. Slaman, W. Tait, and W. H. Woodin. For further information contact H. G. Dales, Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England; email: `pmt6hgd@gps.leeds.ac.uk`; or G. Oliveri, Department of Philosophy, University of Leeds, Leeds LS@9JT, England; email: `oliveri@leeds.ac.uk`; or T. Slaman, Department of Mathematics, University of Chicago, Chicago, Illinois, 60637; email: `ted@math.uchicago.edu`.

- THE WORKSHOP ON HIGHER-ORDER ALGEBRA, LOGIC AND TERM REWRITING (HOA '95) will be held September 21–22, 1995, in Paderborn, Germany. Further information may be obtained from: Bernhard Möller, Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany; email (preferred): `moeller@uni.augsburg.de`; Fax +49-821-5982274.

- A WORKSHOP ON MODERN MATHEMATICAL THOUGHT: ITS HISTORY AND PHILOSOPHY will be held September 21–24, 1995, at the University of Pittsburgh and Carnegie Mellon University. Speakers are K. Manders, H. Bos, J. Tappenden, J. Grey, M. Wilson, J. Lutzen, C. Parsons, M. Friedman, G. Hatfield, H. Karzel, R. Haubrich, U. Majer, H. Stein, M. Hallett, D. Scott, J. Norton, and W. Sieg. Further information may be obtained from the Center for Philosophy of Science, 817 Cathedral of Learning, University of Pittsburgh, Pittsburgh, Pennsylvania 15260; telephone: 412-624-1050; Fax: 412-624-3895.

- THE ANNUAL CONFERENCE OF THE EUROPEAN ASSOCIATION FOR COMPUTER SCIENCE LOGIC (CSL'95) will be held September 22–29, 1995, in Paderborn, Germany. The program includes a tutorial on finite model theory in which J. Flum and E. Grädel will lecture (September 22–24). Chair of the Program Committee is H. Kleine Buning, Fachbereich 17, Mathematik–Informatik, Universität-GH Paderborn, D-33095 Paderborn, Germany; Fax: +49-5251-60-3338; email: `csl95@uni.paderborn.de` Information is available on the World Wide Web at URL `http://www.uni-paderborn.de/fachbereich/AG/agklbue/csl.95/start.html`.

- A Summer School on Semantics and Logics of Computation will be held September 25–29, 1995, at the Isaac Newton Institute for Mathematical Sciences, Cambridge, UK. For further details contact Florence Leroy, Isacc Newton Institute, 20 Clarkson Road, Cambridge CB3 0EH, UK; telephone: +44 1223 335984; Fax: +44 1223 330508; email: `f.leroy@newton.cam.ac.uk`. Details of this event are available via the Newton Institute WWW homepage, at URL `http://www.newton.ac.uk/`.

- The Tbilisi Symposium on Language, Logic and Computation will be held October 19–22, 1995, at Tbilisi State University in Tbilisi, Republic of Georgia. Further information may be obtained via anonymous ftp at the address `ftp.cogsci.ed.ac.uk` in the directory `pub/tbilisi` or by writing to: J. Ginzburg, Human Communication Research Centre, University of Edinburgh, 2 Buccleuch Place, Edinburgh EH8 9LW, Scotland; email: `ginzburg@cogsci.ed.ac.uk`; telephone: +44-131-650-4627; Fax: +44-131-650-4587.

- A Workshop on Temporal Reasoning in Deductive and Object Oriented Databases will be held December 8, 1995, in Singapore. Further information may be obtained on the World Wide Web at URL `http://max.cs.uni-magdeburg.de /institut/db/DOOD-WS.html`, by email at the address `conrad@iti.cs.uni.magdeburg.de`, or by writing to Stefan Conrad (TRDOOD'95), University of Magdeburg, Institute for Technical Information Systems, P. O. Box 4120, D-39016 Magdeburg, Germany.

- The Tenth Amsterdam Colloquium will be held December 18–21, 1995, at the University of Amsterdam. This biennial meeting is intended to foster interdisciplinary research among logicians, philosophers, linguists, and computer scientists. The invited speakers include R. Capenter, G. Chierchia, J. Halpern, H. Kamp, A. Kratzer, F Landman, B. Partee, K. Segerberg, and A. Szabolsci.

- The Third International Workshop on Deontic Logic in Computer Science (DEON '96) will be held January 11–13, 1996, in Lisbon, Portugal. Program Committee Co-Chairs. M. Brown, 541 Hall of Languages, Syracuse University, Syracuse, New York 13244-1170; email: `mabrown@mailbox.syr.edu`; telephone 315- 443-2536; Fax: 315-443-5675; J. Carmo, Department of Mathematics, Instituto Superior Tecnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal; email: `jcc@inesc.pt`; telephone: 351-1-8417141; Fax: 351-91-230243.

- The Congress "Truth: Logic, Representation and World" will be held January 17–20, 1996, in Santiago de Compostela, Spain. Information may be obtained from the Organizing Committee, Dpto. de Lógica y Filosofía de la Ciencia, Fac de Filosofía y CC. de la Educación, Universidad de Santiago de Compostela, Campus Sur, 15706 Santiago de Compostela, Spain, telephone: +34-9-81-521581; Fax: +34-9-81-563100; email: `lflpcmav@usc.es`.

- The Sixth Asian Logic Conference will be held May 27–30, 1996, in Beijing, China. Submissions of papers or abstracts, written in English using the TeX or AMS-TeX system, should be sent (to the address below) before December 1, 1995. For further information contact: The Sixth Asian Logic Conference, Institute of Software, Academia Sinica, P. O. Box 8718, Beijing, China; Fax: (861)-2562533; email. `logic@ox6.ios.ac.cn`.

- The 1996 Federated Logic Conference (FLoC'96) will be held July 27–August 3, 1996, at the College Avenue Campus of Rutgers University, downtown New Brunswick, New Jersey. This consists of the following annual conferences: CADE (Conference on Automated Deduction), CAV (Conference on Computer-Aided Verification), LICS (IEEE Symposium on Logic in Computer Science), and RTA (Conference on Rewriting Techniques and Applications). The meetings will be hosted by DIMACS as part of its Special Year on Logic and Algorithms.

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AMERICAN LOGIC IN THE 1920s

MARTIN DAVIS

In 1934 Alonzo Church, Kurt Gödel, S. C. Kleene, and J. B. Rosser were all to be found in Princeton, New Jersey. In 1936 Church founded The Journal of Symbolic Logic. Shortly thereafter Alan Turing arrived for a two year visit. The United States had become a world center for cutting-edge research in mathematical logic. In this brief survey[1] we shall examine some of the writings of American logicians during the 1920s, a period of important beginnings and remarkable insights as well as of confused gropings.

The publication of Whitehead and Russell's monumental *Principia Mathematica* [18] during the years 1910–1913 provided the basis for much of the research that was to follow. It also provided the basis for confusion that remained a factor during the period we are discussing. In 1908, Henri Poincaré, a famous skeptic where mathematical logic was concerned, wrote pointedly ([13]):

> It is difficult to admit that the word *if* acquires, when written $\supset$, a virtue it did not possess when written *if*.

Principia provided no very convincing answer to Poincaré. Indeed the fact that the authors of *Principia* saw fit to place their first two "primitive propositions"

> ***1.1:** Anything implied by a true proposition is true.
> ***1.2:** $\vdash p \vee p \supset p$

under one and the same heading suggest that they had thought of what they were doing as just such a translation as Poincaré had derided. A radically different view was implied by the American logician Clarence Lewis in his explanation of ***1.1** ([12]):

> The main thing to be noted about this operation of inference is that it is not so much a piece of reasoning as a mechanical, or strictly mathematical, operation for which a rule has been given. No "mental" operation is involved except that required to recognize a

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[1] This paper was written for a Festschrift volume honoring Alonzo Church and is based on a talk given by the author at a symposium honoring Alonzo Church in 1990 at the State University of New York in Buffalo.

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> previous proposition followed by the main implication sign, and to set off what follows that sign as a new assertion.

In the same treatise Lewis put his finger on the source of the confusion:

> … the system takes the laws of the logical relations of propositions for granted in order to prove them.

The decade of the 1920s began with Emil L. Post's seminal doctoral dissertation[2]. Post made Lewis's influence on his thought quite explicit with numerous references to [12]. In his paper, Post studied the propositional calculus[3] of *Principia Mathematica* as a purely "formal" system making a clear distinction between the theorems of the system and theorems about the system.

> We here wish to emphasize that the *theorems* of this paper are *about* the logic of propositions but are *not included* therein. [italics original]

Post introduced the use of "truth tables" as a tool for proving theorems "*about* the logic of propositions." He used the symbols $+, -$ for the truth values and proved that the formulas provable in the *Principia Mathematica* axiomatization are precisely those whose truth tables consist of a column of $+$'s. Post noted that his result showed that the *Principia* propositional calculus is complete and consistent and that it provided an algorithmic procedure for determining whether a given formula was derivable in that system.[4] In Post's proof, he made systematic use of what has come to be called *structural induction*; it is important to realize that since Whitehead and Russell thought of their development as providing a *foundation* for arithmetic, and hence for mathematical induction, using induction to develop their system would have seemed hopelessly circular. In proving his main result, Post observed that the *Principia* rules enabled one to prove for each formula α a formula of the form $\alpha \equiv \beta$ where β is in what is now called *disjunctive normal form* and is obtainable from α by straightforward algebraic operations. Next, for each such β containing at least two propositional variables, one of its variables p could be "factored" yielding a provably equivalent formula of the form:

$$(\delta_1 \wedge p \wedge \neg p) \vee (\delta_2 \wedge p) \vee (\delta_3 \wedge \neg p)$$

where $\delta_1, \delta_2, \delta_3$ do not contain the variable p. The result then follows easily by induction.

Post not only gave a complete analysis of this part of the Whitehead-Russell system, but also explored a number of important generalizations: The two-valued *truth tables* that Post introduced were generalized to tables involving an arbitrary finite number of variables. The rules of inference of the propositional calculus were generalized to rules determined by *productions* of a form so general that (viewed from a later perspective, of course) all recursively enumerable sets of strings were encompassed among the sets of "theorems" of the "logics" that could be defined using such rules. Finally, the $\sim, \vee$ system of *Principia* was generalized to arbitrary systems of truth functions that are complete in the sense that every truth function can be obtained by their suitable iteration; in this last case, only the main result is given.[5]

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[2] [14].

[3] Post did not use this term.

[4] In [1], Paul Bernays stated that he had obtained results similar to Post in his Göttingen *Habilitationsschrift* of 1918.

In Alonzo Church's dissertation [2] of 1926, Church proposed to do for set theory (including Cantor's theory of transfinite numbers) what the creators of non-Euclidean geometry had done for geometry. Lobachevsky had explicitly considered the parallel postulate of Euclid as a mere assumption the consequences of whose denial could be studied, rather than as an immutable a priori property of space. So Church proposed to consider Zermelo's axiom of choice as likewise a mere assumption the consequences of whose denial could be studied. Indeed, Church proposed a number of different propositions each of which contradicted the axiom of choice as possible "alternatives" to Zermelo's assumption. Although Church's dissertation was written in the style of unformalized axiomatics, he was clearly acutely aware of set theory together with logic as a foundation for mathematics. Indeed, he presciently commented:

> It is not improbable that the set of postulates at the basis of our logic is not complete, even if the axiom of choice is included in the set, ...

Church made his line of thought clear by explaining that he believed this to be true

> ... because if it were complete it ought to be possible, in [each] case to construct a particular function F of the kind whose existence is required by the axiom of choice.

Noteworthy contributions to logic and foundations of mathematics were few and far between during the twenties. Among these, we may turn to C. H. Langford. In his [8], [9] Langford developed what has come to be known as *the method of elimination of quantifiers* to solve the decision problem for the first order theory of dense linear orders. However, despite this very important technical contribution, Langford remained badly confused.

For the authors of *Principia*, the logic they were developing and studying was the one true logic. Despite Post's emphasis on using mathematical methods to investigate systems of logics as formal combinatorial objects and Church's willingness to explore variant systems of logic, this attitude persisted and is perhaps seen most sharply in Langford's writings. In 1928 Church wrote:[6]

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[5] The proof was published only two decades later [15]. For further discussion of Post's dissertation and related matters see [6].

> The purpose of this paper is to discuss the possibility of a system of logic in which the law of the excluded middle is not assumed ...

In this article, Church argued strongly for an attitude towards the principles of logic that is open to the possibility of studying variant systems. To this, Langford ([10]) hotly retorted:

> This means that we cannot have alternative logics; for logic is the system of all propositions expressing analytic facts of a certain kind, namely formal analytic facts, and there cannot in the nature of the case be more than one such system, actual or conceivable.

In Europe the year 1928 saw the publication of a little textbook of logic [7] written from the perspective of Hilbert's school that proved to be extremely influential. In particular the propositional calculus and first order logic were developed as formal combinatorial systems with properties that could be investigated by normal mathematical methods. The problem of the completeness of first order logic was presented and provided the young Kurt Gödel with his dissertation topic. The *Entscheidungsproblem,* the decision problem for first order logic, was characterized as *the* fundamental problem of mathematical logic; the fact that this problem is unsolvable was to be proved by Church and by Turing (independently of one another) eight years later. C. H. Langford [11] reviewed this book for the *Bulletin of the American Mathematical Society.* His review not only reflected his commitment to the one true logic, but indeed showed that he quite refused to permit himself to begin to understand what the authors were about.

Hilbert and Ackermann began with a formal treatment of the propositional calculus very much after the manner of Post's dissertation. They used four of the five formulas which the *Principia* development used as "primitive propositions."[7] These were

$$X \vee X \rightarrow X$$
$$X \rightarrow X \vee Y$$
$$X \vee Y \rightarrow Y \vee X$$
$$(X \rightarrow Y) \rightarrow [Z \vee X \rightarrow Z \vee Y]$$

Langford translated these into *Principia* notation, displaying them as

(a): $(p) : p \vee p \cdot \supset \cdot p,$
(b): $(p, q) : p \cdot \supset \cdot p \vee q,$
(c): $(p, q) : p \vee q \cdot \supset \cdot q \vee p,$
(d): $(p, q, r) : \cdot p \supset q \cdot \supset : r \vee p \cdot \supset \cdot r \vee q,$

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[6] in [3]

[7] Paul Bernays had shown that the fifth was superfluous, being derivable from the remaining four.

and observed that they were presented "together with a rule of substitution and a rule of inference.[8]" The parenthetical expressions are (in *Principia* notation), universal quantifiers; Langford uses them to emphasize his conception of these formulas as representing "analytic formal" truths holding with arbitrary propositions substituted for the variables. Langford is dreadfully puzzled over the authors' proofs that these formulas regarded as axioms are "consistent, independent, and complete."

> All we have to do ... in order to see that propositions (**a**) – (**d**), together with the two rules are consistent, is simply to observe that they are all plainly true.

Langford's comments on the proofs of independence are even worse:

> There is no occasion for examining this argument directly; it is quite certainly mistaken, for the sufficient reason that ... (**c**) follows from (**a**), (**b**), and (**d**).

For, as Langford notes, the authors have shown that $p \supset p$ does follow from (**a**), (**b**), and (**d**); insisting that there is no distinction to be made between $p \vee q$ and $q \vee p$ "since only one meaning occurs, namely, 'At least one of the two propositions is true'," Langford asserts that (**c**) is then just an instance of $p \supset p$. Completeness also falls by the wayside since "such true propositions as $\sim (p, q) \cdot p \vee q$ ... appear to be not derivable by the rules given."

It is a great relief to turn from Langford's nonsense to Haskell Curry's work. Curry, studying with Hilbert, turned to a fundamental analysis of the operation of substitution ([4]) previously taken to be primitive. This was the beginning of Curry's work in combinatory logic. Working from an entirely different direction, Church was led in the early 1930s to the λ-calculus, and eventually to the famous "thesis" that bears his name.[9] It was Church's students Kleene and Rosser who were to show that the λ-calculus and combinatory logic were fully equivalent.

Studies like this one can have the effect of bolstering a sense of our superiority to our logical forbears. But this would be a serious mistake. The lessons to be learned are rather that the development of the outlook on our subject that today we take for granted was attained only with great difficulty.

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HOW TO COMPUTE ANTIDERIVATIVES

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This is *not* about the symbolic manipulation of functions so popular these days. Rather it is about the more abstract, but infinitely less practical, problem of the primitive. Simply stated:

Given a derivative $f : \mathbb{R} \rightarrow \mathbb{R}$, how can we recover its primitive?

The roots of this problem go back to the beginnings of calculus and it is even sometimes called "Newton's problem". Historically, it has played a major role in the development of the theory of the integral. For example, it was Lebesgue's primary motivation behind his theory of measure and integration. Indeed, the Lebesgue integral solves the primitive problem for the important special case when $f(x)$ is bounded. Yet, as Lebesgue noted with apparent regret, there are very simple derivatives (e.g., the derivative of $F(0) = 0$, $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$) which cannot be inverted using his integral.

The general problem of the primitive was finally solved in 1912 by A. Denjoy. But his integration process was more complicated than that of Lebesgue. Denjoy's basic idea was to first calculate the definite integral $\int_a^b f(x)\,dx$ over as many intervals (a, b) as possible, using Lebesgue integration. Then, he showed that by using these results, the definite integral could be found over even more intervals, either by using the standard improper integral technique of Cauchy, or an extension technique developed by Lebesgue (see appendix for details). By proving that at least one of these techniques would always succeed, the process could be continued until the definite integral over all possible intervals was obtained. At this point, the antiderivative $F(x) = \int_0^x f(x)\,dx$ (up to a constant) becomes apparent. The trouble with Denjoy's procedure is that it needs to be continued transfinitely and, in fact, may require arbitrarily large countable ordinals to complete. He called his process "totalization". The question was immediately raised (for example in Lusin's thesis) as to whether such use of transfinite numbers was really necessary. Could perhaps a different approach avoid these countable ordinals (or at least arbitrarily large ones) and still recover the primitive?

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In 1915, H. Bauer, using an integral introduced by O. Perron a year earlier, proposed a brand new solution. By utilizing the concept of "major" and "minor" functions, introduced by de la Vallee-Poussin, it was able to avoid any mention of transfinite ordinals. Compared to totalization, this new integral was incredibly simple and much easier to understand and work with (see appendix for details). Accordingly, the Perron-Bauer solution was enthusiastically received by many and gradually became the standard for further investigation. Still, some people (e.g., Looman [11]) complained that by avoiding transfinite ordinals something crucial was lost. Denjoy himself was extremely critical of the Perron-Bauer approach and over the years seemed to become increasingly bitter. Some of his later writings contain scathing attacks on Perron personally as well as his integral.

To understand Denjoy's complaint (about the integral), it helps to consider a third solution to the problem, which is rather silly and reminiscent of a famous joke by the comedian Steve Martin:

> You can be a millionaire and never pay taxes. That's right! You can be a millionaire and never pay taxes. How? It's easy. Two simple steps: First, get a million dollars. Then when they ask you why didn't you pay taxes say "I forgot" and when they say "You forgot!?" say "Well excuuuuuuuuuuse me".

The Steve Martin solution to the primitive problem might go something like this: First, get the right function. Then, show it has the derivative you were searching for.

Now try to ignore the fact that this "Martin integral" is probably the most powerful method of integration known and, in practice, has successfully computed more antiderivatives than all other solutions combined. Consider instead how hard it is to "guess" the antiderivative. For example, try to find the integral of $\sin x/x$. Well, ok you might be able to "guess" some sort of infinite series solution. But things get worse than this. Dougherty and Kechris [8] have shown (using Y. Matiyasevich's work on diophantine representation of recursively enumerable sets [12]) that there are derivatives which are analytically expressible (in terms of an explicit formula using the basic elementary functions sin, cos, exponents, absolute values, etc., and the elementary operations of multiplication, division, composition, and infinite sums) but whose primitive is immensely complicated, so that for example, there is no way to analytically express the primitive. The Martin solution, therefore, cannot seriously be considered a solution at all. But it does seem to illustrate what is the real problem of the primitive and that is—the problem itself. Maybe, the Denjoy-Perron controversy would not have been so bitter had the problem been proposed in a more precise manner. What does it mean to be "given a derivative"? or to "recover a primitive"? This is where the logicians come in.

Let's call a solution "nonconstructive" (which is a polite way of saying that as far as solving the primitive problem, it is just as silly as the Martin integral) if it involves a search over all continuous functions, or something which is morally equivalent to this (e.g., a search over all real numbers). The Perron integral is nonconstructive (see appendix). Nearly a half century later, the Riemann-complete integral was introduced by J. Kurzweil and then rediscovered and developed by R. Henstock. The name comes because it is just a slight variation of the classical Riemann integral, yet it is strong enough to invert derivatives (see appendix). Unlike the Perron integral, it has a natural appeal and really has something substantial to say. Sadly, it too is nonconstructive. In fact, so is just about every other solution which has ever been proposed (see [3]) except those which are based on the original totalization procedure of Denjoy. But does totalization also contain something which makes it nonconstructive? A negative answer was given in the mid-1980s by M. Ajtai. He showed that totalization (or at least a variation of it) follows a very strict and very precise notion of definability (perhaps if Denjoy had known about this he could have expressed his criticism in a more "constructive" manner). Ajtai's result is unpublished but it is referenced in an article by Dougherty and Kechris [8], who outline their own proof of this result. They call this type of definability "Δ^1_1 on the set of derivative codes". But their main thrust was to show that no substantial improvement to this will ever be possible for *any* solution to the primitive problem. So, in a sense, they classified the complexity of the operation of antidifferentiation, answered the question from Lusin's thesis, and proved that Denjoy's solution is and always will be in some sense the best! (something Denjoy probably suspected all along).

But what does any of this have to do with constructiveness? The problem said to "recover" the primitive, not "redefine" it. Well, these two notions can sometimes be very closely related. It follows from the pioneering work of S. Kleene, that there is always a computer type algorithm for carrying procedures which are "hyperarithmetically" defined. It seems natural therefore to try to exhibit such an algorithm. That way, the true essence of both the primitive problem and its solution could be grasped without first taking a year off to study Descriptive Set Theory [14]. Furthermore, since constructiveness is really the whole point, a computer program would be philosophically the most direct way of understanding it.

To accomplish this project, we first need a computer language for carrying out these Kleene-type algorithms. This has been provided by the work of Harel and Kozen and we will discuss it below.

Secondly, because of the sophistication of the techniques of Dougherty and Kechris, our project won't be feasible using their proof, and so we will need a new proof of their result, utilizing only monotone inductions (also explained below). Dougherty and Kechris used non-monotone inductions but they were able to get around this by carefully calculating bounds (using an argument of W. H. Woodin) for the ordinal lengths of their inductions.

Before we begin, let's be clear that we are not talking about a real computer here. Indeed, ordinary finite computers are useless even for the most elementary questions about real numbers. But that's all right. Our goal is not to present an algorithm which can be physically carried out. Rather, it is to use computer programs to illustrate and explain the constructive nature of antidifferentiation. Unfortunately, we will be forced to rely on the mind of the reader to act as our CPU and to do this for a program which we have no way of debugging. The solution may appear at first a little long as computer programs often do, but this is only because we are trying to give all the details. It is hoped that the reader will patiently follow these details in the beginning and that in this way, the notion of constructiveness becomes so ingrained that many of the details in the last part of the program can be easily skipped. We will assume familiarity with Lebesgue measure and integration, the Baire Category theorem, the Heine-Borel Theorem, and the notion of transfinite induction. For a reader who is already well versed in Descriptive Set Theory and only wishes to see the new proof of Ajtai's theorem, see §4.

§1. **Warm up.** If we wish to exhibit an algorithm for finding antiderivatives, ordinary computers are clearly inadequate. For one thing, an ordinary computer can only talk about integers and we need to talk about real numbers and functions. For another, every ordinary computation must be finite. But this does not mean that there can't be a high degree of constructiveness. For example, suppose we are given an infinite sequence of zeros and ones and wish to know if the sequence ever contains a one. Everyone agrees that there is a simple algorithm for doing this, even though it may be impossible to physically carry out. It seems that it would take a computer with a special ability (sometimes called the "infinite mind") to accomplish this task. The "infinite mind" of the computer would allow it to go through an infinite sequence of steps which are already known to be computable and then report if a certain event ever occurred. For reasons which will become apparent later, it is more convenient to imagine that this "infinite mind" checks each of these steps simultaneously rather than sequentially.

The "hyperarithmetical" sets of integers are intuitively the sets whose membership can be decided by such an "infinite minded" computer, and the result of Ajtai, Dougherty, and Kechris (combined with the work of Kleene) surprisingly says that this "infinite mind" is the only extra thing we need to calculate antiderivatives! But first, we have to mention a couple of more things. Since our computer is only allowed to talk about integers (rational numbers and finite strings of integers, etc., are also ok) we have to do all computations using only these objects. For example, we will say that a definite integral is "computed" if we have a program which can tell which rationals lie below the integral and which lie above it. Secondly, we need to be able to use the special "infinite mind" capability in ordinary ways naturally associated with computer programs. For example, using a simple subroutine, the "infinite mind" will also be able to tell if an infinite sequence of zeros and ones has infinitely many ones. But the real power of these computations comes from constructing loops using "**go to**" statements. Harel and Kozen [9] have developed the following computer programming language (called "IND") to make things more explicit. In its simplest form, the language consists of only three types of allowable statements.

```
$l_1$: $y \leftarrow \forall$ (or $y \leftarrow \exists$)
$l_2$: accept (or reject)
$l_3$: if $R(x)$ go to $l_4$
```

(where $R(x)$ is a relation which can be determined with an ordinary computer, i.e., "recursive".)

A program in this language consists of a finite sequence of such labeled statements. Statements of the first type access the "infinite mind" of the computer. Informally, if the statement has the form "$y \leftarrow \forall$" then the computer runs the remainder of the program for all values y of the appropriate data type (see next paragraph). If all of them lead to "**accept**" then the computer is said to accept at this step. If one of them leads to "**reject**" then the computer is said to reject at this step. In all other cases the program is said to have an infinite loop (∞) at this step. The semantics of $y \leftarrow \exists$ are similar and statements of the second and third type are self-explanatory.

The variables in the program will represent either natural numbers, integers, rationals, or finite sequences of these objects. The domain of a variable should be clear from the context. However, just in case, we will use i, j, k, m, n, M in the program to represent positive integers, ε will represent the reciprocal of a positive integer and all other lower case letters will represent rationals. When a variable appears with a bar over it then it will represent a finite sequence. For example, $\bar{p}$ will represent a finite sequence of rationals, $\langle p_1, p_2, \ldots, p_{l(\bar{p})}\rangle$ where $l(\bar{p})$ is the length of the sequence. We make no distinction between numbers and sequences of length one. We also use $\bar{p} \wedge \bar{q}$ to represent the concatenation of two sequences, $\langle p_1, \ldots, p_{l(\bar{p})}, q_1, \ldots, q_{l(\bar{q})}\rangle$. It is clear that all of these objects can be effectively coded by integers.

For convenience, we will often list similar commands on a single line, separated by commas; e.g.,

```
$l_{10}$: $x \leftarrow \forall$, $y \leftarrow \forall$, $z \leftarrow \forall$.
```

Also, we usually put labels only on the statements which are referred to in a different part of the program. In addition, we will use the following "macros" from [9] which are easily computable from the allowable statements:

```
$l_4$: go to $l_t$,
$l_5$: if $R(x)$ accept (or reject), and
$l_6$: $y \leftarrow x$
```

which will be considered abbreviations respectively of:

```
$l_4$: if $0 = 0$ go to $l_t$,
$l_5$: if $R(x)$ go to $l_7$
       go to $l_t$
$l_7$: accept (or reject) (where $l_t$ is the label of the next statement), and
$l_6$: $y \leftarrow \exists$
       if $y \neq x$ reject.
```

As a quick example, suppose we are "given" an infinite sequence of zeros and ones, i.e., $\alpha(1), \alpha(2), \ldots$. In the program we reflect that this sequence is "given" to us by allowing the $R(x)$ in the third type of statement to also be replaced by the relation $\alpha(x) = n$. Now, we can easily design a program which will accept if and only if the given sequence contains infinitely many ones. For example,

```
$l_1$: $n \leftarrow \forall$
       $m \leftarrow \exists$
       if $m < n$ reject
       if $\alpha(m) = 1$ accept
       reject
```

The IND programs provide us with a natural framework in which to do transfinite inductions and also provide the strictest notion of the word "constructive" which allows us to compute antiderivatives. It is at least in some sense possibly much stronger than what Denjoy had in mind when he developed his process of totalization. For example, while our special computer must have an infinite amount of memory available, we are only allowed to use finitely many variables, so that we are only able to store a finite amount of information. Therefore, not every transfinite induction on sets of integers can be programmed this way.

There are several ways to measure the time it takes to run a program in this language. For example, we might try to measure the time elapsed since the beginning of the computation, which we call "forward time", or instead we might try to measure the time from the end of the computation, which we call "backward time". In general these will be different. For example, if we use forward time and we imagine that the branching statements ($x \leftarrow \forall$ and $x \leftarrow \exists$) are handled in parallel then we might say that the running time is the supremum of the lengths of the branches in the resulting computation tree, or at least the part of the tree necessary to determine the output of the program. Since all of our programs are finite in length, the branches of such a tree, and hence the running time, will always be less than or equal to ω. This is not very useful for comparison purposes since most programs will have a running time equal to ω.

Backwards time is defined inductively on the same tree, working from the leaves to the root, assigning either a countable ordinal or else infinity (∞) to each node (see [9] for precise details). Roughly speaking, all the end nodes (accept or reject) are given a time value of 1. The time to completion at a "**go to**" statement is the time at the following node (either the next statement of the program or the destination of the "**go to**"), plus 1. Nodes of the tree associated with an "$x \leftarrow \forall$" statement will have infinitely many immediate successors. The time to completion is then the minimum time it takes for one of these successors to be rejected (in the case that there is such a successor), plus 1, or the supremum of the completion times at all immediate successors (in the case that they all lead to "**accept**"), plus 1, or else infinity (∞) (if neither of the other two cases holds). Nodes associated with statements of the form "$x \leftarrow \exists$" are handled similarly. When we talk about time (we only do this in Proposition 1) we prefer to use this backwards notion. For us it has the advantage of being able to say that if we are in the middle of a program then it takes less time to finish the program than it would to just start the whole thing over from the beginning.

Another convenient thing to remember is that the programming of an inductive procedure is also always backwards. To illustrate this, we consider another example: "Determine the perfect part P of a closed set C". (Recall that a set is "perfect" if it is closed and has no isolated points. The perfect part of a closed set is the same as the set of condensation points, that is, the points in C which have uncountably many elements of C in any neighborhood.) Since we don't allow direct computations with real numbers, let's consider the closed set C as "given" by the relation:

$$R(p,q) \iff (p,q) \text{ is a rational interval in the complement of } C$$

which may be substituted for the $R(x)$ in statements of the third kind (i.e., "**if** $R(x)$ **go to** l_4" can be replaced with "**if** $R(p,q)$ **go to** l_4"). Let P' denote the collection of rational intervals contained in the complement of P. To "produce" the perfect part, P, we mean that we have a program which "accepts" exactly when the input is in P'.

The simplest way to get to the perfect part is through the following induction, which is essentially the same as that of Cantor-Bendixson (see for example [14]): Start with $G^0 =$ the set of rational intervals in the complement of C. Then let

$$(*) \qquad (p,q) \in G^{\alpha+1} \iff (\forall \varepsilon)(\exists r,s)\ 0 < s - r < \varepsilon \\ \& \big(p = r \text{ or } (p,r) \in G^{\alpha}\big) \\ \& \big(s = q \text{ or } (s,q) \in G^{\alpha}\big).$$

If λ is a limit ordinal then $G^{\lambda} = \bigcup_{\alpha<\lambda} G^{\alpha}$. Since every time G is increased it will contain a new rational interval (this is sometimes called the Cantor-Baire stationary principle), there must be some countable ordinal γ where $G^{\gamma+1} = G^{\gamma}$ at which point it is not hard to check that $G^{\gamma} = P'$.

To construct a program for this process, first note that the definition $(*)$ "arithmetically" defines $G^{\alpha+1}$ from G^{α} (i.e., all variables represent integers, rationals, etc., and all relations are computable with an ordinary computer, with the exception of membership in G^{α}). Notice also that the references to G^{α} are "positive" since the complement of G^{α} is not mentioned. Because of this, the induction is "monotone", which means that an increase in G^{α} would cause $G^{\alpha+1}$ to either increase or remain the same. Monotonicity is extremely helpful in programming the induction as we will explain below.

Notice that we can't simply start with G^0 then compute G^1, G^2, etc. Such a process would require both an infinite amount of storage and a computation tree with transfinite branches, neither of which are allowed in our strict notion of constructiveness. Instead, we program it "backwards" using methods outlined in [9]. First, reword $(*)$ as follows:

$$(**) \quad (p,q) \in G \iff (\forall \varepsilon)(\exists r,s)\ 0 < s - r < \varepsilon \\ \& \big(\text{either } p = r \text{ or } (p,r) \text{ is already known to be in } G\big) \\ \& \big(\text{either } s = q \text{ or } (s,q) \text{ is already known to be in } G\big).$$

We now write a program with references to the "old" G (see statements l_2 and l_4) represented by a loop back to the first statement. The role of the variable n is simply to insure that both (p,r) and (s,q) get checked for membership in the "old" G.

```
l_1: if $R(p,q)$ accept
     $\varepsilon \leftarrow \forall$, $r \leftarrow \exists$, $s \leftarrow \exists$
     if $s - r \geq \varepsilon$ or $s \leq r$ reject
     $n \leftarrow \forall$
     if $n = 1$ go to $l_3$
     if $p = r$ accept
     $q \leftarrow r$
l_2: go to $l_4$
l_3: if $s = q$ accept
     $p \leftarrow s$
```

```
$l_4$: go to $l_1$
```

We would like to now check that this program does what it is supposed to do, i.e., it accepts (p, q) if and only if (p, q) is in P'. However, since we will be using this same technique repeatedly, let us talk in a little more generality. Note that in the program above, we combined the two loops into one. That is, in l_2 we didn't say directly to go back to l_1, we saved that for the end.

Therefore, let $\mathcal{P}$ be a program in the language IND, with a finite sequence $\bar{x}$ of input variables, and with an initial statement labeled say "l_1" and with final statement "**go to** l_1". Let P'' be the collection of $\bar{x}$ which are accepted by $\mathcal{P}$. Suppose we change the last statement of $\mathcal{P}$ to the "illegal" statement "**if** $\bar{x} \in A$ **accept**" where A is a set variable and where this command has the obvious meaning. Let's call this "illegal" program $\mathcal{P}_A$. Define $\mathcal{Q}(A) = \{\, \bar{x} \mid \mathcal{P}_A \text{ accepts } \bar{x} \,\}$. Then let $Q^0 = \emptyset$, $Q^{\alpha+1} = \mathcal{Q}(Q^\alpha)$, and when λ is a limit ordinal, $Q^\lambda = \bigcup_{\alpha<\lambda} Q^\alpha$. Note that $\mathcal{Q}$ is a "monotone" operator meaning that if A is increased, $\mathcal{Q}(A)$ must increase or remain the same. Therefore at some (countable) ordinal γ we will have $Q^{\gamma+1} = Q^\gamma$. It is immediate from the definitions that $\mathcal{Q}(P'') = P''$ and that $\mathcal{Q}(Q^\gamma) = Q^\gamma$.

PROPOSITION 1. *If P'' and Q^γ are as above then $P'' = Q^\gamma$.*

PROOF. Let α be the smallest ordinal (if there is one) such that there is some $\bar{x}$ in $Q^\alpha \setminus P''$. By its nature, α must be a successor ordinal. Then $\bar{x} \in \mathcal{Q}(Q^{\alpha-1})$. But $Q^{\alpha-1} \subset P''$ by hypothesis. Therefore, by monotonicity, $\bar{x} \in \mathcal{Q}(P'') = P''$ which is a contradiction.

In the other direction, let α be the least amount of time it takes for $\mathcal{P}$ to accept some $\bar{x}$ which is not in Q^γ (using some backwards notion of time, explained above). Let $A = \{\, \bar{y} \mid \mathcal{P} \text{ accepts } \bar{y} \text{ in time less than } \alpha \,\}$. Then $\bar{x} \in \mathcal{Q}(A)$ and by the induction hypothesis, $A \subset Q^\gamma$. But then by monotonicity, $\bar{x} \in \mathcal{Q}(Q^\gamma) = Q^\gamma$ which is a contradiction. ⊣

The above proposition can be used to check that a program works. Take, for example, our perfect set program given above. Observe that if we replaced the last statement "l_4: **go to** l_1" with the illegal statement "l_4: **if** $(p, q) \in G$ **accept**" then the program becomes an exact translation of $(**)$. Therefore, Q^γ is the same as G^γ and so $P'' = P'$.

In the above program, we have shown that $\mathcal{P}$ accepts all the right inputs and only the right inputs. The algorithm, however, still has a sort of defect. If we input a rational interval which is not in P' then the program will never report this to us, but instead will go into an infinite loop (check this). We might say then, that the perfect part P is really only "half-calculated". In such a situation, the operation is still referred to as "inductive". When we are assured the benefit of an answer for all inputs then it is called "hyperarithmetical". This is analogous to the "semi-recursive" vs. "recursive" relations in ordinary computability theory. It is because of this phenomenon that the use of monotone induction becomes so vital: Suppose for example, in defining $G^{\alpha+1}$ we referred to membership in the complement of G^{α}. Then the program would need a loop which referred back to this complement. This, in turn, would cause all inputs (even those in P') to enter the infinite loop.

Note however, that even though a program may halt on all inputs, this does not mean that it is totally free of infinite loops. This is because we say that a statement of the form "$y \leftarrow \exists$" is accepted if just one value of y leads to acceptance, even though others may lead to an infinite loop. The idea here is that the infinite loops never need to be completely run as long as we think of the "infinite mind" as having the ability to check all the values of y simultaneously.

Harel and Kozen analyzed which relations can be computed by a program in the language IND. Recall that a set of integers is "arithmetical" if and only if it can be defined in the language of arithmetic (using only $+$, $-$, $\times$, $\div$, exp, 1, $=$) (or equivalently, using only recursive relations) and using the usual logical symbols where quantifiers are over integers. It is not hard to check (see [9]) that any arithmetical relation can be programmed in the IND language and in fact can be programmed without using any "loops" (i.e., "**go to**" statements are always forward). For example, suppose $B(x)$ and $C(x)$ are computable from programs starting with labels l_2 and l_3 respectively. Then putting the following three statements before these programs yields a program for the relation $A(x) \leftrightarrow B(x) \,\&\, C(x)$:

```
$l_1$: $m \leftarrow \forall$
    if $m = 1$ go to $l_2$
    go to $l_3$
```

Harel and Kozen prove that the relations which can be programmed in an "inductive" way as defined above, are exactly the ones which can be defined using a transfinite induction $G = \bigcup G^{\alpha}$ where at limit ordinals we take unions and where $G^{\alpha+1}$ is defined "arithmetically" and "positively" from G^{α} as in the previous example. (Work of Kleene classifies these inductive sets as the "Π^1_1 sets", i.e., sets which can be defined using an arithmetical formula preceded by one "$\forall$" quantifier over infinite sequences of integers, so that the hyperarithmetical sets are the same as the "Δ^1_1" sets, i.e., sets which are both Π^1_1 and whose complements are Π^1_1.)

It has long been known that the classes of "inductive" and "hyperarithmetical" sets reflect this kind of algorithmic behavior. This may, in fact, have been Kleene's motivation in introducing them [10]. According to Rogers [17] this behavior has even been used by certain constructivists to claim that the "hyperarithmetical" sets are in some sense the only ones which really exist!

The work of Harel and Kozen makes more explicit the constructive nature of these classes. It also provides an intuitive and philosophically direct method for showing that a relation is in one of these classes—just program it. The naturalness of these classes is also reflected in the many closure properties they enjoy, properties which one would naturally associate with a notion of "computable" (see Rogers [17]).

§2. **Given a derivative.** Before we start our "antiderivative" program we must face the controversial issue of how the function f will be "given" to us. One natural way (and this is the way we will consider) is in terms of the inverse images of rational open intervals. For a derivative, each such inverse image is a countable union of closed sets, each of which may be represented by the collection of (non-degenerate) rational intervals in its complement. This leads us to the following relation:

$$\mathrm{INVIM}(a,b,x,y,n) \iff (x,y) \text{ is in the complement of the } n\text{th closed set in the inverse image of the function on } (a,b).$$

We will therefore consider our derivative as "given" by allowing this relation to be a replacement for $R(x)$ in statements of the type "If $R(x)$ go to l_n". Similarly, we will say that the integral will be "produced" by a program which can compute the relation:

$$\begin{aligned}\mathrm{INT}(p,q,r,n) \iff\ & n = 1 \,\&\, r > F(q) - F(p)\\ & \text{or } n = 2 \,\&\, r < F(q) - F(p),\\ & \text{where } F \text{ is any antiderivative of } f.\end{aligned}$$

The reader may now have a valid complaint. Although the relation INT seems non-controversial, perhaps the relation INVIM is giving away too much information about the function f. For example, consider that in Calculus we usually represent a function as a "machine" or a "black box" which when a value of x is inserted the machine spits out the value of $f(x)$. This sort of representation emphasizes that the inner workings of a function may not be known, and indeed, there may not be any rule or reason as to why a particular function does what it does. It should seem natural then to use a similar idea to represent our derivative. Why then don't we try to use a "black box" instead of the more revealing "inverse images of intervals"? Because doing so would render impossible any sort of countable constructive procedure for calculating the antiderivative. To see this, note that in any such procedure, only countably many values of x would ever be inserted into the "black box". Furthermore, which values to plug in would completely depend on the results of previous inquiries. Therefore, for each derivative there would be a countable set which would completely determine its antiderivative. But this is impossible! For any derivative and any countable set there is another derivative which has exactly the same values on the specified countable set. To see this, just add a derivative which is zero on the given set but not identically zero (such a derivative can be constructed directly, or its existence can be inferred from more general work of Zahorski, see [2], [19]). Nevertheless, classifying the operation of obtaining antiderivatives from a "black box" may still be interesting to study from the standpoint of recursion in higher types.

The use of inverse images can also be partially justified by the fact that in classical calculus, the integrands are really given by a formula. Of course it's not true that every derivative has such a formula, but the idea here is that the function should be presented in a manner which gives insight into how the function values are determined, rather than just as individual function values. The relation INVIM then seems natural because it provides an overall picture of the structure of a function, but in a way which can be used to represent any derivative. It may even seem reasonable to take the opposite point of view. Perhaps the derivative should be given to us in the strongest possible (nontrivial) way! Suppose for example, that method A gives away more information than method B and we know how to compute antiderivatives from method A, then the problem of computing antiderivatives from method B is reduced to the problem of how to get from B to A, which is no longer an integration problem.

Finally, it should be remarked that Dougherty and Kechris [8] consider a derivative given in a slightly different way; as an infinite sequence of continuous functions. We instead use inverse images because it seems more natural and more in line with what Denjoy was doing. Nevertheless, IND calculations with our "Borel code" or with their "Baire code" are completely equivalent. We skip the details. Other methods are also possible. Darji, Evans, and O'Malley have just recently given another very natural way to be "given" a derivative using what they call "first return continuity" (see [5]). It seems likely that their method, as well as others, will turn out to be computationally equivalent. But it is by no means certain that this will be true for every method of representing a derivative conceived in the future.

In the next section we will start our IND program for computing antiderivatives, proving that the operation of antidifferentiation is "inductive". Obtaining a "hyperarithmetical" program (which halts on *all* inputs) is impossible by a result of Dougherty and Kechris [8] who proved that the operation is not Borel. Nevertheless, it will be easy to get a program which halts on all inputs at least when the function given to us is guaranteed to be a derivative. The reason is that in this case, $\mathrm{INT}(p, q, r, 1)$ fails exactly when $\mathrm{INT}(p, q, s, 2)$ holds for every $s < r$, and $\mathrm{INT}(p, q, r, 2)$ fails exactly when $\mathrm{INT}(p, q, s, 1)$ holds for every $s > r$. This allows us to convert a program for $\mathrm{INT}(p, q, r, n)$ into a program for $\neg\mathrm{INT}(p, q, r, n)$. Then running the programs simultaneously, (see [9] for technical details of how to create a single IND program to combine two others) we have a computation which will halt on all derivatives. This level of complexity is what Dougherty and Kechris call "Δ^1_1 on the set of [codes of] derivatives" since although it is not strictly "hyperarithmetical", it always halts as long as we are given the code of a derivative. In other words, since the primitive problem says "Given a derivative ... " we have to assume that they are not tricking us by giving us something which may look like a derivative but really isn't.

§3. Recover the primitive (part 1). We are now ready to start the antiderivative program. We begin with the relation:

$$
\begin{aligned}
(1)\qquad & \text{“Cover}(\bar{p},\bar{q},s,t)\text{”} \\
&\iff \text{“The sequences } (\bar{p},\bar{q}) \text{ of rational intervals} \\
&\qquad \text{form a sequential cover of the rational} \\
&\qquad \text{interval } [s,t]\text{”} \\
&\iff (\exists k) l(\bar{p}) = l(\bar{q}) = k \,\&\, (\forall i \leq k) p_i < q_i \\
&\qquad\qquad \&\, (\forall i < k) p_{i+1} < q_i \,\&\, p_1 < s < t < q_k.
\end{aligned}
$$

Since the relation is arithmetical, it is easy to program both it and its complement. We give the programs here. (From this point on, we will be building our antiderivative program in blocks. Each block will end in an "**accept**", "**reject**", or an unconditional "**go to**" statement. It therefore won't matter which order we assemble the blocks except for the first one, which will begin with the label "l_1" and will be the start of the entire program. This first block will actually be the last one presented. This is why we start here with the label "l_2".)

```
$l_2$: $k \leftarrow \exists$
    if $l(\bar{p}) \neq k$ or $l(\bar{q}) \neq k$ reject
    $i \leftarrow \forall$
    if $i > k$ accept
    if $p_i \geq q_i$ reject
    if $i = k$ accept
    if $p_{i+1} \geq q_i$ reject
    if $p_1 < s < t < q_k$ accept
    reject
```

When a program halts on all inputs, then its complement can be programmed by merely interchanging "$\exists$" with "$\forall$" and also "**accept**" with "**reject**". Thus the following is a program for "$\neg\,$Cover$(\bar{p},\bar{q},s,t)$":

```
$l_3$: $k \leftarrow \forall$
    if $l(\bar{p}) \neq k$ or $l(\bar{q}) \neq k$ accept
    $i \leftarrow \exists$
    if $i > k$ reject
    if $p_i \geq q_i$ accept
    if $i = k$ reject
    if $p_{i+1} \geq q_i$ accept
    if $p_1 < s < t < q_k$ reject
    accept
```

Our program will involve two transfinite inductions. The first will be monotone and will serve to organize the reals into a chain of open sets, with the property that f is bounded on each "new part" of the chain. The second induction will be monotone with respect to the first and will calculate the actual integrals. Each induction will have two stages. We now describe the first stage of the first induction. Let S be a collection of rational intervals. We define

$$(2)\quad (x,y) \in U(S) \iff (\forall s,t)[s,t] \subset (x,y) \rightarrow \\ [(\exists \bar{p},\bar{q})\text{“Cover}(\bar{p},\bar{q},s,t)\text{”} \,\&\, (\forall i \leq l(\bar{p}))(p_i,q_i) \in S].$$

The operation $U(S)$ merely closes S under unions and subintervals, using compactness. The definition is "positive arithmetical" in S and so it is easy to write a program for $U(S)$ where we use the label l_8 to denote the start of a program (not yet given), which accepts an interval (a,b) if and only if $(a,b) \in S$.

```
$l_4$: $s \leftarrow \forall,\ t \leftarrow \forall$
    if $\neg(s < t \,\&\, [s,t] \subset (x,y))$ accept
    $\bar{p} \leftarrow \exists,\ \bar{q} \leftarrow \exists,\ m \leftarrow \forall$
    if $m = 1$ go to $l_2$
$l_5$: $i \leftarrow \forall$
    if $i > l(\bar{p})$ accept
    $a \leftarrow p_i,\ b \leftarrow q_i$
    go to $l_8$
```

Let T be another collection of rational intervals. We now give a program which accepts (x,y) if and only if $(x,y) \notin U(T)$. Here, l_8 is the start of a program (not yet given) which accepts (c,d) if and only if $(c,d) \notin T$.

```
$l_6$: $s \leftarrow \exists,\ t \leftarrow \exists$
    if $\neg(s < t \,\&\, [s,t] \subset (x,y))$ reject
    $\bar{p} \leftarrow \forall,\ \bar{q} \leftarrow \forall,\ m \leftarrow \exists$
    if $m = 1$ go to $l_3$
$l_7$: $i \leftarrow \exists$
    if $i > l(\bar{p})$ reject
    $c \leftarrow p_i,\ d \leftarrow q_i$
    go to $l_8$
```

One may wonder at this point why we are using the same label l_8 to represent both that $(a,b) \in S$ and that $(c,d) \notin T$. Actually, the program starting at l_8 is going to calculate one or the other, depending on an additional parameter $n \in \{1,2\}$. The parameter n will always be 1 when l_4 is reached and will always be 2 when l_6 is reached.

We now proceed to the second stage of the first induction. We define

$$
\begin{aligned}
(3)\quad (a,b) \in \mathcal{B}(S) \iff & (\exists M)(\exists n)(\forall x)(\forall y) \\
& [\mathrm{INVIM}(-M,M,x,y,n)\,\&\,(x,y) \subset (a,b)] \rightarrow (x,y) \in U(S).
\end{aligned}
$$

Note that the operation $\mathcal{B}$ adds to S rational intervals in which $\bigcup S$ covers the complement of one of the closed sets which make up the inverse image of $(-M,M)$ for some M. Thus if $(a,b) \in \mathcal{B}(S)$ then the range of f on $(a,b) \setminus \bigcup S$ is bounded. The operator $\mathcal{B}$ is monotone since $U(S)$ is and membership in $U(S)$ is only mentioned in a "positive" way. It can be iterated in the usual manner: $B^0 = \emptyset$, $B^{\alpha+1} = \mathcal{B}(B^\alpha)$, and if λ is a limit ordinal then $B^\lambda = \bigcup_{\alpha<\lambda} B^\alpha$.

Suppose that S does not contain all rational intervals. Then $\mathcal{B}(S)$ is strictly larger than S. To see this, notice that $\mathbb{R}$ is the union of all of the closed sets making up the inverse images of the intervals $(-M,M)$. Let $C = \mathbb{R} \setminus \bigcup S$. The Baire Category Theorem says that in some nontrivial portion, $(a,b) \cap C$, of C one of these closed sets is dense. But then, since the set is closed, it must contain $(a,b) \cap C$. Therefore, its complement in (a,b) must be covered by $\bigcup S$, and so $(a,b) \in \mathcal{B}(S)$. A similar argument shows that $\bigcup \mathcal{B}(S)$ is dense.

Therefore, $\mathcal{B}$ cannot become stationary until it contains everything, so at some countable ordinal γ, B^γ must be the entire collection of rational intervals. For this reason, it won't be helpful to program membership in B^γ. It will be more useful to have a program to tell in which order the intervals are admitted to B^γ. This is an instance of the Stage Comparison Theorem of Moschovakis, see ([13], p. 27) where the argument for it used below is credited to Kunen.

For each interval (p,q) let $\mathrm{ord}(p,q)$ be the least ordinal α such that $(p,q) \in B^\alpha$. By its nature, $\mathrm{ord}(p,q)$ is a successor ordinal. We abbreviate $(a,b) \leq (c,d)$ if and only if $\mathrm{ord}(a,b) \leq \mathrm{ord}(c,d)$ and $(a,b) < (c,d)$ if and only if $\mathrm{ord}(a,b) < \mathrm{ord}(c,d)$. To program these relations, notice first that

$$
\begin{aligned}
(4)\qquad (a,b) \leq (c,d) &\iff (a,b) \in \mathcal{B}(\{\,(a',b') \mid (a',b') < (c,d)\,\}) \\
\text{and } (a,b) < (c,d) &\iff (c,d) \notin \mathcal{B}(\{\,(c',d') \mid \neg(a,b) \leq (c',d')\,\}).
\end{aligned}
$$

Both of the above equivalences hold because of the monotonicity of $\mathcal{B}$. The second is just the negation of the first with the roles of (a,b) and (c,d)

interchanged. Define the relation

$$(5)\qquad D(a,b,c,d,n) \iff n=1 \text{ and } (a,b)<(c,d) \text{ or } n=2 \text{ and } (a,b)\le(c,d).$$

Using (4), the relation D can be built by transfinite induction using the following operator. Let A be a subset of $\mathbb{Q}^4 \times \{1,2\}$. Then let

$$(6)\quad (a,b,c,d,n)\in\mathcal{D}(A) \iff \\ n=1 \text{ and } (c,d)\notin\mathcal{B}(\{(c',d') \mid (a,b,c',d',2)\notin A\}) \\ \text{or } n=2 \text{ and } (a,b)\in\mathcal{B}(\{(a',b') \mid (a',b',c,d,1)\in A\}).$$

Note that because of the double negation in the first half of the right side, the monotonicity of $\mathcal{D}$ follows from the monotonicity of $\mathcal{B}$. As before, let $D^0=\emptyset$, $D^{\alpha+1}=\mathcal{D}(D^\alpha)$, and for limit ordinals λ, $D^\lambda=\bigcup_{\alpha<\lambda}D^\alpha$. Again, this forms an increasing sequence of sets which becomes stationary at some countable ordinal γ.

Proposition 2 (Moschovakis). *If D and D^γ are as above then $D=D^\gamma$.*

Proof. Let α be the smallest ordinal such that there exists an (a,b,c,d,n) in D where $\operatorname{ord}(a,b)=\alpha$ and $(a,b,c,d,n)\notin D^\gamma$. Let $n=2$ if possible and otherwise let $n=1$.

Case 1. $n=1$.

Since $(a,b,c,d,1)\in D$, we have that $(a,b)<(c,d)$. By induction hypothesis we have that $(a,b,c',d',2)\in D^\gamma$ for every $(c',d')\ge(a,b)$. But

$$\begin{aligned}(c,d)\notin\mathcal{B}(\{(c',d') \mid (c',d')<(c,d)\})\\ =\mathcal{B}(\{(c',d') \mid \neg(c',d')\ge(a,b)\})\\ \supset\mathcal{B}(\{(c',d') \mid (a,b,c',d',2)\notin D^\gamma\})\end{aligned}$$

by monotonicity. Therefore by (6), $(a,b,c,d,1)\in\mathcal{D}(D^\gamma)=D^{\gamma+1}=D^\gamma$ which is a contradiction.

Case 2. $n=2$.

Since $(a,b,c,d,2)\in D$, we have that $(a,b)\le(c,d)$. By induction hypothesis, we have that $(a',b',c,d,1)\in D^\gamma$ for every $(a',b')<(a,b)$. But

$$\begin{aligned}(a,b)\in\mathcal{B}(\{(a',b') \mid (a',b')<(a,b)\})\\ \subset\mathcal{B}(\{(a',b') \mid (a',b',c,d,1)\in D^\gamma\})\end{aligned}$$

by monotonicity. Therefore by (6), $(a,b,c,d,2)\in\mathcal{D}(D^\gamma)=D^{\gamma+1}=D^\gamma$ which is a contradiction.

For the other direction, (4) says that $D = \mathcal{D}(D)$, then induction on α using the monotonicity of $\mathcal{D}$ gives $D^\alpha \subset D$ for all α. $\dashv$

Using the technique already displayed in a previous example and taken from Harel and Kozen [9], we can now transform (3) and (6) into a program which accepts (a, b, c, d, n) if and only if $D(a, b, c, d, n)$. The reason the program does what it is supposed to follows as before, using Propositions 1 and 2:

```
$l_8$: if $n = 1$ go to $l_9$
     $M \leftarrow \exists, n \leftarrow \exists, x \leftarrow \forall, y \leftarrow \forall$
     if $\neg\mathrm{INVIM}(-M, M, x, y, n)$ or $\neg(x, y) \subset (a, b)$ accept
     $n \leftarrow 1$
     go to $l_4$
$l_9$: $M \leftarrow \forall, n \leftarrow \forall, x \leftarrow \exists, y \leftarrow \exists$
     if $\neg\mathrm{INVIM}(-M, M, x, y, n)$ or $\neg(x, y) \subset (c, d)$ reject
     $n \leftarrow 2$
     go to $l_6$
```

§4. Intermission. In the previous section we showed how to generate a transfinite sequence of collections of finite open rational intervals, $\{B^\alpha\}$, with the properties:

(1) If $I \in B^\alpha$ and J is an open rational subinterval of I then $J \in B^\alpha$,
(2) every finite rational open interval in $\bigcup B^\alpha$ is in $B^{\alpha+1}$,
(3) If $I \in B^{\alpha+1}$ then $f(x)$ is bounded on $I \setminus \bigcup B^\alpha$,
(4) $B^\lambda = \bigcup_{\alpha<\lambda} B^\alpha$ if λ is a limit ordinal,
(5) $\bigcup_\alpha B^\alpha$ contains all open rational intervals,
(6) B^0 is empty and for $\alpha > 0$, $\bigcup B^\alpha$ is dense.

We also showed how to effectively compute the relations $\mathrm{ord}(I) < \mathrm{ord}(J)$ (meaning I got into the sequence before$J$) and $\mathrm{ord}(I) \leq \mathrm{ord}(J)$.

Before we continue with the second half of the program, where the integrals are actually calculated, we pause to discuss the overall plan of totalization, why it works, and how we plan to make it programmable.

NOTATION. We denote by $|G|$ the Lebesgue measure of the set G. All of our intervals are finite. For an interval $J = (a, b)$, $(a, b]$, $[a, b)$, or $[a, b]$ we let $F(J)$ denote the integral $F(b) - F(a)$. If $G = \bigcup G_i$ is a countable union of disjoint intervals, then denote by $F(G)$ the sum $\sum F(G_i)$, but we only do this when the sum is known to be absolutely convergent, since in that case, the order of the summation is irrelevant. The relative integral is then defined to be $F(G)/|G|$. If S denotes a bounded measurable set on which f is bounded, then the Lebesgue integral of f over S exists and we call it $L(S)$, and we call $L(S)/|S|$ the relative Lebesgue integral of S.

Suppose now that we have a subrelation $A \subset \mathrm{INT}$. Recall that $\mathrm{INT} = \{ (p,q,r,n) \mid p < q$ are rational and either $n = 1$ and $r > F(q) - F(p)$ or $n = 2$ and $r < F(q) - F(p) \}$ which is the relation we are trying to determine. We denote by D the domain of A so that $D = \{ (p,q) \mid (\exists n, r)(p,q,r,n) \in A \}$ and suppose that the following properties hold:

(i) If $I \in D$ and J is an open rational subinterval of I then $J \in D$.

(ii) If $I \in D$ then $(p,q,r,n) \in A$ if and only if either $n = 1$ and $r > F(I)$ or $n = 2$ and $r < F(I)$.

These properties obviously hold for our first such A, which is the empty set. Our goal is to find a way to expand any such relation, whose domain D is not yet all rational intervals, taking care to preserve the above properties. Once this is established, we can iterate the process knowing that at some closure ordinal, we will arrive at the relation INT. At the limit stages of this iteration we will simply take unions of the previously defined A's. It is easy to see that if the properties (i) and (ii) hold at each previous stage, they will also hold at limit stages. So we only need to concentrate on the successor stages. There are three methods which will help us to add a new rational open interval to D.

The first approach is very easy. If we "know" the integrals on (a_1, a_2), (a_2, a_3), $\ldots$, (a_{n-1}, a_n) then we may find the integral on (a_1, a_n) by simply adding these together. This fact is immediate from the definition.

The second approach is the improper integration technique of Cauchy. If the sequence of open intervals $\{I_n\}$ converges to I, then $F(I) = \lim F(I_n)$. This, of course, is an immediate consequence of the continuity of F.

The third approach is due to Lebesgue. If G is an open subset of I then $F(I) = F(G) + L(I \setminus G)$. This presupposes of course that these terms are well defined, so in particular the Lebesgue integral of f over $I \setminus G$ must exist and the sum $F(G)$ must be absolutely convergent. This equivalence is an elementary consequence of Lebesgue integration and was proved by Lebesgue in 1904 (see [16]). We will refer to it as Lebesgue's Theorem. It will be applied as follows: If $B^\alpha \subset D$ and $I \in B^{\alpha+1}$ then (using (3)) f is bounded on $I \setminus \bigcup D$ so the Lebesgue integral on this set exists. If we are lucky enough to also have a bound, say M, for the relative integrals of the components, J_i, of $I \cap (\bigcup D)$, then $\sum |F(J_i)|$ is no bigger than $M \cdot |I|$. Therefore, in this case, we may apply Lebesgue's Theorem to calculate $F(I) = F\left(I \cap (\bigcup D)\right) + L(I \setminus (\bigcup D))$.

Later accounts of totalization usually combine the first and third approaches (e.g., Saks [18], Natanson [15]). Indeed, the first approach is actually a special case of the third one. However, it will be more convenient for us to treat them separately. Since our purpose is to write a program for this "totalization" process, the first step is to carefully define each of the three extension techniques using only arithmetical quantifiers, and using A

only in a positive way. We start here now the first one:

(7) $(p,q,r,n) \in \mathcal{H}(A) \iff$

there is a partition, $(a_1,\dots,a_j)$ of (p,q) such that for each $i < j$, there is a rational r_i with $(a_i,a_{i+1},r_i,n) \in A$ and either $n = 1$ and $\sum r_i < r$ or $n = 2$ and $\sum r_i > r$.

The operator $\mathcal{H}$ is easily defined in an arithmetical way and since it only mentions A in a positive way, it will be easy to program. Notice that $\mathcal{H}(A)$ contains A, and that the properties (i) and (ii) still hold, where of course, D refers to the new domain, the domain of $\mathcal{H}(A)$. In fact, a slightly stronger form of (i) now holds:

(i′) Any open rational interval which is covered by finitely many elements of D, is in D.

We now assume that (i′) holds and apply the next operator $\mathcal{C}(A)$ defined as follows:

(8) $(p,q,r,n) \in \mathcal{C}(A) \iff$

$(\exists s,t)(\forall \varepsilon)$ there is a rational u, and a rational subinterval (a,b) of (p,q) such that $a - p + q - b < \varepsilon$, $s < r < t,\ (a,b,u,n) \in A$, and either $n = 1$ and $u < s$ or $n = 2$ and $u > t$.

Once again this is easily arithmetical and monotone in A, so it will be easily programmed. It is also readily checked that $\mathcal{C}(A)$ contains A and that the properties (i′) and (ii) still hold when A is expanded to $\mathcal{C}(A)$. In fact, a stronger form of (i′) now holds:

(i″) Any rational interval in $\bigcup D$ is also in D.

Obtaining a suitable definition for the Lebesgue extension will be trickier. The problem is that we have to find a way to talk about components of $\bigcup D$ without mentioning D in a negative way. The solution is to take advantage of the fact that we have already shown how to compute membership in B^α. So when we are tempted to mention D in a negative way, we will try to get by with using B^α instead. To make things easier, we will split this into two propositions. The first of these, Proposition 3, shows how to define the new domain and the second, Proposition 4, tells how to compute the integrals on this new domain. Since we will only apply this third technique after the operators $\mathcal{H}$ and $\mathcal{C}$ have been applied, we can assume that the property (i″) holds. We will first present an easy but useful lemma:

LEMMA. *Let* $\{x_m^n\}$ *be a bounded collection of real numbers, defined whenever* $m \leq n$ *are positive integers. Then for some increasing function* $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ *we have for each* m *that* $\lim_{n\to\infty} x_m^{g(n)}$ *exists.*

PROOF. Let $g_0(n) = n$. Given g_{m-1}, let g_m be a subsequence of g_{m-1} such that $\lim_{n\to\infty} x_m^{g_m(n)}$ exists and g_m, g_{m-1} agree on the first m entries. Since g_n is a subsequence of g_0, it is increasing. Also, for each n, $g_m(n)$ is eventually constant and so $\lim_{m\to\infty} g_m(n)$ exists. Call this limit $g(n)$. Then for each m, $g(n)$ is a subsequence of g_m. Therefore, $g(n)$ is also increasing and since $\lim_{n\to\infty} x_m^{g_m(n)}$ exists, so does $\lim_{n\to\infty} x_m^{g(n)}$. ⊣

PROPOSITION 3. *Suppose* (i″) *and* (ii) *hold,* $D \subset B^\alpha$, *and let* $I \in B^{\alpha+1}$. *Then the following are equivalent:*

(3a) *There is a bound for the relative integrals of the components of* $I \cap (\bigcup D)$.

(3b) *For some finite* M, *every finite collection of rational intervals in* $I \cap (\bigcup B^\alpha)$ *can be covered by a finite collection of disjoint intervals in* $I \cap (\bigcup D)$ *whose relative integrals are in* $(-M, M)$.

PROOF. (3a) → (3b): Suppose every component of $I \cap (\bigcup D)$ has a relative integral between $-M$ and M. Since D contains all the intervals in B^α, any finite collection of rational intervals in $I \cap (\bigcup B^\alpha)$ is covered by a finite collection of components of $I \cap (\bigcup D)$. Each such component can be reduced, if necessary, to a rational subinterval which will then be in D, preserving the property that each has a relative integral between M and $-M$, and also the fact that the original collection of rational intervals remains covered.

(3b) → (3a): Suppose I satisfies property (3b) for some M. Let $M' > M$ so that the range of f on $I \setminus \bigcup B^\alpha$ is also bounded by M' (this is possible by the definition of $B^{\alpha+1}$). Now let J be a component of $I \cap (\bigcup D)$. By ignoring the (two) end components (this won't affect boundedness), we may assume that J is a component of $\bigcup D$. We will finish the proof by showing that the relative integral of J is in $(-M', M')$. By property (3b), any finite number of rational subintervals of $J \cap (\bigcup B^\alpha)$ can be covered by a finite number of disjoint intervals in D whose relative integrals are in $(-M, M)$. But since J is a component of $\bigcup D$ the intervals in such a cover can be assumed to be subintervals of J.

Let $K_1, K_2, \ldots$ be the components of $J \cap (\bigcup B^\alpha)$. Let R_m^n be a rational subinterval of K_m with $|K_m| - |R_m^n| < 1/n$. Let $S^n = \bigcup_{m\leq n} R_m^n$, and C^n be a cover of S^n as promised by (3b). For each $m \leq n$, let C_m^n denote the component of C^n which covers R_m^n. Using the lemma, there is an increasing sequence $g(n)$ such that for each m, the interval $Z_m = \lim_{n\to\infty} C_m^{g(n)}$ exists. Then Z_m must contain all of K_m and so $Z = \bigcup Z_m$ contains all of $J \cap (\bigcup B^\alpha)$. Furthermore, each Z_m has a relative integral in $[-M, M]$ since each $C_m^{g(n)}$ does. Finally, any two intervals Z_i and Z_j are either disjoint or equal: If there is a region of overlap, then there is a region of overlap for $C_i^{g(n)}$ and $C_j^{g(n)}$ for large enough n, causing $C_i^{g(n)} = C_j^{g(n)}$, which forces $Z_i = Z_j$. Therefore, the components of Z are the intervals Z_m, and these have bounded relative integrals, and so Lebesgue's Theorem applies and $F(J) = F(Z) + L(J \setminus Z)$. But the relative integral of Z is in $(-M', M')$ since $M' > M$. Also, by choice of M', the relative Lebesgue integral of $J \setminus Z$ is also in $(-M', M')$. Therefore, so is the relative integral of J. ⊣

PROPOSITION 4. *Suppose properties* (i″) *and* (ii) *hold,* $B^\alpha \subset D$, $I \in B^{\alpha+1}$, $r \in \mathbb{Q}$, *and* $n \in \{1,2\}$. *If I satisfies the property from Proposition* 3, *then the following are equivalent*:

(4a) *Either* $r > F(I)$ *and* $n = 1$ *or* $r < F(I)$ *and* $n = 2$.

(4b) $(\exists$ *finite number* $M)$ $(\exists s < r,\ t > r)$
$(\exists$ *finite union* $A = \bigcup_i A_i$ *of rational subintervals of I in* $\bigcup B^\alpha)$
$(\forall$ *finite union* $B = \bigcup_i B_i$ *of rational subintervals of I in* $\bigcup B^\alpha$ *with* $A \subset B)$
$(\exists$ *finite union* $C = \bigcup_i C_i$ *of disjoint rational intervals in* $I \cap (\bigcup D)$ *with* $B \subset C$ *and with the relative integral of each C_i in* $(-M, M))$
$(\forall$ *finite union, J, of components of C with* $A \subset J)$
$F(J) + L(I \setminus J \setminus \bigcup B^\alpha) < s$ *and* $n = 1$ *or* $F(J) + L(I \setminus J \setminus \bigcup B^\alpha) > t$ *and* $n = 2$.

PROOF. Since both cases $n = 1$ and $n = 2$ are similar, we prove only the case $n = 1$. So that we don't have to keep saying it, all intervals (except those denoted by K or Z) are rational subintervals of I.

(4a) → (4b): First check the case $\alpha = 0$: M may be anything and s may be any rational between $F(I)$ and r. Then A and B will be empty and we let C also be empty forcing J to be empty. Then $F(J) = 0$, f is bounded (hence Lebesgue integrable) on I, $F(I) = L(I) = L(I \setminus J \setminus \bigcup B^\alpha)$ and (4b) follows.

Now let $\alpha > 0$ so that $\bigcup B^\alpha$ is dense. Let M be the bound from property (3a), and increase M so that it is also a bound for the range of f on $I \setminus \bigcup B^\alpha$ (this is possible by definition of $B^{\alpha+1}$), and also larger than $|f(x)|$ calculated at each endpoint of I. Let $r > F(I)$ and $s = (r + F(I))/2$. Choose $A = \bigcup_i A_i$ so that each A_i is in B^α and so that the components of $I \cap (\bigcup D)$ disjoint from A have total measure less than $(r - s)/4M$. Let $B = \bigcup_i B_i$ so that $A \subset B \subset \bigcup B^\alpha$. Choose $C = \bigcup_i C_i$ to be a rational approximation for the union of components of $I \cap (\bigcup D)$ which intersect B, so that:

(a) $B \subset C \subset \bigcup D$, with at most one C_i in each component of $\bigcup D$,
(b) $|I \cap (\bigcup D) \setminus C| < (r - s)/4M$,
(c) the relative integral of each C_i is in $(-M, M)$, and

(d) for each i, if Z_i is the component of $\bigcup D$ containing C_i, then the relative integral of each $Z_i \setminus C_i$ is in $(-M, M)$. This is possible, since for each component (a,b) of $\bigcup D$, there is a rational interval $(c,d) \subset (a,b)$ with $c-a$ and $b-d$ arbitrarily small, and with the relative integrals of (a,c) and (d,b) arbitrarily close to $f(a)$, $f(b)$ respectively. But both $f(a)$ and $f(b)$ are in $(-M, M)$.

Let J be a finite union of components of C, with $A \subset J$. Then:

$$\begin{aligned} &F(J) + L(I \setminus J \setminus \textstyle\bigcup B^\alpha) \\ &\quad = F(J) + L(I \setminus C \setminus \textstyle\bigcup B^\alpha) + L(C \setminus J \setminus \textstyle\bigcup B^\alpha), \end{aligned}$$

since C contains J,

$$\begin{aligned} &= F(J) + F(I \setminus C) + F(C \setminus J) - F(I \setminus C) - F(C \setminus J) \\ &\qquad + L(I \setminus C \setminus \textstyle\bigcup B^\alpha) + L(C \setminus J \setminus \textstyle\bigcup B^\alpha) \\ &= F(I) - F(I \setminus C) - F(C \setminus J) + L(I \setminus C \setminus \textstyle\bigcup B^\alpha) \\ &\qquad + L(C \setminus J \setminus \textstyle\bigcup B^\alpha) \\ &= F(I) - F((I \setminus C) \cap (\textstyle\bigcup D)) - L(I \setminus C \setminus (\textstyle\bigcup D)) \\ &\qquad - F(C \setminus J) + L(I \setminus C \setminus \textstyle\bigcup B^\alpha) + L(C \setminus J \setminus \textstyle\bigcup B^\alpha), \end{aligned}$$

by Lebesgue's Theorem on $I \setminus C$,

$$\begin{aligned} &\leq F(I) + \left|F((I \setminus C) \cap (\textstyle\bigcup D))\right| \\ &\qquad + \left|L(I \setminus C \setminus \textstyle\bigcup B^\alpha) - L(I \setminus C \setminus (\textstyle\bigcup D))\right| \\ &\qquad + |F(C \setminus J)| + |L(C \setminus J \setminus \textstyle\bigcup B^\alpha)| \\ &= F(I) + \left|F((I \setminus C) \cap (\textstyle\bigcup D))\right| \\ &\qquad + \left|L(I \setminus C \setminus (\textstyle\bigcup B^\alpha) \cap (\textstyle\bigcup D))\right| \\ &\qquad + |F(C \setminus J)| + |L(C \setminus J \setminus \textstyle\bigcup B^\alpha)|, \end{aligned}$$

since $B^\alpha \subset D$,

$$\begin{aligned} &< F(I) + M|(I \setminus C) \cap (\textstyle\bigcup D)| \\ &\qquad + M\left|I \setminus C \setminus (\textstyle\bigcup B^\alpha) \cap (\textstyle\bigcup D)\right| + M|C \setminus J| \\ &\qquad + M|C \setminus J \setminus \textstyle\bigcup B^\alpha|, \end{aligned}$$

by (d), (c) and the fact that the range of f on $I \setminus \bigcup B^\alpha$ is bounded by M,

$$\begin{aligned} &\leq F(I) + 2M\left|(I \setminus C) \cap (\textstyle\bigcup D)\right| + 2M|C \setminus J| \\ &\leq F(I) + \frac{r-s}{2} + \frac{r-s}{2}, \end{aligned}$$

by (b) and since any component of $C \setminus J$ is contained in a component of $\bigcup D$ which is disjoint from A,

$$= F(I) + r - s = s < r.$$

(4b) → (4a): Given M and $s < r$ which make (4b) hold, choose $M' > M$ such that the range of f on $I \setminus \bigcup B^{\alpha}$ is bounded by M' (possible by definition of $I \in B^{\alpha+1}$). Let $K_1, K_2, \ldots$ be an enumeration of the components of $I \cap (\bigcup B^{\alpha})$. After $A = \bigcup_i A_i$ is given, let R^n_m be a rational subinterval of K_m, containing $A \cap K_m$, and such that $|K_m \setminus R^n_m| < 1/n$. Let $S^n = R^n_1 \cup R^n_2 \cup \cdots \cup R^n_n$. When n is large enough so that S^n contains A, let C^n be a cover of S^n as promised by (4b) with $B = S^n$, and let C^n_m be the component of C^n which contains R^n_m.

Using the lemma, for some sequence $g(n)$ and each m, the limit interval $Z_m = \lim_{n\to\infty} C^{g(n)}_m$ exists. As before, any two intervals Z_i and Z_j are either disjoint or equal. Also, for each m, Z_m contains K_m and has a relative integral in $[-M, M]$. Then $Z = \bigcup Z_m$ contains $I \cap (\bigcup B^{\alpha})$ and it follows that Lebesgue's Theorem may be applied to get $F(I) = F(Z) + L(I \setminus Z)$.

Let $Z' = Z_1 \cup \cdots \cup Z_k$ be a finite collection of components of Z so that Z' contains A and $|Z \setminus Z'| < (r-s)/2M'$. It follows that $|F(Z) - F(Z')| < (r-s)/2$. Then let $J = \bigcup\{\, C^p_m \mid m \le k \,\}$ for some p in the range of g, large enough so that:

(1) $A \subset J$

(2) $|(Z \setminus J) \cup (J \setminus Z)| < (r-s)/2M'$

(3) $|F(Z) - F(J)| < (r-s)/2$.

Applying (4b) (with $C = C^p$) we get that$F(J) + L(I \setminus J \setminus \bigcup B^{\alpha}) < s$. Then,

$$\begin{aligned} F(I) &= F(Z) + L(I \setminus Z) \\ &= F(Z) + L(I \setminus Z \setminus \textstyle\bigcup B^{\alpha}), \end{aligned}$$

since Z contains $I \cap (\bigcup B^{\alpha})$,

$$\begin{aligned} &= F(Z) + L(I \setminus J \setminus \textstyle\bigcup B^{\alpha}) - L(Z \setminus J \setminus \bigcup B^{\alpha}) \\ &\qquad + L(J \setminus Z \setminus \textstyle\bigcup B^{\alpha}) \\ &\le F(Z) + L(I \setminus J \setminus \textstyle\bigcup B^{\alpha}) \\ &\qquad + \left|L((Z \setminus J) \cup (J \setminus Z) \setminus \textstyle\bigcup B^{\alpha})\right| \\ &\le F(Z) + L(I \setminus J \setminus \textstyle\bigcup B^{\alpha}) \\ &\qquad + M'\left|(Z \setminus J) \cup (J \setminus Z) \setminus \textstyle\bigcup B^{\alpha}\right|, \end{aligned}$$

by choice of M',

$$\begin{aligned}&\le F(Z) + L(I \setminus J \setminus \bigcup B^\alpha)\\ &\qquad + M'|(Z \setminus J) \cup (J \setminus Z)|\\ &< F(J) + \frac{r-s}{2} + L(I \setminus J \setminus \bigcup B^\alpha) + \frac{r-s}{2},\end{aligned}$$

by (2) and (3)

$$< s + (r - s) = r,$$

and this concludes the proof of Proposition 4. ⊣

We can now use these two propositions to define a Lebesgue extension operator. Since (4b) easily implies (3b) we only need to mention (4b):

(9) $(p, q, r, n) \in \mathcal{L}(A) \iff$
$(p, q, r, n) \in A$, or else $(p, q) \in B^{\alpha+1}$ for some α
with $B^\alpha \subset D$ and (4b) holds (with references to F
replaced by the corresponding references to A).

We see from the propositions that this definition correctly applies the third extension technique. We also see that the operator $\mathcal{L}$ is monotone in A, that $A \subset \mathcal{L}(A)$ and that although (i′) and (i″) may no longer hold after $\mathcal{L}$ is applied, certainly (i) and (ii) still hold. The process of iterating the operators can therefore continue. To carry out the totalization, we therefore define a new operator $\mathcal{I}(A) = \mathcal{L}(\mathcal{C}(\mathcal{H}(A)))$ and iterate this in the usual way. In the next section we will show how to program this induction. But for now, it is still incumbent upon us to show that the process succeeds in extending the domain of A. This argument is due to Denjoy.

Suppose $A \subset \mathrm{INT}$, $A \neq \mathrm{INT}$, and A satisfies (i) and (ii). It may be that none of the operators $\mathcal{H}$, $\mathcal{C}$, and $\mathcal{L}$ individually extend the domain D, but the combination $\mathcal{I} = \mathcal{L}(\mathcal{C}(\mathcal{H}(A)))$ certainly will. To see this, suppose $\mathcal{H}(A) = A$. Then (i′) holds. Suppose also that $\mathcal{C}(A) = A$. Then (i″) also holds. Let P be the complement of $\bigcup D$. If P were empty then A would be INT and we would be done. So P is a nonempty closed set. Let J be in $B^{\alpha+1}$ with α minimal such that $J \cap P$ is nonempty. We will show under these conditions that the operator $\mathcal{L}$ will add a new subinterval to D.

Now by the definition of derivative, given any $\varepsilon < 1$, for each $x \in P$ there is a maximum number $\delta(x) > 0$ such that the difference quotient of F (i.e., relative integral) over any interval of size less than $\delta(x)$ which contains x must be within ε of $f(x)$. By the Baire Category Theorem, there must be a rational interval $K \subset J$ and a number $\delta > 0$ such that $K \cap P$ is nonempty and $\delta(x) > \delta$ on a dense subset $P' \subset K \cap P$. By shrinking K if necessary, we can also assume that $|K| < \delta$. Now since $K \in B^{\alpha+1}$ there is an interval $(-M, M)$ which contains the range of $f(x)$ on $K \setminus (\bigcup B^{\alpha})$. Since P is disjoint from $\bigcup B^{\alpha}$, we have $K \cap P \subset K \setminus \bigcup B^{\alpha}$. Then any subinterval of K containing an element of P' must have a relative integral which is in $(-M - \varepsilon, M + \varepsilon)$. Since any component of $K \cap (\bigcup D)$ will have elements of P' arbitrarily close by, its relative integral must be in $[-M - \varepsilon, M + \varepsilon]$. Therefore, K satisfies Proposition (3a) and is therefore in Dom $\mathcal{L}(A)$. Finally since $K \cap P \neq \emptyset$, K can't be in D and we are done.

§5. Recover the primitive (part 2). We will now continue with the program started in §3 where now it is our goal to write a program for the induction $\mathcal{I} = \mathcal{L}(\mathcal{C}(\mathcal{H}(A)))$ mentioned in the last section. This is easily accomplished using the methods already established. We will do it in blocks, making sure everything is either arithmetical or uses a positively arithmetically defined operator, and also perhaps references to previous programs. The proof that each program does what it is supposed to is either trivial or (if it contains a loop) follows easily from Proposition 1. Each block will be given a name in quotation marks, and a definition. Then the relation will be programmed and, when possible, a program for its negation will follow.

"UB($\bar{c}, \bar{d}, p, q$)"

$\iff$ $(\bar{c}, \bar{d})$ is a collection of rational subintervals of $(p, q) \cap (\bigcup B^{\alpha-1})$, where $\alpha = \operatorname{ord}(p, q)$,

$$\iff (\exists j) l(\bar{c}) = l(\bar{d}) = j \;\&\; (\forall i \leq j) p \leq c_i < d_i \leq q \;\&\; (\forall i \leq j)(\forall s, t)(c_i < s < t < d_i) \rightarrow \big[(\exists \bar{p}, \bar{q}) \text{“Cover}(\bar{p}, \bar{q}, s, t)\text{”} \;\&\; \big(\forall i \leq l(\bar{p})\big) \operatorname{ord}(p_i, q_i) < \operatorname{ord}(p, q)\big]$$

```
$l_{10}$: $j \leftarrow \exists$
      if $\neg(l(\bar{c}) = l(\bar{d}) = j)$ or $\neg(\forall i \leq j) p \leq c_i < d_i \leq q$ reject
      $i \leftarrow \forall, s \leftarrow \forall, t \leftarrow \forall$
      if $i > j$ accept
      if $\neg(c_i < s < t < d_i)$ accept
      $\bar{p} \leftarrow \exists, \bar{q} \leftarrow \exists$
      $m \leftarrow \forall$
      if $m = 1$ go to $l_2$
      $i \leftarrow \forall$
      if $i > l(\bar{p})$ accept
      $n \leftarrow 1, a \leftarrow p_i, b \leftarrow q_i, c \leftarrow p, d \leftarrow q$
      go to $l_8$
```

$\neg$UB($\bar{c}, \bar{d}, p, q$)

```
$l_{11}$: $j \leftarrow \forall$
      if $\neg(l(\bar{c}) = l(\bar{d}) = j)$ or $\neg(\forall i \leq j) p \leq c_i < d_i \leq q$ accept
```

```
    $i \leftarrow \exists, s \leftarrow \exists, t \leftarrow \exists$
    if $i > j$ reject
    if $\neg(c_i < s < t < d_i)$ reject
    $\bar{p} \leftarrow \forall, \bar{q} \leftarrow \forall$
    $m \leftarrow \exists$
    if $m = 1$ go to $l_3$
    $i \leftarrow \exists$
    if $i > l(\bar{p})$ reject
    $n \leftarrow 2, a \leftarrow p, b \leftarrow q, c \leftarrow p_i, d \leftarrow q_i$
    go to $l_8$
```

"WIT(M, m, p, q)"

$\iff$ (M, m) witnesses that (p, q) belongs to B^{α}, where $\alpha = \operatorname{ord}(p, q)$,

$\iff$ the mth closed set in the inverse image of $(-M, M)$ contains $(p, q) \setminus \bigcup\{\, (x, y) \mid \operatorname{ord}(x, y) < \operatorname{ord}(p, q) \,\}$,

$\iff$ $(\forall c, d)\big[(c, d) \subset (p, q)\ \&\ \text{INVIM}(-M, M, c, d, m)\big] \rightarrow$ "UB(c, d, p, q)".

```
$l_{12}$: $c \leftarrow \forall, d \leftarrow \forall$
      if $\neg(p \leq c < d \leq q)$ accept
      if INVIM$(-M, M, c, d, m)$ go to $l_{13}$
      accept
$l_{13}$: $\bar{c} \leftarrow c, \bar{d} \leftarrow d$
      go to $l_{10}$
```

"$\neg$WIT(M, m, p, q)"

```
$l_{14}$: $c \leftarrow \exists, d \leftarrow \exists$
      if $\neg(p \leq c < d \leq q)$ reject
      if INVIM$(-M, M, c, d, m)$ go to $l_{15}$
      reject
$l_{15}$: $\bar{c} \leftarrow c, \bar{d} \leftarrow d$
      go to $l_{11}$
```

Our next block characterizes an important open set which we call $G(n, a, b, p, q, \bar{e}, \bar{g})$ which is defined to be the union of the following:

(i) the complement of the union of the first n closed sets in the inverse image under f of the interval (a, b),
(ii) the union of the intervals in $B^{\alpha-1}$ where $\alpha = \operatorname{ord}(p, q)$, and
(iii) the union of the sequence $(\bar{e}, \bar{g})$ of rational intervals.

"$G(s, t, n, a, b, p, q, \bar{e}, \bar{g})$"

$\iff$ "$[s, t]$ is a subset of G",

$\iff$ $(\exists \bar{p}, \bar{q})$ "Cover$(\bar{p}, \bar{q}, s, t)$" & $(\forall i \leq l(\bar{p}))$ either $(\forall k \leq n)$INVIM(a, b, p_i, q_i, k), or "UB(p_i, q_i, p, q)" or $(\exists j)(p_i, q_i) \subset (e_j, g_j)$.

```
$l_{16}$: $\bar{p} \leftarrow \exists$, $\bar{q} \leftarrow \exists$
      $m \leftarrow \forall$
      if $m = 1$ go to $l_{17}$
      go to $l_2$
$l_{17}$: $i \leftarrow \forall$
      if $i > l(\bar{p})$ accept
      $k \leftarrow \forall$
      if $k > n$ accept
      if $\mathrm{INVIM}(a, b, p_i, q_i, k)$ accept
      $j \leftarrow \exists$
      if $(p_i, q_i) \underline{\subset} (e_j, g_j)$ accept
      $\bar{c} \leftarrow p_i$, $\bar{d} \leftarrow q_i$
      go to $l_{10}$
```

"$\neg G(s, t, n, a, b, p, q, \bar{e}, \bar{g})$"

```
$l_{18}$: $\bar{p} \leftarrow \forall$, $\bar{q} \leftarrow \forall$
      $m \leftarrow \exists$
      if $m = 1$ go to $l_{19}$
      go to $l_3$
$l_{19}$: $i \leftarrow \exists$
      if $i > l(\bar{p})$ reject
      $k \leftarrow \exists$
      if $k > n$ reject
      if $\mathrm{INVIM}(a, b, p_i, q_i, k)$ reject
      $j \leftarrow \forall$
      if $(p_i, q_i) \underline{\subset} (e_j, g_j)$ reject
      $\bar{c} \leftarrow p_i$, $\bar{d} \leftarrow q_i$
      go to $l_{11}$
```

Let $\mathbb{Q}(m)$ denote some canonical bijection from the positive integers to the rationals, and let the "*m*th component" of an open set be the one (if any) which includes $\mathbb{Q}(m)$ but does not include $\mathbb{Q}(j)$ for any $j < m$. The "*m*th component" of an open set may not exist, but at least if we list them this way, no component is counted twice. Our next block estimates the size of the *m*th component of $G \cap (p, q)$.

"$\mathrm{LEN}(r, n, a, b, p, q, \bar{e}, \bar{g}, m)$"

$\iff$ "$r = 0$ or else the *m*th component of $G(n, a, b, p, q, \bar{e}, \bar{g}) \cap (p, q)$ exists and has length greater than r",

$\iff$ $r = 0$ or $(\exists s, t) r < t - s \;\&\; p \leq s < \mathbb{Q}(m) < t \leq q$
$\&$ "$G(s, t, n, a, b, p, q, \bar{e}, \bar{g})$"
$\&$ $(\forall j < m)$ if $\mathbb{Q}(j) < \mathbb{Q}(m)$
then "$\neg G(\mathbb{Q}(j), \mathbb{Q}(m), n, a, b, p, q, \bar{e}, \bar{g})$"

and if $\mathbb{Q}(m) < \mathbb{Q}(j)$
then "$\neg G\big(\mathbb{Q}(m), \mathbb{Q}(j), n, a, b, p, q, \bar{e}, \bar{g}\big)$".

```
$l_{20}$: if $r = 0$ accept
     $k \leftarrow \forall$
     if $k = 1$ go to $l_{21}$
     $s \leftarrow \exists, t \leftarrow \exists$
     if $r \geq (t - s)$ or $\neg\big(p \leq s < \mathbb{Q}(m) < t \leq q\big)$ reject
     go to $l_{16}$
$l_{21}$: $j \leftarrow \forall$
     if $j \geq m$ accept
     if $\mathbb{Q}(j) > \mathbb{Q}(m)$ go to $l_{22}$
     $s \leftarrow \mathbb{Q}(j), t \leftarrow \mathbb{Q}(m)$
     go to $l_{18}$
$l_{22}$: $s \leftarrow \mathbb{Q}(m), t \leftarrow \mathbb{Q}(j)$
     go to $l_{18}$
```

"$\neg\mathrm{LEN}(r, n, a, b, p, q, \bar{e}, \bar{g}, m)$"

```
$l_{23}$: if $r = 0$ reject
     $k \leftarrow \exists$
     if $k = 1$ go to $l_{24}$
     $s \leftarrow \forall, t \leftarrow \forall$
     if $r \geq (t - s)$ or $\neg\big(p \leq s < \mathbb{Q}(m) < t \leq q\big)$ accept
     go to $l_{18}$
$l_{24}$: $j \leftarrow \exists$
     if $j \geq m$ reject
     if $\mathbb{Q}(j) > \mathbb{Q}(m)$ go to $l_{25}$
     $s \leftarrow \mathbb{Q}(j), t \leftarrow \mathbb{Q}(m)$
     go to $l_{16}$
$l_{25}$: $s \leftarrow \mathbb{Q}(m), t \leftarrow \mathbb{Q}(j)$
     go to $l_{16}$
```

"$\mathrm{LEBM}(r, n, a, b, p, q, \bar{e}, \bar{g})$"

$\iff$ "$r >$ the Lebesgue measure of $(p, q) \setminus G(n, a, b, p, q, \bar{e}, \bar{g})$",

$\iff$ "the Lebesgue measure of $(p, q) \cap G(n, a, b, p, q, \bar{e}, \bar{g})$ is $> q - p - r$",

$\iff$ "some finite union of components of $(p, q) \cap G(n, a, b, p, q, \bar{e}, \bar{g})$ has measure greater than $q - p - r$",

$$\iff (\exists i)(\exists \bar{r})\big[l(\bar{r}) = i \;\&\; \textstyle\sum \bar{r} > q - p - r \;\&\; (\forall j \leq i)\ \mathrm{LEN}(r_j, n, a, b, p, q, \bar{e}, \bar{g}, j)\big].$$

```
$l_{26}$: $i \leftarrow \exists, \bar{r} \leftarrow \exists$
     if $l(\bar{r}) \neq i$ or $\sum \bar{r} \leq q - p - r$ reject
     $j \leftarrow \forall$
     if $j > i$ accept
```

```
      $r \leftarrow r_j, m \leftarrow j$
      go to $l_{20}$
```

“$\neg$LEBM$(r, n, a, b, p, q, \bar{e}, \bar{g})$”

```
$l_{27}$: $i \leftarrow \forall, \bar{r} \leftarrow \forall$
      if $l(\bar{r}) \neq i$ or $\sum \bar{r} \leq q - p - r$ accept
      $j \leftarrow \exists$
      if $j > i$ reject
      $r \leftarrow r_j, m \leftarrow j$
      go to $l_{23}$
```

“LEBI$(r, p, q, \bar{e}, \bar{g})$”

$\iff$ “$r >$ the Lebesgue integral of f on $(p,q) \setminus \bigcup B^{\alpha-1} \setminus \bigcup\{(\bar{e},\bar{g})\}$, where $\alpha = \mathrm{ord}(p,q)$”,

$\iff$ $(\exists M, m)$ “WIT(M, m, p, q)”

$$\begin{aligned}
&\& (\exists \bar{p}, \bar{q}, \bar{u}, k) l(\bar{p}) = l(\bar{q}) = l(\bar{u}) = k \\
&\& p_1 = -M, q_k = M, \\
&\& (\forall i < k)[p_i < q_i = p_{i+1} < q_{i+1}] \\
&\& (\forall i \leq k)(\exists n) \text{“}\neg\mathrm{LEBM}(u_i, n, p_i, q_i, p, q, \bar{e}, \bar{g})\text{”} \\
&\& (\exists x)(\forall n) \text{“}\mathrm{LEBM}(x, n, -M, M, p, q, \bar{e}, \bar{g})\text{”} \\
&\& M \cdot x + \textstyle\sum \bar{u}(\bar{q} - M) < r.
\end{aligned}$$

The idea here is that we break the range $(-M, M)$ into small intervals (p_i, q_i). The inverse image of each of these intervals on our domain $(p,q) \setminus (\bigcup B^{\alpha-1}) \setminus \bigcup\{(\bar{e},\bar{g})\}$ is then under-estimated (using $\bar{u}$) by considering only the first j closed sets of this inverse image. We then subtract M from the function, causing it to be negative on its domain. When u_i is multiplied by $(q_i - M)$ this estimate is negative, but not as negative as it should be, and so when these are added we get an over-estimate, $\sum \bar{u}(\bar{q} - M)$, for the integral of $f(x) - M$. Then we add back in $M \cdot x$ where x is an over-estimate for the entire domain, finally obtaining an over-estimate for the integral of f. We then check to see if this can be made less than r.

```
$l_{28}$: $M \leftarrow \exists, m \leftarrow \exists$
      $z \leftarrow \forall$
      if $z = 1$ go to $l_{12}$
      $\bar{p} \leftarrow \exists, \bar{q} \leftarrow \exists$
      $\bar{u} \leftarrow \exists, k \leftarrow \exists$
      if $\neg\big(l(\bar{p}) = l(\bar{q}) = l(\bar{u}) = k \;\&\; p_1 = -M \;\&\; q_k = M\big)$ reject
      if $\neg(\forall i < k)[p_i < q_i = p_{i+1} < q_{i+1}]$ reject
      $z \leftarrow \forall$
      if $z = 1$ go to $l_{29}$
      $i \leftarrow \forall$
      if $i > k$ accept
      if $\neg(p_i < q_i = p_{i+1} < q_{i+1})$ reject
```

```
        $n \leftarrow \exists$
        $r \leftarrow u_i, a \leftarrow p_i, b \leftarrow q_i$
        go to $l_{27}$
    $l_{29}$: $x \leftarrow \exists$
        $n \leftarrow \forall$
        if $M \cdot x + \sum \bar{u}(\bar{q} - M) \geq r$ reject
        $r \leftarrow x, a \leftarrow -M, b \leftarrow M$
        go to $l_{26}$
```

"$\neg$LEBI$(r, p, q, \bar{e}, \bar{g})$"

```
    $l_{30}$: $M \leftarrow \forall, m \leftarrow \forall$
        $z \leftarrow \exists$
        if $z = 1$ go to $l_{14}$
        $\bar{p} \leftarrow \forall, \bar{q} \leftarrow \forall, \bar{u} \leftarrow \forall, k \leftarrow \forall$
        if $\neg(l(\bar{p}) = l(\bar{q}) = l(\bar{u}) = k \,\&\, p_1 = -M \,\&\, q_k = M)$ accept
        if $\neg(\forall i < k)[p_i < q_i = p_{i+1} < q_{i+1}]$ accept
        $z \leftarrow \exists$
        if $z = 1$ go to $l_{31}$
        $i \leftarrow \exists$
        if $i > k$ reject
        if $\neg(p_i < q_i = p_{i+1} < q_{i+1})$ accept
        $n \leftarrow \forall$
        $r \leftarrow u_i, a \leftarrow p_i, b \leftarrow q_i$
        go to $l_{26}$
    $l_{31}$: $x \leftarrow \forall$
        $n \leftarrow \exists$
        if $M \cdot x + \sum \bar{u}(\bar{q} - M) \geq r$ accept
        $r \leftarrow x, a \leftarrow -M, b \leftarrow M$
        go to $l_{27}$
```

"$(p, q, r, n) \in \mathcal{H}(A)$"

$$\begin{aligned} \iff\ & (\exists \bar{a}, \bar{r}, j) l(\bar{a}) = l(\bar{r}) = j \,\&\, j > 1 \\ & \&\, a_1 = p, a_j = q \text{ and } (\forall i < j)(a_i < a_{i+1}) \\ & \&\, \left(n = 1 \,\&\, \sum \bar{r} < r \text{ or } n = 2 \,\&\, \sum \bar{r} > r\right) \\ & \&\, (\forall i < j)(a_i, a_{i+1}, r_i, n) \in A. \end{aligned}$$

Here, A denotes the subrelation of INT which is "already" determined ("already" means in a backwards sense; A hasn't actually been mentioned in the program, but represents those "simpler" integrals which now need to be determined). It is in this block then, that the loop for the induction occurs. Notice the final statement "go to l_1". This refers to the start of the entire program which will appear in our last block.

```
    $l_{32}$: $\bar{a} \leftarrow \exists, \bar{r} \leftarrow \exists, j \leftarrow \exists$
        if $\neg(l(\bar{a}) = l(\bar{r}) = j \,\&\, j > 1 \,\&\, a_1 = p \,\&\, a_j = q)$ reject
    if $(\exists i < j) a_i \geq a_{i+1}$ reject
    if $\neg(n = 1 \,\&\, \sum \bar{r} < r$ or $n = 2 \,\&\, \sum \bar{r} > r)$ reject
    $i \leftarrow \forall$
    if $i \geq j$ accept
    $p \leftarrow a_i, q \leftarrow a_{i+1}, r \leftarrow r_i$
    go to $l_1$
```

"$(p,q,r,n) \in \mathcal{C}(\mathcal{H}(A))$"

$$\begin{aligned} \iff (\exists s,t)(\forall \varepsilon)(\exists u,a,b)\ & p < a < b < q,\ a - p + q - b < \varepsilon,\ s < r < t, \\ & (\text{either } n = 1 \,\&\, u < s \text{ or } n = 2 \,\&\, u > t), \\ & \&\, \text{“}(a,b,u,n) \in \mathcal{H}(A)\text{”}. \end{aligned}$$

```
$l_{33}$: $s \leftarrow \exists, t \leftarrow \exists$
    $\varepsilon \leftarrow \forall$
    $u \leftarrow \exists, a \leftarrow \exists, b \leftarrow \exists$
    if $\neg(p < a < b < q \,\&\, a - p + q - b < \varepsilon \,\&\, s < r < t)$ reject
    if $\neg(n = 1 \,\&\, u < s$ or $n = 2 \,\&\, u > t)$ reject
    $p \leftarrow a, q \leftarrow b, r \leftarrow u$
    go to $l_{32}$
```

The next relation "COVER" is capitalized to distinguish it from our previously defined "Cover":

"COVER$(\bar{a}, \bar{b}, \bar{c}, \bar{d})$"

$$\begin{aligned} &\iff (\bar{c}, \bar{d}) \text{ covers } (\bar{a}, \bar{b}), \\ &\iff (\exists j,k) l(\bar{a}) = l(\bar{b}) = j,\ l(\bar{c}) = l(\bar{d}) = k, \\ &\qquad\qquad \&\, (\forall i \leq j)(\exists l \leq k)(c_l \leq a_i < b_i \leq d_l). \end{aligned}$$

```
$l_{34}$: $j \leftarrow \exists, k \leftarrow \exists$
    if $\neg(l(\bar{a}) = l(\bar{b}) = j \,\&\, l(\bar{c}) = l(\bar{d}) = k)$ reject
    if $(\forall i \leq j)(\exists l \leq k)(c_l \leq a_i < b_i \leq d_l)$ accept
    reject
```

"$\neg$COVER$(\bar{a}, \bar{b}, \bar{c}, \bar{d})$"

```
$l_{35}$: $j \leftarrow \forall, k \leftarrow \forall$
    if $\neg(l(\bar{a}) = l(\bar{b}) = j \,\&\, l(\bar{c}) = l(\bar{d}) = k)$ accept
    if $\neg(\forall i \leq j)(\exists l \leq k)(c_l \leq a_i < b_i \leq d_l)$ reject
    accept
```

"DISJ$(\bar{e}, \bar{g}, p, q)$"

$$\begin{aligned} &\iff (\bar{e}, \bar{g}) \text{ are disjoint subintervals of } (p,q), \\ &\iff (\exists j)(l(\bar{e}) = l(\bar{g}) = j) \\ &\qquad\qquad \&\, (\forall k,m \leq j)\big(e_m \notin (e_k, g_k) \,\&\, g_m \notin (e_k, g_k) \\ &\qquad\qquad \&\, p \leq e_m < g_m \leq q\big) \end{aligned}$$

```
$l_{36}$: $j \leftarrow \exists$
    if $\neg(l(\bar{e}) = l(\bar{g}) = j)$ reject
```

```
        if $(\forall k,m \le j)\big(e_m \notin (e_k,g_k) \,\&\, g_m \notin (e_k,g_k)$
                $\& \, p \le e_m < g_m \le q\big)$ accept
        reject
```

“$\neg$DISJ$(\bar{e},\bar{g},p,q)$”

```
    $l_{37}$: $j \leftarrow \forall$
        if $\neg(l(\bar{e}) = l(\bar{g}) = j)$ accept
        if $\neg(\forall k,m \le j)\big(e_m \notin (e_k,g_k) \,\&\, g_m \notin (e_k,g_k)$
                $\& \, p \le e_m < g_m \le q)$ reject
        accept
```

“SUBCOLL$(\bar{u},\bar{v},\bar{e},\bar{g})$”

$\Longleftrightarrow$ the collection of intervals $(\bar{u},\bar{v})$ is a subset of the collection $(\bar{e},\bar{g})$, with left endpoints in increasing order and with no duplication,

$$\begin{aligned}\Longleftrightarrow\ & (\exists j,k)\big(l(\bar{u}) = l(\bar{v}) = j \le l(\bar{e}) = l(\bar{g}) = k\big)\\ &\qquad \& \, (\forall i \le j)(\exists m \le k)(u_i = e_m, v_i = g_m)\\ &\qquad \& \, (\forall i < j)u_i < u_{i+1}.\end{aligned}$$

```
    $l_{38}$: $j \leftarrow \exists, k \leftarrow \exists$
        if $\neg(l(\bar{u}) = l(\bar{v}) = j \le l(\bar{e}) = l(\bar{g}) = k)$ reject
        if $(\forall i \le j)(\exists m \le k)(u_i = e_m, v_i = g_m) \,\&\, (\forall i < j)u_i < u_{i+1}$ accept
        reject
```

“$\neg$SUBCOLL$(\bar{u},\bar{v},\bar{e},\bar{g})$”

```
    $l_{39}$: $j \leftarrow \forall, k \leftarrow \forall$
        if $\neg(l(\bar{u}) = l(\bar{v}) = j \le l(\bar{e}) = l(\bar{g}) = k)$ accept
        if $\neg(\forall i \le j)(\exists m \le k)(u_i = e_m, v_i = g_m)$ or $\neg(\forall i < j)u_i < u_{i+1}$
            reject
        accept
```

“PROP4(p,q,r,n)”

$$\begin{aligned}\Longleftrightarrow\ & (\exists M,s,t)(s < r < t) \,\&\, (\exists \bar{a},\bar{b}) \text{ “UB}(\bar{a},\bar{b},p,q)\text{”}\\ &\quad \& \, (\forall \bar{c},\bar{d})\big(\text{“UB}(\bar{c},\bar{d},p,q)\text{” } \& \text{“COVER}(\bar{a},\bar{b},\bar{c},\bar{d})\text{”}\big) \rightarrow\\ &\qquad (\exists \bar{e},\bar{g})\big[\text{“DISJ}(\bar{e},\bar{g},p,q) \,\&\, \text{“COVER}(\bar{c},\bar{d},\bar{e},\bar{g})\text{”}\\ &\qquad\qquad \& \, \big(\forall k \le l(\bar{e})\big)\\ &\qquad\qquad\qquad \big[(\exists x,y)\big(-M(g_k - e_k) < y < x < M(g_k - e_k)\big)\\ &\qquad\qquad\qquad\qquad \& \text{“}(r_k,s_k,x,1) \in C(H(A))\text{”}\\ &\qquad\qquad\qquad\qquad \& (r_k,s_k,y,2) \in C(H(A))\text{”}\big]\\ &\qquad\qquad \& \, (\forall \bar{u},\bar{v})\big(\text{“COVER}(\bar{a},\bar{b},\bar{u},\bar{v})\text{”}\\ &\qquad\qquad\qquad \& \text{“SUBCOLL}(\bar{u},\bar{v},\bar{e},\bar{g})\text{”}\big) \rightarrow\\ &\qquad\qquad\qquad \big[(\exists \bar{x},m)(l(\bar{x}) = l(\bar{u}) = m)\\ &\qquad\qquad\qquad\qquad \& \, (\forall i \le m) \text{ “}(u_i,v_i,x_i,n) \in C(H(A))\text{”}\\ &\qquad\qquad\qquad\qquad \& \, (\exists y)\big(n = 1 \,\&\, \text{“LEBI}(y,p,q,\bar{u},\bar{v})\text{”}\end{aligned}$$

$$
\begin{aligned}
&\qquad \& y + \textstyle\sum \bar{x} < s) \\
&\text{or } \big(n = 2 \,\&\, \text{“}\neg\mathrm{LEBI}(y, p, q, \bar{u}, \bar{v})\text{”} \\
&\qquad \& y + \textstyle\sum \bar{x} > t\big)\big]\big].
\end{aligned}
$$

```
$l_{40}$: $M \leftarrow \exists, s \leftarrow \exists, t \leftarrow \exists$
      if $\neg(s < r < t)$ reject
      $\bar{a} \leftarrow \exists, \bar{b} \leftarrow \exists$
      $z \leftarrow \forall$
      if $z = 1$ go to $l_{41}$
      $\bar{c} \leftarrow \bar{a}, \bar{d} \leftarrow \bar{b}$
      go to $l_{10}$
$l_{41}$: $\bar{c} \leftarrow \forall, \bar{d} \leftarrow \forall$
      $z \leftarrow \exists$
      if $z = 1$ go to $l_{35}$, if $z = 2$ go to $l_{42}$
      go to $l_{11}$
$l_{42}$: $\bar{e} \leftarrow \exists, \bar{g} \leftarrow \exists$
      $z \leftarrow \forall$
      if $z = 1$ go to $l_{43}$, if $z = 2$ go to $l_{44}$, if $z = 3$ go to $l_{46}$
      go to $l_{36}$
$l_{43}$: $\bar{a} \leftarrow \bar{c}, \bar{b} \leftarrow \bar{d}, \bar{c} \leftarrow \bar{e}, \bar{d} \leftarrow \bar{g}$
      go to $l_{34}$
$l_{44}$: $k \leftarrow \forall$
      if $k > l(\bar{e})$ accept
      $x \leftarrow \exists, y \leftarrow \exists$
      if $\neg(-M(g_k - e_k) < y < x < M(g_k - e_k))$ reject
      $z \leftarrow \forall$
      if $z = 1$ go to $l_{45}$
      $p \leftarrow r_k, q \leftarrow s_k, r \leftarrow x, n \leftarrow 1$
      go to $l_{33}$
$l_{45}$: $p \leftarrow r_k, q \leftarrow s_k, r \leftarrow y, n \leftarrow 2$
      go to $l_{33}$
$l_{46}$: $\bar{u} \leftarrow \forall, \bar{v} \leftarrow \forall$
      $z \leftarrow \exists$
      if $z = 1$ go to $l_{39}$, if $z = 2$ go to $l_{47}$
      $\bar{c} \leftarrow \bar{u}, \bar{d} \leftarrow \bar{v}$
      go to $l_{35}$
$l_{47}$: $\bar{x} \leftarrow \exists, m \leftarrow \exists$
      if $\neg(l(\bar{x}) = l(\bar{u}) = m)$ reject
      $z \leftarrow \forall$
      if $z = 1$ go to $l_{48}$
      $i \leftarrow \forall$
      if $i > m$ accept
      $p \leftarrow u_i, q \leftarrow v_i, r \leftarrow x_i$
```

```
        go to $l_{33}$
$l_{48}$: $y \leftarrow \exists$
        if $\neg(n = 1$ or $n = 2)$ reject
        if $n = 2$ go to $l_{49}$
        if $\neg(y + \sum \bar{x} < s)$ reject
        $r \leftarrow y, \bar{e} \leftarrow \bar{u}, \bar{g} \leftarrow \bar{v}$
        go to $l_{28}$
$l_{49}$: if $\neg(y + \sum \bar{x} > t)$ reject
        $r \leftarrow y, \bar{e} \leftarrow \bar{u}, \bar{g} \leftarrow \bar{v}$
        go to $l_{30}$
```

$$\text{“}(p,q,r,n) \in \mathcal{L}(\mathcal{C}(\mathcal{H}(A)))\text{”} \iff \text{“}(p,q,r,n) \in \mathcal{C}(\mathcal{H}(A))\text{” or } \Big[\big[(\forall c,d)\text{“UB}(c,d,p,q)\text{”} \rightarrow (\exists m,u)\text{“}(c,d,u,m) \in \mathcal{C}(\mathcal{H}(A))\text{”}\big] \; \& \text{“PROP4}(p,q,r,n)\text{”}\Big].$$

```
$l_1$: $z \leftarrow \exists$
        if $z = 1$ go to $l_{33}$
        $t \leftarrow \forall$
        if $t = 1$ go to $l_{40}$
        $c \leftarrow \forall, d \leftarrow \forall$
        $z \leftarrow \exists$
        if $z = 1$ go to $l_{50}$
        $\bar{c} \leftarrow c, \bar{d} \leftarrow d$
        go to $l_{11}$
$l_{50}$: $m \leftarrow \exists, u \leftarrow \exists$
        $p \leftarrow c, q \leftarrow d, r \leftarrow u, n \leftarrow m$
        go to $l_{33}$
```

Appendix §A. The solutions of Denjoy, Perron-Bauer, and Kurzweil-Henstock. We outline here the three most famous solutions to the problem of the primitive, mentioned in the introduction.

A.1. Denjoy's solution. Let D be a collection of open real intervals I, on which the definite integral $F(I)$ is known. Suppose also that whenever $J \subset I \in D$ then $J \in D$. There are three techniques which may be used to find a definite integral $F(J)$ where $J \notin D$:

(a) (trivial) If J is partitioned by the intervals $I_1, \ldots, I_n \in D$ then $F(J) = \sum F(I_n)$.

(b) (Cauchy) If $J = \lim_{n\to\infty} I_n$ where each $I_n \in D$ then $F(J) = \lim_{n\to\infty} F(I_n)$. This is a direct consequence of the continuity of F.

(c) (Lebesgue) If f is bounded on $J \setminus \bigcup D$ then the Lebesgue integral $L(J \setminus \bigcup D)$ exists. If we also have a bound for the difference quotients $F(I)/|I|$ over all components I of $J \cap (\bigcup D)$ then the corresponding sum $\sum F(I)$ over these components is absolutely convergent, and will be denoted by $F(J \cap (\bigcup D))$. It then follows by a theorem of Lebesgue that $F(J) = F(J \cap (\bigcup D)) + L(J \setminus \bigcup D)$. Application of this technique, of course, will require that every component of $J \cap (\bigcup D)$ be already in D.

First note that if $F(J)$ can be found by any of the above techniques, then so can $F(K)$ for any $K \subset J$. Therefore, each provides a method of expanding D.

Next we show that at least one of the three techniques will properly expand D unless, of course, D already contains all open intervals. To this end, suppose neither (a) nor (b) expands D. Then $\mathbb{R} \setminus \bigcup D$ could have no isolated points. Furthermore, any interval covered by finitely many elements of $\bigcup D$ would already be in D. If in addition, (b) fails then any component of $\bigcup D$ is already in D. Under this assumption we show that (c) will not fail.

Using $\varepsilon = 1$ in the definition of derivative, we know that for each x there is a $\delta(x) > 0$ such that the difference quotient $F(I)/|I|$ is within 1 of $f(x)$ whenever I contains x and $|I| < \delta(x)$. An easy argument using the continuity of F shows that the function $\delta(x)$ can be chosen to be upper-semicontinuous, meaning that each of the sets $\{ x \mid \delta(x) \geq \delta \}$ is closed. It follows by the Baire Category Theorem that for some $\delta > 0$, the set $\{ x \mid \delta(x) \geq \delta \}$ contains a nontrivial portion, J, of $\mathbb{R} \setminus \bigcup D$. Assume without loss of generality that $|J| < \delta$. Then for each x in $J \setminus \bigcup D$, $f(x)$ is within 1 of $F(J)/|J|$ implying that f is bounded on $J \setminus \bigcup D$. Furthermore, if I is a component of $J \cap (\bigcup D)$ and x is an endpoint of I in J, then the difference quotient $F(I)/|I|$ is within 1 of $f(x)$, hence within 2 of $F(J)/|J|$. Therefore such difference quotients are also bounded. It follows that $F(J)$ can be calculated by (c).

Finally, since one of the techniques always succeeds, we may iterate them, taking unions at limit ordinals, until D contains all open intervals. At this point, the definite integral for any interval is known and hence so is the primitive (up to a constant).

Note: Several variations of this technique are possible. For example, Dougherty and Kechris first obtain a transfinite sequence $D_0, D_1, \ldots$ of open sets, with f being bounded on each $D_{\alpha+1} \setminus D_\alpha$. Then they apply Denjoy's procedure to find the integral on each piece. Our computer program is modeled after this Dougherty-Kechris variation.

A.2. Perron-Bauer solution. Let f be a derivative and $\varphi(x)$, $\psi(x)$ be continuous functions with $\varphi(0) = \psi(0) = 0$. Then φ is called a "majorant" (or "major function") of f if

$$\underline{D}\varphi(x) = \liminf_{h \to 0} \frac{\varphi(x+h) - \varphi(x)}{h}$$

is always at least as big as $f(x)$. Similarly, $\psi(x)$ is called a "minorant" of f if

$$\bar{D}\psi(x) = \limsup_{h\to 0} \frac{\psi(x+h)-\psi(x)}{h}$$

is always at least as small as $f(x)$. We show that when $x \geq 0$, $\varphi(x) \geq \psi(x)$. To see this note that $\underline{D}(\varphi - \psi) \geq \underline{D}(\varphi) - \bar{D}(\psi) \geq 0$. Hence, $\varphi - \psi$ is monotonically increasing. Then since $\varphi(0) - \psi(0) = 0$, $\varphi(x) - \psi(x) \geq 0$ when $x \geq 0$.

Let F be the unique primitive of f with $F(0) = 0$. Since F is both a majorant and a minorant, we have for any majorant φ, and any minorant ψ, and any $x \geq 0$, that $\varphi(x) \geq F(x) \geq \psi(x)$. It follows that for $x \geq 0$, $F(x)$ can be found by taking the infimum of the majorants or the supremum of the minorants.

Note: Denjoy ([7], pp. 677–683) likened this to the following situation: Suppose two people wanted to know how to get an airplane to fly twice the speed of sound. The first person carefully details the design of the wings, the construction of the engine, etc. The second person finds a much easier approach. Just take the infimum of all the airplanes which go more than twice the speed of sound and the supremum of all the ones which go less than twice the speed of sound, and there it is!

A.3. Kurzweil-Henstock solution. Let us first recall from elementary calculus, the definitions behind the Riemann integral:

DEFINITION. A finite sequence of points $x_0 < x_1 < \cdots < x_n$ is called a partition of $[a,b]$ if $x_0 = a$ and $x_n = b$. The mesh of the partition is the maximum value of $x_i - x_{i-1}$. A set of tags for a partition is a collection of elements c_i with $c_i \in [x_{i-1}, x_i]$. Given a function f and a tagged partition, the corresponding Riemann sum is $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$. Finally, I is called the Riemann integral of f on $[a,b]$ if and only if $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall$ tagged partition of $[a,b]$ with mesh $< \delta)$ the corresponding Riemann sum is within $\varepsilon(b-a)$ of I.

The Riemann integral cannot be used to invert every derivative, but it will invert a continuous one. To see this, let $F'(x) = f(x)$ where f is continuous. By the definition of derivative, given x and $\varepsilon > 0$ there is a $\delta > 0$ such that $|F(y) - F(z) - f(x)(y-z)| < \varepsilon(y-z)$ whenever $z \leq x \leq y$ and $y - z < \delta$. Fix ε and for each x, let $\delta(x)$ denote the largest value of δ for which this definition holds. Also, let $\delta'(x)$ be similarly defined using $\varepsilon/2$. Then by the continuity of F and f we have that for all x' in some neighborhood of x, $\delta(x') \geq \delta'(x)$. Such neighborhoods form an open covering of $[a,b]$ and so by compactness, there is some δ such that $\delta(x) \geq \delta > 0$ for each x in $[a,b]$. Then, if $x_0 < x_1 < \cdots < x_n$ is a partition of $[a,b]$ with mesh less than δ, and if $\{c_i\}$ is a collection of tags, with $c_i \in (x_{i-1}, x_i)$ then $F(x_i) - F(x_{i-1})$

is within $\varepsilon(x_i - x_{i-1})$ of $f(c_i)(x_i - x_{i-1})$. Summing over i, we get that $\sum f(c_i)(x_i - x_{i-1})$ is within $\varepsilon(b-a)$ of $F(b) - F(a)$ and we are done.

If $f(x)$ is not continuous, then we can't always find a uniform δ. Kurzweil (and later Henstock) made the observation that in this case we can just leave $\delta(x)$ a function of x (called a "gauge" function) and then replace the condition that the mesh be less than δ to the requirement that $x_i - x_{i-1} < \delta(c_i)$ (this requirement on a tagged partition is called "δ-fine"). The same proof now shows that this new definition of integral will invert any derivative. In fact, the proof is much simpler because we no longer have to show that $\delta(x)$ is bounded above zero. This improved version is sometimes called the Riemann-complete integral.

Note: It may be hard to believe that Riemann himself was not aware of this "improvement". But if he was, why didn't he state his integral in this more powerful way? Perhaps one explanation is that despite the strong similarity, there is a profound difference between the original Riemann integral and the Riemann-complete version. Riemann's integral can be effectively used to obtain approximations to the increment $F(b) - F(a)$. One just chooses partitions with smaller and smaller mesh, and is guaranteed that the corresponding Riemann sums will get closer and closer to the correct value. In the Riemann-complete integral we don't know (without prior knowledge of the gauge function) if our tagged partitions are ever going to be δ-fine, and without this knowledge we are totally at a loss. There is no way to tell if our approximations will be getting better and better or if they are getting worse and worse.

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FREGE'S THEOREM AND THE PEANO POSTULATES

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Two thoughts about the concept of number are incompatible: that any zero or more things have a (cardinal) number, and that any zero or more things have a number (if and) only if they are the members of some one set. It is Russell's paradox that shows the thoughts incompatible: the sets that are not members of themselves cannot be the members of any one set. The thought that any (zero or more) things have a number is Frege's; the thought that things have a number only if they are the members of a set may be Cantor's and is in any case a commonplace of the usual contemporary presentations of the set theory that originated with Cantor and has become ZFC.

In recent years a number of authors have examined Frege's accounts of arithmetic with a view to extracting an interesting subtheory from Frege's formal system, whose inconsistency, as is well known, was demonstrated by Russell. These accounts are contained in Frege's formal treatise *Grundgesetze der Arithmetik* and his earlier exoteric book *Die Grundlagen der Arithmetik*. We may describe the two central results of the recent re-evaluation of his work in the following way: Let *Frege arithmetic* be the result of adjoining to full axiomatic second-order logic a suitable formalization of the statement that the Fs and the Gs have the same number if and only if the Fs and the Gs are equinumerous. Then (1) Frege himself succeeded in deriving arithmetic from Frege arithmetic and (2) Frege arithmetic is equi-consistent with full second-order arithmetic (and is thus consistent, with moral certainty). So Frege derived arithmetic from a single consistent and obvious-seeming principle, if not from logic alone, as he had hoped he had. A number of the articles responsible for the re-evaluation of Frege's formal work are collected in William Demopoulos' anthology, *Frege's Philosophy of Mathematics*.[1]

Frege, we now see, thus provided a consistent theory of the natural numbers altogether different from those of Dedekind, Russell and Whitehead, Zermelo, von Neumann, and the several founders of combinatory logic and the lambda-calculus. Particularly noteworthy is the difference between

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[1] William Demopoulos, ed., *Frege's Philosophy of Mathematics*, Harvard University Press, Cambridge, Massachusetts, 1995.

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Frege's account and that found in *Principia Mathematica*: for Frege, numbers are certain objects that lie at the bottom level of a type-theoretical hierarchy, while for Russell and Whitehead, they are situated two levels above the bottom of a similar hierarchy.

Here we investigate the Fregean conception of number; we are going to show that on Fregean definitions of *zero*, *(immediately) precedes*,[2] and *natural number*, but without use of the aforementioned principle, the Peano postulates contain a notable redundancy, whose discovery can, with only a small amount of exaggeration, be attributed to Frege, since the proof of redundancy is an adaptation of Frege's proof that the series of natural numbers is infinite. Thus the study of the details of Frege's work has yielded a result concerning the extremely familiar Peano axiomatization. We begin the investigation by recalling an old observation of Henkin's concerning a dependency among the Peano postulates.[3]

Suppose that the Dedekind-Peano postulates are formulated as usual in a second-order language containing a constant **0** (for zero) and a one-place function sign **s** (for successor):

(i) $\forall x\ \mathbf{0} \neq \mathbf{s}x$
(ii) $\forall x \forall y (\mathbf{s}x = \mathbf{s}y \rightarrow x = y)$
(iii) Mathematical induction: $\forall F(F\mathbf{0} \wedge \forall x(Fx \rightarrow F\mathbf{s}x) \rightarrow \forall x Fx)$.

Henkin observed that (iii) *implies the disjunction of* (i) *and* (ii).

We may see this as follows: by (iii), the range of the first order variables is the class of denotations of **0**, **s0**, **ss0**, ... Either these denotations are all distinct, in which case (i) and (ii) both hold, or they are not and for some least m, $\mathbf{s}^m\mathbf{0}$ denotes the same object as $\mathbf{s}^n\mathbf{0}$, for some $n > m$. If $m = 0$, (i) fails but (ii) holds; if $m > 0$, (ii) fails but (i) holds.

It is easy to construct models in which each of the seven conjunctions $\pm$(i) $\pm$ (ii) $\pm$ (iii) other than $-$(i) $-$ (ii) $+$ (iii) holds; so no other dependencies among (i), (ii), and (iii) await discovery.

Our main interest is in the Fregean conception of number and we shall accordingly follow Frege in not *assuming* that the relation *precedes* is the graph of a total function on the natural numbers, or even a functional relation. Accordingly, we formulate the Peano postulates in the language $\{\mathbf{0}, P, N\}$, with P a binary predicate letter for 'precedes' and N a unary predicate letter for 'is a natural number', as the universal closures of the following seven formulas:

(a) $N\mathbf{0}$

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[2] We use 'precedes' to mean 'immediately precedes' (rather than 'is less than').

[3] Leon Henkin, "On mathematical induction", *American Mathematical Monthly* 67, April 1960, 323-338.

(b) $Nm \wedge mPn \rightarrow Nn$
(c) $Nm \wedge mPn \wedge mPn' \rightarrow n = n'$
(d) $Nm \rightarrow \exists n \;\; mPn$
(e) $Nn \rightarrow \neg nP\mathbf{0}$
(f) $Nm \wedge Nm' \wedge mPn \wedge m'Pn \rightarrow m = m'$
(g) $F\mathbf{0} \wedge \forall m \forall n(Fm \wedge mPn \rightarrow Fn) \rightarrow Fn.$

The background logic of our investigation will be axiomatic second-order logic.

(g) is of course a formulation of mathematical induction. It is immediately implied by the stronger-looking

(h) $F\mathbf{0} \wedge \forall m \forall n(Nm \wedge Fm \wedge mPn \rightarrow Fn) \rightarrow Fn.$

But (h) is equivalent to (g) in the presence of (a) and (b): Let $Gn \equiv Nn \wedge Fn$. Then (a), (b) and the antecedent of (h) yield $G\mathbf{0}$ and $\forall m \forall n(Gm \wedge mPn \rightarrow Gn)$; so by (g), Gn, whence Fn.

The conjunction of (a), (b) and (g) is equivalent to the celebrated Dedekind-Frege-Russell definition of N from $\mathbf{0}$ and P:

Def N $\quad \forall x(Nx \leftrightarrow \forall F(F\mathbf{0} \wedge \forall m \forall n(Fm \wedge mPn \rightarrow Fn) \rightarrow Fx)).$

We first examine the dependencies among (c), (d), (e) and (f) under the assumption of Def N. It will help matters if we give (c), (d), (e), and (f) more suggestive designations. Thus we dub (c) **Functionality**, (d) **Totality**, (e) **Exiledom** (the sequence of natural numbers never returns to its point of origin), and (f) **Injectivity**.

We shall use: $\vdash^*$ to denote derivability from Def N (in axiomatic second-order logic).

Formalizing Henkin's argument, we see that $\vdash^* \mathbf{F} \wedge \mathbf{T} \rightarrow \mathbf{E} \vee \mathbf{I}$.

In a similar vein, we may see that $\vdash^* \mathbf{F} \wedge \neg\mathbf{T} \rightarrow \mathbf{E} \wedge \mathbf{I}$, by formalizing the following argument: If **F** holds, every object satisfying Nx bears (the) P(-relation) to at most one object; if ¬**T** holds, some object satisfying Nx bears P to no object. By (a), (b) and (g), the objects satisfying Nx are just the denotation a_0 of $\mathbf{0}$, the unique object a_1, if it exists, to which a_0 bears P, the unique object a_2, if it exists, to which a_1 bears P, ... Now if $i < j$ and a_j exists, then a_i exists, but a_j does not bear P to a_i; otherwise we have a loop and so **T** holds; nor can every a_i exist, for then, since every a_i bears P to a_{i+1}, **T** again holds. So the a_is eventually give out without ever repeating. But then **E** and **I** both hold.

So we have

$$\vdash^* \neg(\mathbf{F} \wedge \mathbf{T} \wedge \neg\mathbf{E} \wedge \neg\mathbf{I}) \quad \text{(Henkin)},$$

as well as

$$\vdash^* \neg(\mathbf{F} \wedge \neg\mathbf{T} \wedge \mathbf{E} \wedge \neg\mathbf{I}),$$
$$\vdash^* \neg(\mathbf{F} \wedge \neg\mathbf{T} \wedge \neg\mathbf{E} \wedge \mathbf{I}), \quad \text{and}$$
$$\vdash^* \neg(\mathbf{F} \wedge \neg\mathbf{T} \wedge \neg\mathbf{E} \wedge \neg\mathbf{I}).$$

We shall discuss the twelve other possibilities below. One of them, however, deserves mention here: It is not the case that $\vdash^* \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \to \mathbf{T}$, for (a), (b), (g), **F**, **E** and **I** are all true and **T** false in the model consisting of a single element which does *not* bear P to itself; of course, that single element will be the denotation of **0**, and it will therefore satisfy Nx.

The use of the constant **0**, rather than a monadic predicate letter Z (for 'is *a* zero') in the formulation of **E** and Def N is important. In particular, the assumption that there are not two zeros has significant consequences. For suppose we reformulate the Peano Postulates in the language $\{Z, P, N\}$ as the universal closures of:

(a′) $Zn \to Nn$
EZ $\exists n Zn$
UZ $Zm \wedge Zn \to m = n$
(b) $Nm \wedge mPn \to Nn$
F $Nm \wedge mPn \wedge mPn' \to n = n'$
T $Nm \to \exists n\, mPn$
E′ $Zm \wedge Nn \to \neg nPm$
I $Nm \wedge Nm' \wedge mPn \wedge m'Pn \to m = m'$
(g′) $\forall F(\forall x(Zx \to Fx) \wedge \forall d \forall a(Fd \wedge dPa \to Fa) \to \forall x(Nx \to Fx))$

and, correspondingly, redefine N:

Redef N $\forall n(Nn \leftrightarrow \forall F(\forall x(Zx \to Fx) \wedge \forall d \forall a(Fd \wedge dPa \to Fa) \to Fn))$.

We can derive (a′) ∧ (b) ∧ (g′) from Redef N, and thus also derive (h′), i.e.,

$$\forall F(\forall x(Zx \to Fx) \wedge \forall d \forall a(Nd \wedge Fd \wedge dPa \to Fa) \to \forall x(Nx \to Fx)).$$

But not only can $\mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \to \mathbf{T}$ not be derived (in axiomatic second-order logic) from Redef $N \wedge$ EZ $\wedge$ UZ, as the model with a single element a such that Za, $\neg aPa$, and Na shows, neither can any of

$$\mathbf{F} \wedge \mathbf{T} \to \mathbf{E}' \vee \mathbf{I},$$
$$\mathbf{F} \wedge \neg\mathbf{T} \to \mathbf{E}' \vee \mathbf{I},$$
$$\mathbf{F} \wedge \neg\mathbf{T} \to \mathbf{E}' \vee \neg\mathbf{I}, \quad \text{and}$$

$$\mathbf{F} \wedge \neg\mathbf{T} \to \neg\mathbf{E}' \vee \mathbf{I}$$

be derived from Redef $N \wedge$ EZ, as the following easy countermodels respectively show:

$$\langle\{a,b,c\},\{a,b\},\{aa,bc,cc\}\rangle,$$
$$\langle\{a,b,c,d\},\{a,b,c\},\{aa,bd,cd\}\rangle,$$
$$\langle\{a,b,c\},\{a,b\},\{aa,bc\}\rangle, \quad \text{and}$$
$$\langle\{a,b,c\},\{a,b\},\{ac,bc\}\rangle.$$

(a, b, c, d are any four objects. The members of the second set in each model are the objects satisfying Z there. The members of the third are the ordered pairs satisfying P. In each model all objects satisfy N.)

So, as will appear later, **F**, **T**, **E**, **I** are totally independent if Redef N and EZ but nothing else is assumed.

We now add to the treatment of number given so far the distinctly Fregean ingredient. It will be of interest to stick with the language $\{Z, P, N\}$ for the time being.

Frege held that numbers are objects, but that what he called "a statement of number", i.e., a statement saying how many things there are of a certain kind,[4] contains an assertion about a concept. Concepts, according to Frege, are denotations of predicates, or values of monadic second-order variables; objects are denoted by terms that may appear on either side of the sign of identity. To express this aspect of Frege's account of number, we introduce a sign $^{\#}$ for a totally defined function. $^{\#}$ is read "the number of things that . . . " and is a function-sign of "mixed type": when attached to a monadic second-order variable F, $^{\#}$ produces a term $^{\#}F$ of the same type as a first-order variable n. So $n = {}^{\#}F$ is well-formed. We emphasize that $^{\#}$ is to denote a total function and thus that $\forall F \exists! n\; n = {}^{\#}F$ will be valid. We assume that the axioms and rules of second-order logic guarantee in some manner the derivability of $\forall F \exists! n\; n = {}^{\#}F$.

We now define Z and P:

Def Z $\quad \forall n(Zn \leftrightarrow \exists F(n = {}^{\#}F \wedge \forall x \neg Fx))$;

Def P $\quad \forall m \forall n(mPn \leftrightarrow \exists F \exists G \exists y(Gy \wedge \forall x(Fx \leftrightarrow Gx \wedge x \neq y) \wedge m = {}^{\#}F \wedge n = {}^{\#}G)$.

Def P is a slight reformulation of Frege's definition of 'precedes'.

We shall use: $\vdash'$ to denote derivability from Def Z, Def P, and Redef N.

According to Def Z, a zero is the number belonging to some empty concept.

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[4] One of Frege's examples is "Jupiter has four moons".

By Def Z alone, $\vdash'$ EZ, for $\exists F \forall x(Fx \leftrightarrow x \neq x)$ is one of the comprehension axioms of second-order logic, and therefore $\vdash' \exists F \forall x \neg Fx$, whence $\vdash' \exists n \exists F(n = {}^{\#}F \wedge \forall x \neg Fx)$, i.e., $\vdash' \exists n Zn$. Also by Def Z alone,

$$\vdash' \mathrm{UZ} \leftrightarrow \forall F \forall G(\forall x \neg Fx \wedge \forall x \neg Gx \rightarrow {}^{\#}F = {}^{\#}G),$$

whose right side may be read: Empty concepts have the same number.

"Empty concepts ... " is an immediate consequence of *Functional Extensionality (FE)*

$$\forall x(Fx \leftrightarrow Gx) \rightarrow {}^{\#}F = {}^{\#}G,$$

which could well be considered a truth of logic, as it is valid under any extensional semantics for languages containing the function sign $^{\#}$. For if $\forall x(Fx \leftrightarrow Gx)$ is true, then the same class of objects will be assigned to the variables F and G, and then ${}^{\#}F = {}^{\#}G$ will be true.[5] The converse of FE is inconsistent, as the argument of Russell's paradox shows. But FE is itself an immediate consequence of *Hume's principle*

$${}^{\#}F = {}^{\#}G \leftrightarrow F \approx G,$$

where $F \approx G$ is some standard second-order formula for equinumerosity. By now it is well known than Hume's principle is consistent. However, Hume's principle holds in no finite domain and cannot be considered a truth of logic.

Our principal result is that **T** can be derived from **F**, **E**$'$, and **I** in $\vdash'$, i.e., that $\vdash' \mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow \mathbf{T}$, which is perhaps surprising in view of the earlier observation that $\nvdash^* \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \rightarrow \mathbf{T}$.

To prove this theorem we define the ancestral R^* of an arbitrary relation R:

$$xR^*y \equiv \forall F(\forall a(xRa \rightarrow Fa) \wedge \forall d \forall a(Fd \wedge dRa \rightarrow Fa) \rightarrow Fy).$$

We write $xR^{*=}y$ to mean $xR^*y \vee y = x$.

Among the consequences of the definitions of R^* and $R^{*=}$ are:

(1) $\forall x \forall y(xRy \rightarrow xR^*y)$,
(2) $\forall x \forall y(xR^*y \wedge yR^*z \rightarrow xR^*z)$,
(3) $\forall x \forall y(xR^*y \rightarrow \exists z(xR^{*=}z \wedge zRy))$,
(4) $\forall x \forall y(Fx \wedge xRy \rightarrow Fy) \rightarrow \forall x \forall y(Fx \wedge xR^{*=}y \rightarrow Fy)$

and

(5) $\forall x \forall y(xR^{*=}y \leftrightarrow \forall F(Fx \wedge \forall d \forall a(Fd \wedge dRa \rightarrow Fa) \rightarrow Fy))$.

Frege's definition of 'natural number' was $\forall n(Nn \leftrightarrow \mathbf{0}P^{*=}n)$, rather than Def N, but by (5), the definitions are equivalent.

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[5] FE is further discussed in my "On the proof of Frege's theorem", to appear in a *Festschrift* for Paul Benacerraf edited by Adam Morton and Stephen Stich

We now prove that $\vdash' \mathbf{F} \land \mathbf{E}' \land \mathbf{I} \to \mathbf{T}$.

Lemma 1. $\vdash' \mathbf{E}' \land \mathbf{I} \to (Nx \to \neg xP^*x)$.

Proof. Suppose Zx and xP^*x. By (a′), Nx. By (3), for some z, $xP^{*=}z$ and zPx. By (b′) and (4), Nz, contra $\mathbf{E}'$. So $\forall x(Zx \to \neg xP^*x)$.

Suppose Nd, dPa and aP^*a. By (b′), Na. By (3), for some z, $aP^{*=}z$ and zPa. By (1) and (2), zP^*z. By (b′) and (4), Nz. By $\mathbf{I}$, $z = d$. Thus dP^*d. So $\forall d \forall a(Nd \land \neg dP^*d \land dPa \to \neg aP^*a)$.

By (h′) (and the comprehension scheme of second-order logic), the lemma follows. (The proof does not cite Def Z or Def P, by the way.) ⊣

Lemma 2. $\vdash' \mathbf{E}' \to (Za \to \exists G(aP^{\#}G \land \forall x(Gx \leftrightarrow Nx \land xP^{*=}a)))$.

Proof. Suppose Za. Then by Def Z, for some F, $\forall x \neg Fx$ and $a = {}^{\#}F$. By second-order logic, for some G, $\forall x(Gx \leftrightarrow Nx \land xP^{*=}a)$. To show $aP^{\#}G$ it is enough, by Def P, to show that for some y, Gy and $\forall x(Fx \leftrightarrow Gx \land x \neq y)$. Let $y = a$. Since Za, Na; and trivially, $aP^{*=}a$. So Ga.

Suppose Gx and $x \neq a$. Then Nx and xP^*a. By (3), for some z, $xP^{*=}z$ and zPa. Since Nx and $xP^{*=}z$, Nz, by (b′) and (4). So Nz, zPa, and Za, contra $\mathbf{E}'$. Since $\forall x \neg Fx$, $\forall x(Fx \leftrightarrow Gx \land x \neq y)$. ⊣

Lemma 3. $\vdash' \mathbf{F} \land \mathbf{E}' \land \mathbf{I} \to (Nd \land dPa \land dP^{\#}F \land \forall x(Fx \leftrightarrow Nx \land xP^{*=}d) \to \exists G(aP^{\#}G \land \forall x(Gx \leftrightarrow Nx \land xP^{*=}a)))$.

Proof. Suppose Nd, dPa, $dP^{\#}F$, and $\forall x(Fx \leftrightarrow Nx \land xP^{*=}d)$. By second-order logic, for some G, $\forall x(Gx \leftrightarrow Nx \land xP^{*=}a)$. We must show $aP^{\#}G$.

By $\mathbf{F}$, $a = {}^{\#}F$. So it is enough to show ${}^{\#}FP^{\#}G$. Since Nd and dPa, Na; and trivially $aP^{*=}a$. So by Def P it suffices to show

$$\forall x(Nx \land xP^{*=}a \land x \neq a \leftrightarrow Nx \land xP^{*=}d).$$

Assume Nx.

Suppose $xP^{*=}a$ and $x \neq a$. Then xP^*a and by (3), for some z, $xP^{*=}z$ and zPa. Since Nx and $xP^{*=}z$, Nz by (b′) and (4). By $\mathbf{I}$, $z = d$, whence $xP^{*=}d$.

Conversely, suppose $xP^{*=}d$. Since dPa, xP^*a, whence $xP^{*=}a$. If $x = a$, then xP^*x, contra Lemma 1. So $x \neq a$. ⊣

Theorem 1. $\vdash' \mathbf{F} \land \mathbf{E}' \land \mathbf{I} \to \mathbf{T}$.

Proof. By comprehension, (h′), and Lemmas 2 and 3,

$$\vdash' \mathbf{F} \land \mathbf{E}' \land \mathbf{I} \to (Nm \to \exists G(mP^{\#}G \land \forall x(Gx \leftrightarrow Nx \land xP^{*=}m))).$$

Thus $\vdash' \mathbf{F} \land \mathbf{E}' \land \mathbf{I} \to (Nm \to \exists n\, mPn)$, i.e., $\vdash' \mathbf{F} \land \mathbf{E}' \land \mathbf{I} \to \mathbf{T}$. ⊣

Let us now write $\vdash$ to mean derivability from "Empty concepts have the same number", Def $\mathbf{0}$, Def P and Def N, where Def $\mathbf{0}$ is $\forall n(n = \mathbf{0} \leftrightarrow \exists F(n = {}^{\#}F \wedge \forall x \neg Fx))$.

We then have

$$\vdash \forall n(Zn \leftrightarrow n = \mathbf{0}) \rightarrow \text{Def } Z \wedge \text{Redef } N.$$

Combining our results, we have

$$\begin{aligned} &\vdash \mathbf{F} \wedge \mathbf{T} \rightarrow \mathbf{E} \vee \mathbf{I}, \\ &\vdash \mathbf{F} \wedge \neg\mathbf{T} \rightarrow \mathbf{E} \wedge \mathbf{I}, \quad \text{and} \\ &\vdash \forall n(Zn \leftrightarrow n = \mathbf{0}) \rightarrow (\mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow \mathbf{T}). \end{aligned}$$

Substituting $\cdot = \mathbf{0}$ for $Z\cdot$, we have

$$\vdash \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \rightarrow \mathbf{T}.$$

It follows (by propositional logic) that

$$\vdash \mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I}).$$

We now show that this result is best possible in the sense that no stronger truth-functional compound of **F**, **T**, **E** and **I** can be derived.

Let

$$\begin{aligned} [1] &= \{\mathbf{F}, \mathbf{T}, \mathbf{E}, \mathbf{I}\}, \\ [2] &= \{\mathbf{F}, \mathbf{T}, \mathbf{E}, \neg\mathbf{I}\}, \\ [3] &= \{\mathbf{F}, \mathbf{T}, \neg\mathbf{E}, \mathbf{I}\}, \\ [4] &= \{\neg\mathbf{F}, \mathbf{T}, \mathbf{E}, \mathbf{I}\}, \\ [5] &= \{\neg\mathbf{F}, \mathbf{T}, \mathbf{E}, \neg\mathbf{I}\}, \\ [6] &= \{\neg\mathbf{F}, \mathbf{T}, \neg\mathbf{E}, \mathbf{I}\}, \\ [7] &= \{\neg\mathbf{F}, \mathbf{T}, \neg\mathbf{E}, \neg\mathbf{I}\}, \\ [8] &= \{\neg\mathbf{F}, \neg\mathbf{T}, \mathbf{E}, \mathbf{I}\}, \\ [9] &= \{\neg\mathbf{F}, \neg\mathbf{T}, \mathbf{E}, \neg\mathbf{I}\}, \\ [10] &= \{\neg\mathbf{F}, \neg\mathbf{T}, \neg\mathbf{E}, \mathbf{I}\}, \quad \text{and} \\ [11] &= \{\neg\mathbf{F}, \neg\mathbf{T}, \neg\mathbf{E}, \neg\mathbf{I}\}. \end{aligned}$$

Since $(\mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I}))$ is truth-functionally equivalent to $\bigwedge[1] \vee \cdots \vee \bigwedge[11]$, in order to show that $\vdash A$ for no truth-functional compound A of **F**, **T**, **E** and **I** stronger than $(\mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I}))$, it will suffice to provide a model $\langle D, f \rangle$ of each $[i]$, $1 \leq i \leq 11$. Henceforth we write: mPn to mean: P applies to m, n. In each of the eleven models, $f : \mathcal{P}D \rightarrow D$; and so "empty concepts" and indeed FE will hold; $\mathbf{0}$ will denote $f\emptyset$ and so Def $\mathbf{0}$ will hold; mPn if and only if there exist A, B and y in B such that $m = fA$, $n = fB$, $A \subseteq B$

and $A = B - \{y\}$, and so Def P will hold; and N will apply to $f\emptyset$ and those elements of D to which $f\emptyset$ bears P^* and so Def N will hold.

[1]: Let $D = N = \{\text{the naturals}\}$. Let $fA = n + 1$ if $|A| = n$, $fA = 0$ if $|A|$ is infinite. Then if $m > 0$, mPn if and only if $n = m + 1$ and $0Pn$ if and only if $n = 0$.

[2]: Let $D = \{0, 1\}$. Let $f\emptyset = 0$ and $f\{0\} = f\{1\} = f\{0, 1\} = 1$. Then $0P1$, $1P1$, not: $0P0$, and not: $1P0$.

[3]: Let $D = \{0\}$. Let $f\emptyset = f\{0\} = 0$. Then $0P0$.

[4]: The model for [4] is the most complicated one. Let $D = N$, $E = \{\text{the evens}\}$, $O = \{\text{the odds}\}$. Let $f\emptyset = 1$, $fE = 1$, $fA = 2n + 1$ if $|A| = n$, $fA = 2n$ if for some $B \subseteq O$, $0 < |B| = n$ and $A = E \cup B$, and $fA = 0$ otherwise. Then we have $1P3P5P7P \cdots$ and $1P2P4P6P \cdots$. Moreover, $0P0$ and $0P1$, and $0P2$, $0P4$, $0P6, \ldots$, and P holds between no other pairs except those indicated.

F fails because $1P3$ and $1P2$. **T** clearly holds. But **E** and **I** hold too, because the objects to which N applies in this model are the positive integers, and the only pairs of positive integers among which P holds are those indicated in: $1P3P57P \cdots$ and $1P2P46P \cdots$.

[5]: Let $D = \{0, 1, 2, 3\}$. Let $f\emptyset = 0$, $f\{0\} = 1$, $f\{1\} = 2$ and $fA = 3$ otherwise. Then $0P1$, $0P2$, $1P3$, $2P3$, $3P3$ and nothing else holds.

[6]: Let $D = N$. Let $fA = n$ if $|A| = n$, $fA = 0$ if A is infinite. Then $\mathbf{0}$ denotes 0; $0P0$, $0P1$, $1P2$, $2P3, \ldots$ and nothing else holds.

[7]: Let $D = \{0, 1\}$. Let $f\emptyset = 0$, $f\{0\} = 1$, $f\{1\} = 0$ and $f\{0, 1\} = 0$. Then $0P0$, $0P1$, $1P0$, but not: $1P1$.

[8]: Let $D = N$. Let $f\emptyset = f(N - \{0\}) = 2$, $fN = 1$, $fA = n + 2$ if $|A| = n$, and $fA = 0$ otherwise. Then $0P0$, $0P2$, $0P1$, $2P1$, $2P3P4P \cdots$ and nothing else holds. Here $\mathbf{0}$ denotes 2 and the objects to which N applies are $2, 3, 4, \ldots$ and also 1. Since $2P1$ and $2P3$, **F** fails; since for all a, not: $1Pa$, **T** fails. But **E** and **I** hold, since although $0P2$ and $0P1$, N does not apply to 0.

[9]: Let $D = \{0, 1, 2, 3\}$. Let $f\emptyset = 0$, $f\{0\} = 1$, $f\{1\} = 2$, $f\{0, 1\} = 1$, $f\{0, 2\} = 2$, $f\{0, 1, 2\} = 1$, $f\{1, 2, 3\} = 2$, $f\{1, 2\} = 2$, $f\{1, 3\} = 1$, $f\{0, 1, 2, 3\} = 3$, and $fA = 1$ otherwise. Then $0P1$, $0P2$, $1P1$, $2P2$, $1P2$, $2P1$, $1P3$, $2P3$ and nothing else holds.

[10]: Let $D = \{0, 1\}$. Let $f\emptyset = 0$, $f\{0\} = f\{1\} = 0$, and $f\{0, 1\} = 1$. Then $0P0$ and $0P1$, but not: $1P0$ and not: $1P1$.

[11]: Let $D = \{0, 1, 2\}$. Let $f\emptyset = 0$, $f\{0\} = 1$, $f\{1\} = 0$, $f\{2\} = 0$, $f\{0, 1\} = 0$, $f\{0, 2\} = 0$, $f\{1, 2\} = 1$ and $f\{0, 1, 2\} = 2$. Then $0P0$, $0P1$, $1P0$, $0P2$, $1P2$, and nothing else holds.

So if $\vdash A$, A a truth-functional compound of **F**, **T**, **E**, **I**, then $\vdash (\mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I})) \rightarrow A$.

In any model for the language $\{^{\#}\}$ of the form $\langle\{a\}, f\rangle$, it must be that aPa, since $f\emptyset = f\{a\} = a$. Thus the model described far above consisting of a single element not bearing P to itself, which shows that

$$\nvdash^{*}\mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \rightarrow \mathbf{T},$$

cannot be converted to a model for $\{^{\#}, \mathbf{0}, P, N\}$ of $\{\mathbf{F}, \neg\mathbf{T}, \mathbf{E}, \mathbf{I}\}$ (and Def $\mathbf{0}$, Def P, Def N and "Empty concepts … ").

The central argument of Frege's *Foundations of Arithmetic* is a sketch of a proof that every natural number precedes some object (which will of course itself be a natural number). The argument sketched by Frege proceeds by showing that $\mathbf{0}P^{\#}[x : xP^{*=}\mathbf{0}]$ and that if d is a natural number[6], dPa and $dP^{\#}[x : xP^{*=}d]$, then $aP^{\#}[x : xP^{*=}a]$. Our proof that $\vdash' \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \rightarrow \mathbf{T}$ is a modification of Frege's argument: instead of showing that if n is a natural number, $nP^{\#}[x : xP^{*=}n]$, we in effect show from **F**, **E** and **I** that any natural number n is the number of some concept G such that $\forall x(Gx \leftrightarrow 0P^{*=}xP^{*=}n)$. Frege's proof appealed to Hume's principle, which, as Frege showed, is also sufficient for **F**, **E**, and **I**; our derivation of $(\mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I}))$ appeals only to "empty concepts have the same number" and the definitions from $^{\#}$ of $\mathbf{0}$, P, and N.

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[6] Frege realized that he needed this condition on d only in *Grundsetze der Arithmetik* and not in *Die Grundlagen der Arithmetik* For more details on his error, see "*Die Grundlagen der Arithmetik*, §§82–3, by George Boolos and Richard G. Heck, Jr., to appear in *Philosophy of Mathematics Today*, edited by Matthias Schirn.

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OPTIMAL PROOFS OF DETERMINACY

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Introduction. In this paper I shall present a method for proving determinacy from large cardinals which, in many cases, seems to yield optimal results. One of the main applications extends theorems of Martin, Steel and Woodin about determinacy within the projective hierarchy. The method can also be used to give a new proof of Woodin's theorem about determinacy in $L(\mathbb{R})$.

The reason we look for optimal determinacy proofs is not only vanity. Such proofs serve to tighten the connection between large cardinals and descriptive set theory, letting us bring our knowledge of one subject to bear on the other, and thus increasing our understanding of both. A classic example of this is the Harrington-Martin proof that Π^1_1-determinacy implies $(<\omega^2 - \Pi^1_1)$-determinacy.[1] This is an example of a *transfer theorem*, which assumes a certain determinacy hypothesis and proves a stronger one. While the statement of the theorem makes no mention of large cardinals, its proof goes through $0^\#$, first proving that

$$\Pi^1_1\text{-determinacy} \Longrightarrow 0^\# \text{ exists},$$

and then that

$$0^\# \text{ exists} \Longrightarrow (<\omega^2 - \Pi^1_1)\text{-determinacy}.$$

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[1] Recall that a set A of *reals* (points in the Baire space $\mathbb{R} = {}^{\mathbf{N}}\mathbf{N}$) is Π^1_n if it can be defined by a Π^1_n formula, i.e.,

$$(*) \qquad A = \{x \in \mathbb{R} \mid \forall y_1 \exists y_2 \ldots \forall/\exists y_n R(x, y_1, \ldots, y_n)\},$$

where the relation R is arithmetical; it is $\Pi^1_n(z)$ (Π^1_n relative to z) if $(*)$ holds with some R which is arithmetical in the real $z \in \mathbb{R}$; and it is $\boldsymbol{\Pi}^1_n$ (boldface Π^1_n) if it is $\Pi^1_n(z)$ for some z. A set A is $(\alpha - \Pi^1_1)$ for an ordinal $\alpha \leq \omega^2$ if there exists a recursive sequence $\langle B_\gamma \mid \gamma < \alpha \rangle$ of Π^1_1 sets such that

$$x \in A \iff \text{the least } \gamma \text{ such that } [\gamma = \alpha \text{ or } x \notin B_\gamma] \text{ is odd};$$

and A is $(<\alpha - \Pi^1_1)$ if it is $(\beta - \Pi^1_1)$ for some $\beta < \alpha$. The relativized classes $(<\alpha - \Pi^1_1)(z) = (<\alpha - \Pi^1_1(z))$ and the boldface classes $(<\alpha - \boldsymbol{\Pi}^1_1)$ are defined similarly

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More recent examples of the connection between large cardinals and descriptive set theory include Steel's proof that

$$\mathrm{AD}^{L(\mathbb{R})} \Longrightarrow \mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{GCH},$$

see [9], and several results of Woodin about models of AD^+, a strengthening of the axiom of determinacy AD which Woodin has introduced. These proofs not only use large cardinals, but also reveal a deep, structural connection between descriptive set theoretic notions and notions related to large cardinals. Steel proved $\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{GCH}$ by showing that, in fact, $\mathrm{HOD}^{L(\mathbb{R})}$ is a *core model* below $\Theta^{L(\mathbb{R})}$. Many of Woodin's results about AD^+ rely on the relation between models of AD^+ and models of choice with infinitely many Woodin cardinals. From any ZFC model which satisfies "*there exist infinitely many Woodin cardinals*", it is possible to derive in a canonical way a model of AD^+, and conversely, every model of AD^+ is a derived model, see [10]. In [4] and [3] it is shown not just that $0^{\#}$ exists $\Longrightarrow$ $(<\omega^2-\Pi^1_1)$-determinacy, but also that $0^{\#}$ and $(<\omega^2-\Pi^1_1)$ sets are very closely connected: $0^{\#}$ is recursively isomorphic to the "complete" $\Game(<\omega^2-\Pi^1_1)$ real, where $\Game$ is the *game quantifier* of [7, Section 6D]. (We will recall the definition of the game quantifier at the end of this Introduction.)

Probably the most important result of the present paper is Lemma 1.7 which gives a general method for proving many determinacy results. This is formulated in Section 1, which also includes an intuitive survey of the notions needed for it. It is hoped that any interested logician should find Section 1 readable.

The main result of Section 2 is Theorem 2.5, an application of Lemma 1.7 which strengthens the following theorem in [5].

THEOREM (Martin-Steel). *If there exist n Woodin cardinals and a measurable cardinal above them, then Π^1_{n+1}-determinacy holds.*

Theorem 2.5 is different from the Martin-Steel result in two ways. First, it proves more determinacy, showing that all $\Game^{(n)}(<\omega^2-\Pi^1_1)$ sets are determined. Secondly, the hypothesis of Theorem 2.5 is weaker and is, in fact, optimal because of the following theorem of Woodin [10].

THEOREM (Woodin). *If Π^1_{n+1}-determinacy holds, then for every real z, there exists a model M satisfying the assumptions of Theorem 2.5 with $z \in M$.*

Woodin's theorem and Theorem 2.5 combined yield the following transfer result

THEOREM. *Π^1_{n+1}-determinacy $\Longrightarrow$ $\Game^{(n)}(<\omega^2-\Pi^1_1)$-determinacy.*

For $n = 0$, this is the Harrington-Martin Theorem mentioned above; for odd n, it was established by Kechris and Woodin by a purely descriptive set

$$\begin{array}{c|ccccccc} \mathrm{I} & x_0 & & x_2 & & \cdots & & \\ \hline \mathrm{II} & & x_1 & & x_3 & & \cdots & \end{array}$$

FIGURE 1. The standard game $G(P)$ with payoff P.

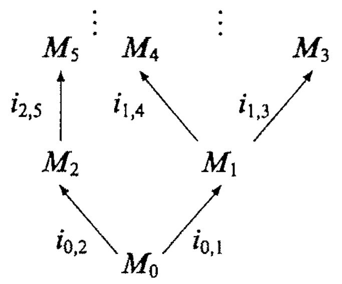
theoretic argument, see [2]; and for even $n > 0$, there is no known proof which does not involve large cardinals.

The game quantifier. For each set $P \subseteq \mathbb{R}$, a *run* of the *standard* (perfect information, two-person) *game* $G(P)$ *with payoff* P is illustrated in Figure 1: the players I and II alternate choosing natural numbers $x_0, x_1, x_2, \ldots$, forever, and in the end I wins the run if the jointly produced real $x = \langle x_0, x_1, x_2, \ldots \rangle$ is in P, otherwise II wins. We say that I *wins* $G(P)$ (or P) if I has a winning strategy for $G(P)$—and similarly for II. Finally, for each set $A \subseteq \mathbb{R} \times \mathbb{R}$ and each real x, we set

$$(\Game y)A(x,y) \iff \text{I wins the game } G(\{y \mid \langle x,y \rangle \in A\}).$$

A set of reals is in $\Game^{(n)}(<\omega^2 - \Pi^1_1)$ if it can be defined from a $(<\omega^2 - \Pi^1_1)$ set using n successive applications of the game quantifier $\Game$. It is easy to see that every Π^1_{n+1} set is $\Game^{(n)}(<\omega^2 - \Pi^1_1)$, but the converse is false.

§1. A few preliminaries and a lemma. To keep the level of technical details as low as possible, I will not give a precise definition of iteration trees. The reader who would like to see the actual definition should consult [6]. For our purposes, it will be enough to know that iteration trees are certain trees of models (of ZFC) and embeddings between them: we think of the nodes



of the tree as representing the models, while the branches represent the embeddings, e.g., $i_{0,2} : M_0 \to M_2$. We will only deal with trees which are finite or have height ω.

In real life, there are many restrictions on the structure of iteration trees. The embeddings must be the ultrapower embeddings by some *extenders*—more precisely, if M_ℓ and M_k are directly connected by a branch of the tree, then the embedding $i_{\ell,k}$ must be the ultrapower embedding $i_{\ell,k} : M_\ell \to \mathrm{Ult}(M_\ell, E_{k-1}) = M_k$, using some extender E_{k-1} in M_{k-1}.[2] Given a finite tree with a last model M_n, we can extend the tree by picking some extender $E_n \in M_n$ and applying it to the model M_k for some $k \leq n$, to get $i_{k,n+1} : M_k \to \mathrm{Ult}(M_k, E_n) = M_{n+1}$. There are still more details omitted, the most important one being that we require $V^{M_k}_{\kappa+1} = V^{M_n}_{\kappa+1}$, where $\kappa = \mathrm{crit}(E_n)$, i.e., M_k and M_n have the same subsets of the critical point κ of the extender. This agreement allows us to apply E_n to the model M_k.

Let T be some iteration tree of height ω, and let b be a branch of T. (Informally, $b = \langle M_{k_j} \mid j \in \omega \rangle$, and the tree has maps $i_{k_j,k_{j+1}} : M_{k_j} \to M_{k_{j+1}}$, for all $j < \omega$.) We let M_b be the direct limit of $\langle M_{k_j}, i_{k_j,k_{j'}} \rangle$, and $i_{k_j,b} : M_{k_j} \to M_b$ be the direct limit maps; and let $i_b = i_{0,b}$. It is a fact that all models along a finite tree are well founded. A natural question to ask about the tree T is whether any of the direct limit models is also well founded, i.e., *is there some infinite branch b of T such that M_b is well founded?* (One reason for considering this question and its importance will become clear in Section 2.) Let us call a model M *1-iterable* if *for every iteration tree T with $M_0 = M$, there exists an infinite branch b of T such that M_b is well founded.* For simplicity, we will say that T is a *tree on M*, if T is an iteration tree with $M_0 = M$.

Note that we do not require the trees T to be in M or to be somehow definable over M. Each finite, initial segment of T will be definable over M from a finite sequence of extenders, but the *infinite* sequence of extenders which defines the whole of T need not be in M. If M_b is well founded, we can ask whether M_b itself is 1-iterable. In general, let us say that M is *n-iterable* (with $n \geq 1$), if *for every iteration tree T on M, there exists an infinite branch b of T such that M_b is $(n-1)$-iterable.*

As mentioned above, in this paper we will stick to trees of height ω, with ω-closed extenders. So in "real world" terms, our definition of n-iterability is the following:

DEFINITION 1.1. A model M is 0-iterable if it is well founded; and M is *n-iterable* $(n \geq 1)$ if for every height-ω iteration tree T on M which uses only extenders whose ultrapowers are ω-closed in the models in which they appear, there exists an infinite branch b of T such that M_b (the direct limit along b) is $(n-1)$-iterable.

The following theorem of [6] shows that there are many models which are n-iterable:

THEOREM 1.2 (Martin-Steel). *Let M be countable, let $\pi : M \to V_\theta$ be an elementary embedding of M into a rank of V, and let T be an iteration tree on*

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[2] For technical reasons, let us use only extenders whose ultrapowers are ω-closed. This is not part of the standard definition, but it is useful for our purposes

M; then there exists an infinite branch b of T and a map $\sigma : M_b \to V_\theta$, such that the diagram

$$\begin{tikzcd}
V_\theta & \\
M \arrow[u, "\pi"] \arrow[r, "i_b"'] & M_b \arrow[ul, "\sigma"']
\end{tikzcd}$$

commutes.

Proof. See [6]. ⊣

Corollary 1.3 (Martin-Steel). *Each model M as in Theorem 1.2 is n-iterable, for every $n \in \omega$.*

The models we will work with are class models and therefore not countable. However, every model M we will use will be of the form $L(V_\lambda^M)$, so that M satisfies the claim that "it is the constructible universe L, built over a rank-initial segment of the universe, V_λ". The following result gives one method of obtaining such models which are n-iterable.

Theorem 1.4 (Martin-Steel). *Let $M = L(V_\lambda^M)$ with V_λ^M countable, suppose that $(V_\lambda^M)^\#$ exists and can be elementarily embedded into $V_\theta^\#$ for some θ, say $\pi : (V_\lambda^M)^\# \to V_\theta^\#$, and let T be an iteration tree on M; then there exists an infinite branch b of T and an embedding $\sigma : i_b\left((V_\lambda^M)^\#\right) \to V_\theta^\#$, such that the diagram*

$$\begin{tikzcd}
V_\theta^\# & \\
(V_\lambda^M)^\# \arrow[u, "\pi"] \arrow[r, "i_b"'] & i_b\left((V_\lambda^M)^\#\right) \arrow[ul, "\sigma"']
\end{tikzcd}$$

commutes.

From this we can see that $i_b\left((V_\lambda^M)^\#\right) = (V_{i_b(\lambda)}^{M_b})^\#$, and by a proof similar to that of Corollary 1.3 we get:

Corollary 1.5 (Martin-Steel). *Every model M as in Theorem 1.4 is n-iterable, for every $n < \omega$.*

So, for example, if $V_\theta^\#$ exists and $L(V_\theta) \models$ "There exist n Woodin cardinals", then there exists an n-iterable class model with n Woodin cardinals: simply take $M = L(V_{\bar\theta}^M)$ to be the Skolem hull of the indiscernibles for V_θ. Thus the existence of an n-iterable model with n Woodin cardinals follows from, e.g., the existence of n Woodin cardinals and a measurable above them all. (The converse, of course, is false.)

Let us for now assume that M is a class model such that

I	x_0, T_0, h_0		x_2, T_1, h_1		x_4, T_2, h_2		$\cdots$
II		x_1		x_3		$\cdots$	

FIGURE 2. The game $\widehat{G}(\dot{A})$.

- $M \models$ "δ is Woodin", and
- (In V) there exists some g which is M-generic $/\mathrm{col}(\omega, \delta)$,

where $\mathrm{col}(\omega, \delta)$ is the usual forcing poset for making δ countable. So $g : \omega \twoheadrightarrow \delta$ is a surjection, and in particular, δ is countable in V. We begin by considering games with payoff in $M^{\mathrm{col}(\omega,\delta)}$. Let $\dot{A}$ be a name for a set of reals in $M^{\mathrm{col}(\omega,\delta)}$, so that $\dot{A}_g$ is a set of reals in $M[g]$, the model with the generic g adjoined. Let $G(\dot{A}_g)$ be the standard game illustrated in Figure 1, with payoff $\dot{A}_g$. This need not be determined, since $M[g]$ satisfies, after all, the Axiom of Choice, but, as it turns out, we can obtain by changing it slightly a game which is (roughly) determined and which is quite useful in proving many forms of determinacy.

DEFINITION 1.6. In the game $\widehat{G}(\dot{A})$, the players I and II alternate moves as in Figure 2, where each x_i must be an integer and the auxilliary objects played by Player I must satisfy the following:

- Each T_i is a finite iteration tree, and $i < j \Longrightarrow T_j$ extends T_i.
- Each h_i is a function with $\mathrm{dom}(h_i) = T_i$ (so, informally, for every model N in the tree T_i, $h_i(N)$ is defined), and $i < j \Longrightarrow h_i \subset h_j$.

At the end of a run, let

$$x = \langle x_0, x_1, \ldots \rangle,$$
$$T = \bigcup_{i<\omega} T_i$$
$$h = \bigcup_{i<\omega} h_i,$$

so that T is then an iteration tree of height ω, and h is a map with $\mathrm{dom}(h) = T$. Now h induces a (continuous) map $\tilde{h}$ with $\mathrm{dom}(\tilde{h}) = \{\text{infinite branches of } T\}$, by

$$\tilde{h}(b) = \bigcup_{N \in b} h(N), \quad \text{for } b \text{ an infinite branch of } T.$$

I wins the game $\widehat{G}(\dot{A})$ if whenever b is an infinite branch of T such that M_b is well founded, then the following two conditions are satisfied:

(i) $\tilde{h}(b)$ is M_b-generic $/\mathrm{col}(\omega, i_b(\delta))$.
(ii) $x \in i_b(\dot{A})_{\tilde{h}(b)}$.

Compared with $G(\dot{A}_g)$, the game $\widehat{G}(\dot{A})$ is easier for I in two ways. First, I is allowed to move $\dot{A}$ by using an iteration tree—replacing the name $\dot{A}$ by $i_b(\dot{A})$. Secondly, I is allowed to choose the generic object $\tilde{h}(b)$ which is used to interpret the name $i_b(\dot{A})$.

Lemma 1.7. *For M as above, $\dot{A}$ a name for a set of reals in $M^{\mathrm{col}(\omega,\delta)}$, and g M-generic* $/\mathrm{col}(\omega, \delta)$:

either (I) : I *wins* $\widehat{G}(\dot{A})$ *via a strategy* $\widehat{\sigma} \in M[g]$, *which wins against all of* II*'s plays in* V (not only plays in $M[g]$)*;*

or (II) : II *wins the game* $G(\dot{A}_g)$ *via a strategy* $\sigma \in M[g]$, *which need only work against* I*'s plays in* $M[g]$.

In Sections 2 and 3, I will use this Lemma to prove several determinacy results. But before that, let us notice one important corollary.

Corollary 1.8. *Suppose M is* 1*-iterable,* $M \models$ "δ is Woodin" *and* (*in* V) *there exists some g which is M-generic* $/\mathrm{col}(\omega,\delta)$*; then every real z* (*in* V) *can be made generic over an iterate of M, i.e., for any real z, there exists an iteration tree T* (*of height* ω) *on M such that for any infinite branch b of T with a well founded direct limit model M_b, there is an h, M_b-generic* $/\mathrm{col}(\omega, i_b(\delta))$ *with* $z \in M_b[h]$.

To prove this we apply the Lemma with $\dot{A} = \{\text{all reals in } M^{\mathrm{col}(\omega,\delta)}\}$.

Compared with Woodin's iterations to make reals generic (see the forthcoming [3]), this method has the disadvantage of using a forcing which is not δ-c.c. ($\mathrm{col}(\omega,\delta)$), and the more serious disadvantage of requiring full information about the real z. For example, this method cannot be used to make a Turing degree generic without first picking a representative of the degree. The main advantage of this method is that it uses only iteration trees of height ω, and so it can be used to make reals generic over models which are just 1-iterable.

§2. The projective hierarchy. Let us now see how to use Lemma 1.7 to establish the determinacy of various classes of projective sets. As a simple first example:[3]

Theorem 2.1. *If M is* 1*-iterable,* $M \models$ "δ is Woodin" *and* (*in* V) *there exists a g which is M-generic* $/\mathrm{col}(\omega,\delta)$, *then* Π^1_2*-determinacy holds* (*in* V).

Remark. The existence of such M is a large cardinal assumption. As mentioned in Section 1, it is weaker than "there exists a Woodin cardinal and a measurable cardinal above it".

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[3] A set A is Σ^1_n if its complement $\mathbb{R} \setminus A$ is Π^1_n, it is Δ^1_n if it is both Σ^1_n and Π^1_n; and similarly for the relativized and the boldface notions A set is *projective* if it is Σ^1_n for some n,

Proof. Let A be a Σ^1_2 set of reals, say

$$x \in A \iff V \models \phi[x],$$

where ϕ is some Σ^1_2 formula. To show that the standard game $G(A)$ with payoff set A is determined, we apply Lemma 1.7 to the canonical name $\dot{A}$ for $A \cap M[g]$, i.e.,

$$\dot{A} = \{\langle \dot{y}, p\rangle \mid p \Vdash^{M}_{\mathrm{col}(\omega,\delta)} \phi(\dot{y})\}.$$

Case I. I wins $\widehat{G}(\dot{A})$, say by $\widehat{\sigma} \in M[g]$. Let σ be the strategy for I obtained by ignoring the auxilliary moves T_ι, h_ι in $\widehat{G}(\dot{A})$.

Claim: *σ is a winning strategy for* I *in $G(A)$.*

Proof. We have to show that if x is a play by σ, then $x \in A$. Now since x is by σ, there exists a corresponding play x, T, h by $\widehat{\sigma}$. As M is 1-iterable, the tree T has a well founded infinite branch b, and since $\widehat{\sigma}$ is winning for I in $\widehat{G}(\dot{A})$:

(i) $\tilde{h}(b)$ is M_b-generic $/\mathrm{col}(\omega, i_b(\delta))$, and
(ii) $x \in i_b(\dot{A}_{\tilde{h}(b)})$.

But

$$\begin{aligned} i_b(\dot{A}) = \{\langle \dot{y}, p\rangle \mid\ & p \text{ is a condition in } \mathrm{col}(\omega, i_b(\delta)),\\ & \dot{y} \text{ is a name for a real in } M_b^{\mathrm{col}(\omega, i_b(\delta))}, \text{ and}\\ & M_b \models \text{“}p \Vdash \phi(\dot{y})\text{”}\}, \end{aligned}$$

so by (ii), $M_b[\tilde{h}(b)] \models \phi[x]$. Since ϕ is Σ^1_2, it is absolute for $M_b[\tilde{h}(b)]$ and V, so $V \models \phi[x]$ and $x \in A$. ⊣ [Claim]

Case II. II wins $G(\dot{A}_g)$ via a strategy $\sigma \in M[g]$, which works against all plays in $M[g]$.

Claim: *σ is, in fact, a strategy for* II *in $G(A)$, which works against all plays (in V).*

Proof. If not, then there is (in V) a play x by σ which is won by I, i.e., $x \in A$. So

$$V \models (\exists x)[\phi[x] \,\&\, x \text{ is by } \sigma].$$

Since ϕ is Σ^1_2, this entire statement is Σ^1_2, hence absolute. So

$$M[g] \models (\exists x)[\phi[x] \,\&\, x \text{ is by } \sigma];$$

but then for some x in $M[g]$, x is by σ, and $x \in A$—which is a contradiction, since σ is winning for II in $M[g]$. ⊣ [Claim]

Since by Lemma 1.7 either Case I or Case II must hold, the game $G(A)$ is determined. ⊣

So we can very easily use Lemma 1.7 if A is absolute enough between $M[g]$, V and between $M_b[h]$, V. Notice that we did not, in fact, use iterability in Case II. The following observation of John Steel shows how we can, to a certain extent, avoid iterability in Case I as well.

Lemma 2.2. *Let M be as in Theorem* 2.1, *except that M need not be* 1-*iterable, let*

$$A = \{x \mid V \models \phi[x]\}$$

be a Σ^1_2 set of reals, let

$$\dot{A} = \{\langle \dot{y}, p\rangle \mid p \Vdash^M_{\mathrm{col}(\omega,\dot{\delta})} \phi(\dot{y})\},$$

and assume that I *wins $\widehat{G}(\dot{A})$; then there exists $\bar{\sigma} \in M$* (not $M[g]$), *a strategy for player* I, *such that if $x \in M$* (again, not $M[g]$) *is a play by $\bar{\sigma}$, then $x \in A$.*

Proof. We avoid using iterability by passing to a submodel of M. More precisely, *working in M*, let $\bar{M}$ be a countable, transitive model which can be embedded into V^M_θ for some large enough θ. Say $\pi : \bar{M} \to V^M_\theta$, and assume that $\delta, \dot{A} \in \mathrm{range}(\pi)$, $\delta = \pi(\bar{\delta})$, $\dot{A} = \pi(\bar{\dot{A}})$. Now, by Corollary 1.3,

$$M \models \text{“}\bar{M} \text{ is iterable”}.$$

Furthermore

$$M \models \text{“there exists } \bar{g} \text{ which is } \bar{M}\text{-generic } /\mathrm{col}(\omega, \bar{\delta})\text{”},$$

and by the elementarity of π,

$$\bar{M} \models \text{“I wins } \widehat{G}(\bar{\dot{A}})\text{”}.$$

As in the proof of Theorem 2.1, we can obtain $\bar{\sigma} \in \bar{M}[\bar{g}] \subset M$, such that if $x \in M$ is a play by $\bar{\sigma}$, then there exist T, $\tilde{h}$ with the properties that:

- T is an iteration tree on $\bar{M}$.
- If b is an infinite branch of T such that $\bar{M}_b$ is well founded, then
  - $\tilde{h}(b)$ is $\bar{M}_b$-generic $/\mathrm{col}(\omega, i_b(\bar{\delta}))$, and
  - $x \in \bar{M}_b[\tilde{h}(b)]$ and $\bar{M}_b[\tilde{h}(b)] \models \phi[x]$.

Since $\bar{M}$ is 1-iterable (in M), we can find a branch b such that $\bar{M}_b$ is well founded, and so $\bar{M}_b[\tilde{h}(b)] \models \phi[x]$. But ϕ is Σ^1_2, and so reflects upwards from set models, so $M \models \phi[x]$.

(It is essential here that ϕ is Σ^1_2. A Π^1_2 statement, for example, would not reflect from $\bar{M}_b[\tilde{h}(b)]$ to M, since $\bar{M}_b[\tilde{h}(b)]$ does not contain all the ordinals.) ⊣

Lemma 2.2 together with the absoluteness of Σ^1_2 statements for class models gives the following result:

COROLLARY 2.3. *If* $M \models$ "δ is Woodin" *and* (in V) *there exists g which is M-generic* $/\mathrm{col}(\omega, \delta)$, *then* Δ^1_2*-determinacy holds in* V. Notice that we do not assume here that M is iterable.

This was first proved by Woodin, see [10]. With some extra effort it seems possible to replace in this Corollary the requirement

"(in V) there exists g which is M-generic $/\mathrm{col}(\omega, \delta)$"

with the slightly weaker

"δ is countable (in V)".

It is not known if "$\delta \leq \omega_1^V$" could suffice, but it is known that nothing weaker than "$\delta \leq \omega_1^V$" is enough. This follows from the following result of Woodin:

THEOREM (Woodin). *If* Δ^1_2*-determinacy holds, then there exists a class inner model M and an ordinal* $\delta \leq \omega_1^V$, *such that* $M \models$ "δ is Woodin".

It is simple to adapt the proof of Corollary 2.3 to give:

THEOREM 2.4. *Suppose* $M \models$ "δ is Woodin", *assume that* (in V) *there exists a g which is M-generic* $/\mathrm{col}(\omega, \delta)$, *and let A be a set of reals such that there exist trees S and* S^* *in M which project to A and* $\mathbb{R} \setminus A$ *respectively; then A is determined, and the player who wins* $G(A)$ *has a winning strategy in M*. Here S and S^* are usual trees, i.e., $S, S^* \subseteq (\omega \times \lambda)^{<\omega}$ for some ordinal λ, $A = \{x \in \mathbb{R} \mid (\exists f)(\forall n < \omega)[\langle x_0, f_0, x_1, f_1, \dots, x_n, f_n \rangle \in S]\}$, and $\mathbb{R} \setminus A = \{x \in \mathbb{R} \mid (\exists f)(\forall n < \omega)[\langle x_0, f_0, x_1, f_1, \dots, x_n, f_n \rangle \in S^*]\}$.

Theorem 2.4 is more general than Corollary 2.3. Indeed it is very simple to derive Corollary 2.3 from Theorem 2.4, by using the Shoenfield trees for A and $\mathbb{R} \setminus A$ when A is Δ^1_2.

Let us now move further up the projective hierarchy. We assume from now on that $(\forall x \in \mathbb{R})[x^\#$ exists]. By a result of Martin [4], a set $A \subseteq \mathbb{R}$ is $\Game(<\omega^2 - \Pi^1_1)(z)$ if and only if there is some k and some formula ϕ such that

$$x \in A \iff L[x] \models \phi[x, z, c_0, \dots, c_{k-1}],$$
$$\text{whenever } c_0, \dots, c_{k-1} \text{ are indiscernibles for } x \text{ and } z.$$

THEOREM 2.5. *Suppose M is n-iterable* $(n < \omega)$, $z \in M \cap \mathbb{R}$, $M \models$ "$\delta_1 < \dots < \delta_n$ are Woodin" *and assume that* (in V) *there exists a g which is M-generic* $/\mathrm{col}(\omega, \delta_n)$; *then* $\Game^{(n)}(<\omega^2 - \Pi^1_1)(z)$*-determinacy holds* (in V).

THEOREM 2.6. *If* $B \subseteq \mathbb{R}$ *is in* $\Game^{(n+1)}(<\omega^2 - \Pi^1_1)(z)$, *then there exists a formula* ϕ *and some* $k \in \omega$, *such that for every real* x,

$$x \in B \iff M \models \phi[x, z, c_0, \dots, c_{k-1}],$$

whenever M is an n-iterable model such that

- $M \models$ "$\delta_1 < \cdots < \delta_n$ are Woodin",
- $x, z \in M$,
- (in V) *there exists a g which is M-generic* $/\mathrm{col}(\omega, \delta_n)$.
- $c_0, \ldots, c_{k-1}$ *are indiscernibles for* $V^M_{\delta_n}$, *and*

Theorems 2.5 and 2.6 are proved together by induction on n. Theorem 2.6 for the integer n supplies us with the absoluteness needed to apply Lemma 1.7 in proving Theorem 2.5 for the integer $n+1$. Note the similarity between Theorem 2.6 and Martin's result mentioned above; for $n = 0$, Theorem 2.6 *is* Martin's result.

§3. The full axiom of determinacy.

Finally, using the ideas of this paper, we can also give a new and simpler proof of the following theorem of Woodin.

THEOREM 3.1 (Woodin). *Let δ be a limit of Woodin cardinals* (in V), *let H be V-generic* $/\mathrm{col}(\omega, <\delta)$, *and let $\mathbb{R}^*$ be the set of reals in the symmetric collapse, i.e.,*

$$\mathbb{R}^* = \bigcup_{\alpha<\delta} \mathbb{R}^{V[H\cap \mathrm{col}(\omega,<\alpha)]};$$

then $L(\mathbb{R}^*) \models$ AD.

We need some terminology. A *coarse premouse* is a structure $\mathcal{M} = \langle M, \in, \delta^{\mathcal{M}} \rangle$ satisfying the fragment of ZFC specified in [6, Definition 1.1]. We say that $\mathcal{M}$ is an *ω-Woodin mouse* just in case $\mathcal{M} \models$ "$\delta^{\mathcal{M}}$ is a limit of Woodin cardinals". We say $\mathcal{M}$ is *ω-iterable* just in case II wins the weak iteration game of length ω on $\mathcal{M}$ (cf. [6, §5], where the game is called $WG(\mathcal{M}, \omega)$). The proof of 1.3 easily shows that if $\langle V_\alpha, \in, \delta \rangle$ is a coarse premouse and $\mathcal{M}$ is a countable elementary submodel of $\langle V_\alpha, \in, \delta \rangle$, then $\mathcal{M}$ is ω-iterable. Finally, if $\mathcal{M}$ is a coarse premouse, we say that $\mathbb{R}^*$ is (the set of reals of) a symmetric collapse of $\mathcal{M}$, just in case for some H which is M-generic $/\mathrm{col}(\omega, <\delta^{\mathcal{M}})$,

$$\mathbb{R}^* = \bigcup_{\alpha<\delta^{\mathcal{M}}} \mathbb{R} \cap M[H \cap \mathrm{col}(\omega, <\alpha)].$$

We use the following absoluteness lemma which is a slight variant of a theorem due to Steel and Woodin, see [8, §4].

LEMMA 3.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be countable, ω-iterable, ω-Woodin mice. Let $\mathbb{R}^*_{\mathcal{M}}$ and $\mathbb{R}^*_{\mathcal{N}}$ be the symmetric collapses of $\mathcal{M}$ and $\mathcal{N}$ respectively. Suppose θ is a sentence and*

$$\alpha_0 = \text{the least } \alpha \in OR^{\mathcal{M}} \text{ such that } L_\alpha(\mathbb{R}^*_{\mathcal{M}}) \models \theta,$$

$$\beta_0 = \text{the least } \beta \in OR^{\mathcal{M}} \text{ such that } L_\beta(\mathbb{R}^*_{\mathcal{N}}) \models \theta.$$

(We suppose there are such α and β.) *Let $\phi(v)$ be a formula and $x \in \mathbb{R}^*_{\mathcal{M}} \cap \mathbb{R}^*_{\mathcal{N}}$; then*

$$L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}}) \models \phi[x] \iff L_{\beta_0}(\mathbb{R}^*_{\mathcal{N}}) \models \phi[x].$$

Proof (Sketch). The idea is to simultaneously build plays of the weak iteration games $WG(\mathcal{M}, \omega)$ and $WG(\mathcal{N}, \omega)$, in a "back-and-forth" fashion. These yield iteration maps $i : \mathcal{M} \to \mathcal{P}$ and $j : \mathcal{N} \to \mathcal{Q}$, and we can arrange that there are symmetric collapses $\mathbb{R}^*_{\mathcal{P}}$ of $\mathcal{P}$ and $\mathbb{R}^*_{\mathcal{Q}}$ of $\mathcal{Q}$ such that $\mathbb{R}^*_{\mathcal{P}} = \mathbb{R}^*_{\mathcal{Q}}$. Corollary 1.8 (or, alternatively, Woodin's genericity theorem) makes the successor step of the construction possible (indeed trivial). The homogeneity of the collapse poset and the elementarity of i and j now yield the desired conclusion. ⊣

Proof of 3.1 (Sketch). If 3.1 fails, then a simple Löwenheim-Skolem argument produces a countable, ω-iterable, ω-Woodin mouse $\mathcal{M}$, such that for some symmetric collapse $\mathbb{R}^*_{\mathcal{M}}$ of $\mathcal{M}$, and some $\alpha \in OR^{\mathcal{M}}$, $L_\alpha(\mathbb{R}^*_{\mathcal{M}}) \models \neg\mathrm{AD}$. Let α_0 be the least such α, and let θ be the sentence "$\neg$AD". Every set in $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}})$ is ordinal definable over $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}})$ from some real; to simplify notation, let us assume the real is 0 in our case; that is, there is a $B \subseteq \mathbb{R}^*_{\mathcal{M}}$, ordinal definable over $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}})$ such that $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}}) \models$ "B is not determined". By minimizing the ordinal parameters, we may assume that B is, in fact, definable over $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}})$ from no parameters. Say $B = \{x \mid L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}}) \models \phi[x]\}$.

Let $\delta = \delta^{\mathcal{M}}$, and let $\delta_0 < \delta$ be the least Woodin cardinal of M. Let $\dot{A}$ be the following name for a set of reals in $M^{\mathrm{col}(\omega,\delta_0)}$:

$$\begin{aligned} \dot{A} = \{\langle \dot{x}, p\rangle \mid\ & \dot{x} \text{ is a name for a real in } M^{\mathrm{col}(\omega,\delta_0)}, \\ & p \in \mathrm{col}(\omega, \delta_0), \text{ and} \\ & p \Vdash_{\mathrm{col}(\omega,\delta_0)} \text{“} \Vdash_{\mathrm{col}(\omega,<\delta)} \text{“}L_{\alpha_0}(\mathbb{R}^*) \models \phi[\check{\dot{x}}], \\ & \qquad \text{where } \alpha_0 \text{ is the least } \alpha \text{ such that } L_{\alpha_0}(\mathbb{R}^*) \models \neg\mathrm{AD}\text{” ”}\}. \end{aligned}$$

We now use Lemma 1.7 with $\dot{A}$. We may assume Case I holds, as otherwise we can just replace $\dot{A}$ with its complement. (Notice that Case II cannot hold for both, as then we could just pit the relevant strategies against each other and get a contradiction.)

Let $g \in \mathbb{R}^*_{\mathcal{M}}$ be M-generic $/\mathrm{col}(\omega, \delta_0)$. Lemma 1.7 gives us a strategy $\hat{\sigma} \in M[g]$ which is winning for I in $\widehat{G}(\dot{A})$. From $\hat{\sigma}$ we can obtain a strategy σ by ignoring I's auxilliary moves. Since $\sigma \in \mathbb{R}^*_{\mathcal{M}}$, σ is not winning for I in $G(B)$, and so we can fix a play x by σ such that $x \in \mathbb{R}^*_{\mathcal{M}} \setminus B$. Let x, T, h be the corresponding play by $\hat{\sigma}$.

Since $\mathcal{M}$ is ω-iterable, we have a cofinal branch b of T such that $\mathcal{M}_b$ is ω-iterable. Since $\hat{\sigma}$ is winning, $x \in i_b(\dot{A})_{\hat{h}(b)}$, where $\hat{h}(b)$ is the M_b-generic on $\mathrm{col}(\omega, i_b(\delta_0))$ produced by h. Inspecting $\dot{A}$, we see that this means that for some symmetric collapse $\mathbb{R}^*_{\mathcal{M}_b}$ of $\mathcal{M}_b$, there is a β such that $L_\beta(\mathbb{R}^*_{\mathcal{M}_b}) \models \neg\mathrm{AD}$, and for β_0 the least such β, $L_{\beta_0}(\mathbb{R}^*_{\mathcal{M}_b}) \models \phi[x]$. But $x \notin B$, so $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}}) \models \neg\phi[x]$, and this contradicts Lemma 3.2. ⊣

The proof of 3.2 also shows that if $\mathcal{M} = \langle M, \in, \delta \rangle$ is an ω-Woodin mouse with $OR \subseteq M$ and V_δ^M countable, and if $\mathcal{M}$ is (forced to be) ω-iterable in $V^{\mathrm{col}(\omega, 2^{\aleph_0})}$, then $L(\mathbb{R}) \equiv L(\mathbb{R}^*_{\mathcal{M}})$ for any symmetric collapse $\mathbb{R}^*_{\mathcal{M}}$ of M. So the existence of such an $\mathcal{M}$ implies $\mathrm{AD}^{L(\mathbb{R})}$. The existence of such an $\mathcal{M}$ follows, using the proof of 1.2, from the existence of ω Woodin cardinals with a measurable above them all.

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IN MEMORIAM: RAPHAEL MITCHEL ROBINSON

About a month after his 83rd birthday Raphael Robinson was almost wholly incapacitated by a massive stroke, and 8 weeks later, on January 27, 1995, he died of ensuing complications. Mathematics was his life. He was always working on problems—those brought to him in journals or by colleagues, and others that he invented. Just three days before his death he received word that a paper of his, originating in a published problem, was accepted for publication. His 64 publications spanned a full 6 decades, and included significant papers on number theory, combinatorics, complex analysis, and geometry, in addition to logic and set theory.

Robinson was born on November 2, 1911, in National City, California. He came to Berkeley to study mathematics, obtaining an A.B. in 1932, an M.A. in 1933, and his Ph.D. in December, 1934. His dissertation was in complex analysis. He took 12 graduate courses in a variety of fields, but not one was close to logic or set theory—which was later the setting of about a quarter of his publications.

After completing his studies Robinson served as Instructor for two years at Brown University, then returned for good to Berkeley in 1937, reaching the rank of full Professor in 1949. He taught a great variety of courses and was known as an excellent teacher, but was most interested in problem-solving and research and in 1972, at age 61, he elected early retirement, at considerable financial sacrifice, in order to concentrate on these pursuits.

In Fall 1939 Robinson had a student named Julia Bowman in his number theory class; she had just transferred to Berkeley for her senior year, during which his was one of five mathematics courses that she took. Raphael and Julia began to take long walks during which he explained the various fields of mathematics that interested him. When she obtained the A.B. degree in 1940 he urged her to remain at Berkeley for graduate study, which she did. The walks continued, and their friendship evolved into courtship. She obtained her M.A. in June 1941, and they were married in December of that year. During the next 44 years their shared mathematical interests enriched the work of each; although they never published a joint paper they often wrote on the same topic and in several areas developed ideas introduced earlier by the other. At the end of her life, after she had become the first woman mathematician elected to the National Academy of Sciences and the first woman president of the American Mathematical Society, Julia told her sister Constance Reid, the widely known biographer of mathematicians,

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that she doubted she would have become a mathematician if it had not been for Raphael, adding that he continued to teach her throughout their life together, encouraging and supporting her in many ways.

Robinson's seventeen papers that fall within the area of foundations of mathematics begin with two on axiomatic set theory. In [1937] he published a simplification of von Neumann's axiom system based on *function*, rather than *set*, as the primitive undefined concept. Then, in [1945], he solved a problem raised by Tarski, who had arrived in Berkeley in 1942 and had taken on Julia Robinson as one of his first Ph.D. students. Raphael's paper showed how the notion of a *finite sequence of classes* can be defined in Gödel–Bernays set theory, even though it contains classes that are not members of any class.

Robinson's subsequent foundational work falls mainly into three overlapping areas: Recursive functions and RE sets, decision problems, and problems of definability.

1. Recursive functions and RE sets. In [1947] and [1955], Robinson characterized the set of primitive recursive functions (PR) in a variety of ways, and in [1948] he gave an elegant construction of a function definable by double recursion which is *not* in PR, but can be used to generate PR in a certain way. In [1956] Robinson improved a characterization by Martin Davis of RE sets by Diophantine equations preceded by existential and one bounded universal quantifiers, by reducing to 4 the number of existential quantifiers needed; and then in [1972] he improved this representation by referring to new work of Julia Robinson and Matijasevič.

2. Decision problems. Probably the result of Robinson most widely used by logicians to show the undecidability of many first-order theories is his elegant demonstration that a subset of Peano arithmetic (now known as Robinson's arithmetic Q) consisting of seven simple sentences is essentially undecidable; he first presented this at the 1950 International Congress of Mathematicians [1952], and it was later included in the book [1953]. In [1951], Robinson used the methods of that book to show the undecidability of many classes of rings; in [1964], using methods developed by Julia Robinson the preceding year, he showed that any transcendental extension of a real field is undecidable, in contrast to Tarski's early showing that each real closed field is decidable. In the following decade, Robinson took up a class of decision problems originating with Hao Wang concerning the possibility of completely tiling the plane by translations of a given finite array of tiles (subject to possible conditions defined in terms of their colored edges). The question whether there is such a *periodic* tiling of the plane is clearly decidable, but in [1971] Robinson produced a set of 6 polygonal tiles which can tile the whole plane, but never in a periodic way, and he produced an array of 36 square tiles with colored edges, for which he proved that there is no decision method to tell which partial tilings obtained by finitely many translations of the array can be extended to a tiling of the whole plane.

3. Problems of definability. For many of the structures shown by Robinson to have an undecidable theory, a crucial step of his proof involved showing the definability (in that first-order theory), of the fragment of Peano arithmetic described in [1952] and [1953]. In [1951a] and [1951b] he gave the simplest arithmetical definition of $\mathbb{N}$ in the theory of $\mathbb{Z}$, and discussed the relation between the arithmetical definability of an element x of an algebraic number field F and the invariance of x under automorphisms of F. Then, in [1958], he turned to questions of definability in the monadic second-order theory of some fragments of the theory of $\mathbb{N}$, showing that $+$ is definable in terms of successor and doubling, and then deriving the undecidability of this theory as a corollary.

Robinson's seventeen papers on foundations of mathematics rarely touch on general questions. Each concerns particular problems which he solves by original ideas that often emerge from his deep knowledge of classical analysis, algebra, and above all number theory. He was truly a pioneer in the applications of classical mathematics to problems of metamathematics. In this he reminds me of another pioneer who by chance had the same surname—Abraham Robinson.

In conclusion, I wish to mention what I believe to be Raphael Robinson's most widely known result, even though it is not really in the area of logic and set theory. In [1947] he greatly improved the Banach–Tarski Paradox of 1924, by showing how to partition the points of a solid 3-dimensional sphere into 5 subsets, one of which contains only a single point, such that these sets can be moved by translations and rotations so as to form two solid spheres of the same radius. He proved that 5 is the minimum number that can accomplish this, even if reflections are allowed as well as translations and rotations. He also showed that the surface of the 3-dimensional sphere can be partitioned into 4 subsets, which can be recombined to form two such spherical surfaces of the same radius.

The Robinsons were close friends of mine. Together with four other couples and the first Math Department secretary, the Robinsons and Henkins formed a set of "Foster Parents" who helped support, and exchanged monthly letters with, a sequence of impoverished children who grew to maturity in distant parts of the world. Julia was outgoing, Raphael reticent, but they shared political and social views. On behalf of several of these Julia joined in organized activism, which was not Raphael's style. However, when 5 members of the Math Department were fired in the 1950 Loyalty Oath controversy that racked the University, Raphael joined a group of colleagues who created a "mathfund" to which they pledged 10% of their salaries to support the non-signers, and he served as the Treasurer for the fund.

In his last 10 years, after Julia's death from cancer, Raphael lived alone, taking care of their modest home, shopping for and preparing food. He continued to participate in university colloquia and other mathematics meetings in California, giving talks at some. After his stroke he received much care from Julia's sister, Constance Reid. Shortly before his death he asked her to let people know his wish that those wanting to contribute in remembrance of him, should make the contribution to the Julia Robinson Fund which he had established after her death, to support graduate students in mathematics at the University of California, Berkeley.

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LEON HENKIN

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1994 SPRING MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

Kansas City, Missouri, May 6–7, 1994

The Spring Meeting of the Association was held in Kansas City, Missouri, on May 6–7, 1994, in conjunction with the Central Division of the American Philosophical Association The program included a panel discussion, two invited lectures, nine contributed papers, and four papers presented by title The program committee consisted of Larry Moss and Colin McLarty (chair).

The panel on *Category Theory and the Foundations of Mathematics and Logic* consisted of John Bell, Peter Freyd, and Vaughn Pratt; Colin McLarty was the moderator.

The following invited talks were presented·

Shaughan Lavine (Columbia University), *Understanding the infinite*

L. Thorne McCarty (Rutgers University), *Indefinite information in commonsense reasoning.*

Abstracts of contributed talks given (in person or by title) by members of the Association for Symbolic Logic follow.

For the Program Committee
COLIN MCLARTY, CHAIR

Abstracts of contributed talks

▶ STEVEN MICHAEL AWODEY, *Axiom of choice and excluded middle in categorical logic.*

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A notion of constructive, higher-order logic which includes the axiom of choice (AC) but not excluded middle (EM) cannot be modeled in the standard way in topoi, by a well-known result of Diaconescu. Certain predicative type theories are related to locally cartesian closed (LCC) categories the way higher-order logic is to topoi. A proof of AC is given in the LCC category setting, thus valid in any such type theory Since topoi are LCC this provides a way of modeling constructive logic with AC in topoi where EM still fails in general. It is further shown that EM holds just in case AC holds under the standard interpretation of higher-order logic in topoi, and that this in fact obtains just when the two interpretations agree, i e., when they give all the same logical truths.

▶ JON BARWISE, *Substructural logics and a model of information flow.*

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Informally, an information network is a constellation of components connected together in ways that systematically supports the extraction of information about components from

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information about other components. In a recent paper, "On the logic of information flow," Gabbay, Haronis, and I have explored Gentzen systems based on a formalization of this idea of an information network. Two of these systems bear striking similarities to the Lambek Calculus (LC). Indeed, they can be thought of as systems where the circle of the LC splits into an application operator and a composition operator. In this talk I will define and illustrate information networks and exhibit the complete calculus for these networks. This work grew out of noting remarkable formal similarities between my work on situation theory and Gabbay's work on labeled deductive systems, and also out of joint work of mine with Jerry Seligman on imperfect information flow.

▶ STEPHEN HARNISH, *An empirical set theory: Analyzing fractals via quantified resolutions.*

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Fractal images and many natural phenomena exhibit "complex structure" at "varying levels of detail." One way of making this observation precise follows. Assume "images" are arbitrary subsets of $\mathbf{R}^n$ or some other metric space, M. We'll consider simple conceptions of "image resolution" by introducing operators which simplify or "resolve" the images at certain levels of resolution. These operators have algebraic properties reminiscent of universal and existential quantification. Finally, we'll examine several schemes for representing the structure of images by directed graphs and interpreting the directed graphs as nonwellfounded sets.

An empirically chosen universe of sets results from the choices of the metric space, the scheme for graphical representation, and the criterion for set equality within a particularly nonwellfounded set theory. I'll focus on the case where M is the real plane, $\mathbf{R}^2$, and the decreasing-existential scheme of graphical representation is employed. The resulting graphs are trees with countable branching.

▶ ROBERT LUBARSKY, *Kripke models for independence results in IKP.*

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Classically, Σ_1-Dependent Choice, Resolvability, and Π_2-Reflection are all inequivalent over KP, but equivalent in L. However, using only intuitionistic logic, the inequivalence carries over even into L. The constructions all hinge on turning the classical forcing arguments for independence into constructions of Kripke models, and moreover into constructions of certain ordinals in Kripke models. The details of one of the proofs were given.

▶ CHRISTOPHER MENZEL, *Singular propositions and modal logic.*

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By *Russellian semantics* I mean the conjunction of the following three theses: (1) There is no more to the semantics of a name than the object referred to. (2) The meaning of a sentence F is an abstract entity, the *proposition* $[F]$ it expresses. (3) Propositions are structurally complex, that is, roughly, (i) they have an internal structure that corresponds directly to the syntactic structure of the sentences that express them, and (ii) the metaphysical constituents of that structure are the semantic values of the corresponding syntactic constituents of those sentences.

Consider a sentence containing a name—the sentence 'Quine is a philosopher', say. If

Russellian semantics is correct, it follows that the proposition this sentence expresses contains *Quine himself* as a constituent in that part of its metaphysical structure corresponding to the name 'Quine'. But this raises a puzzle. Intuitively,

$$\text{Quine might not have existed.} \qquad (*)$$

That is, the proposition [Quine does not exist] could have been true. But if Russellian semantics is correct, there would have *been* no such proposition as [Quine does not exist] if Quine hadn't existed, hence it wouldn't have been there to be true; rather, there would have been no information about Quine whatever. Since of course [Quine does not exist] also fails to be true when Quine does exist, it follows that, necessarily, whether he exists or not the proposition that he doesn't is not true. Thus, Quine's nonexistence is not really possible, and hence, contrary to our initial intuition, (∗) isn't true after all.

In this talk I argue for a logical solution to this puzzle that does minimal violence to classical quantified modal logic while preserving both the structured view of propositions and the intuition that (∗) is true. The framework for this solution is a first-order modal language enriched with complex predicates/terms that, intuitively, denote structured properties, relations, and propositions.

► JAMES OTTO, *The linear time hierarchy* (Δ_0) *via tiers and monoidal categories.*

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We find the functions computable in linear time among the primitive recursive functions. We do this via tiers, to restrict the definability of nested loops, and via weakening product to tensor, to make duplication of variables definable only by explicit loops.

We then find that the linear time hierarchy relations, which Woods [9] has shown are the bounded arithmetic relations (Δ_0), are the tier 0 quantifications of the linear time relations, and that the recursively enumerable relations are the tier 1 existential quantifications of the linear time relations.

Our characterization of the linear time functions differs from similar work of Bloch's [3] by not allowing diagonal inside loops.

Our characterization of the linear time hierarchy relations differs from Wrathall's [11] in replacing bounded quantifications with (not explicitly bounded) tier 0 quantifications. Bellantoni [1] has a related characterization of the polynomial time hierarchy functions.

As is usual in recursion theory, we characterize the linear time functions as the smallest set of maps closed under certain rules. However the maps are in *set*^{2} rather than in *set*. This is our understanding [7] of tiers [2, 5].

As our rules include composition and identities, we collect them into a doctine $\mathcal{D}$ (= restricted class) of categories $\mathcal{C}$. However, rather than have $\mathcal{C}$ be FP (= having finite products) as is often implicit in recursion theory, we take $\mathcal{C}$ to be monoidal [6] In particular $\mathcal{C}$ is closed under tensor $\otimes$ rather than under product $\times$ We further take $\mathcal{C}$ to have 2 tiers of dyadic numbers and to have flat recursion [5, 4] and left very safe recursion [3, 8].

The image in *set*^{2} of the initial category $\mathcal{I}$ in our doctrine $\mathcal{D}$ is then both what is definable in *set*^{2} by our doctrine and the smallest set of maps in *set*^{2} closed under our rules. Our rules restrict the rules of Román [10] and Paré-Román [8] for numbers in FP categories and monoidal categories.

[1] S. Bellantoni, ***Ph.D. thesis,*** Department of Computer Science, University of Toronto, 1992.

[2] S. Ballantoni and S Cook (editors), ACM STOC Proceedings, 1992.

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[4] R. Cockett, conversation, 1993.

[5] J. Marion and D. Leivant, e-mail, 1992.

[6] S. MacLane, ***Categories for the working mathematician***, Springer-Verlag, 1971.
[7] J. Otto, *Kalmar, linear space, and P*, 1993; Notes on Pspace, 1993.
[8] R. Paré and L. Román, ***Studia Logica*** (1989).
[9] J. Paris, A. Wilkie, and G. Wilmers (editors), ***Logic colloquium 84***, 1986
[10] L. Román, ***Journal of Pure and Applied Algebra*** (1989).
[11] C. Wrathall, ***SIAM Journal on Computing*** (1978)

► MASARU SHIRAHATA, *The extension of linear set theory with equality.*

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In this paper, we study set theory based on linear logic with the equality relation. In particular, we present the system AZF with equality, which is an extension of the author's system LZF and a conservative extension of Zermelo-Fraenkel set theory without the regularity or ZF$^-$

The motivation to formulate set theory in linear logic is that we can have the unrestricted comprehension in such a system without being inconsistent. However, Grishin showed that the equality relation induces the contraction over equations and, combined with the extensionality, the contraction over any formulas. He concluded from this that linear set theory with the extensionality axiom is inconsistent

Given this result, we formulate AZF with the equality relation but without the extensionality. AZF has the unrestricted comprehension, and the contraction over equations as a rule of inference from the beginning. We then show by the cut-elimination argument that AZF is a conservative extension of ZF$^-$ and therefore consistent.

With equality at hand, it becomes possible to define in general basic mathematical constructions such as ordered pairs, relations and functions. We work out such constructions in some detail and see the peculiarities unique to linear set theory, as well as the limitation imposed by the lack of extensionality. Finally, to clear the way for the future work, we show that Grishin's result does not hold if we formulate the extensionality with the exponentials, which were not available to him.

► LAWRENCE NEFF STOUT, *Generalizing categorial logic—What can be gained?*

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Category theory can be applied to logic at several different levels: proof theory at the propositional level can be viewed in terms of cartesian closed categories; first order logic can be obtained using locally cartesian closed categories to get universal and existential quantification; higher order logic can be obtained using subobject classifiers. Each level can be generalized.

We can obtain propositional logics with alternative forms of conjunction by considering monoidal closed structures. The first order logic can arise from factorization systems (as in Monro's article *Logic, sheaves, and factorization systems*, ***The Journal of Symbolic Logic***, vol. 58, 1993) coupled with a local monoidal closed structure. Monoidal systems can introduce noncommutativity or nilpotence. Factorization systems can introduce nondistributivity.

Existential quantification comes from factorization, universal quantification comes from the local monoidal structure. If the factorization system respects the monoidal structure a satisfactory typed proof theory can be given for first order logic. This logic is best thought of as dealing with inferences about predicates rather than statements about individuals It can take into account limitations on resources and multiple use of the same data.

Subobject classification is a very strong condition even with the subobjects are generalized using factorization systems, essentially forcing the logic to resemble topos logic. Weakening the conditions on subobject classification by allowing nonunique classifying maps and a generic subobject which is not the terminal object will allow for higher order logic with more variety. The uniqueness condition can be replaced by functorial canonical choices often arising from such additional structure as inf-complete orders on hom sets.

Abstracts presented by title

► GUNWON LEE, *ROM: A realization of model.*

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A model described in terms of a theory in a language can be realized. It presupposes that the model in principle is a conceptual model or a conventional device as the models of metaphors in John R. Searle, which I think is not free from the ontological commitments. When a mathematical model given as a theory in the number theories in general has been realized as a physical machine, such as the finite states machine; finite state can be said to be a ROM. A symbolically realizable model of G. Lee; SROM is a model which can be realized as a model usable for the word processing in principle. However if we are restricting the model as the physically realized model PROM, then PROM is a subset of SROM. SROM is not free from the ontological commitment unless we specify the ROM as in practice as in S. Kleene as e-realization, I think. However the fuzziness of the physicality remains as the open question.

► ZHOU XUNWEI, *Predicate logic and set theory are the same discipline.*

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In classical logic, logic and set theory are regarded as different disciplines, yet their theorems are strikingly similar. Let's compare theorems in logic and set theory:

	Logic	Set theory
1	$\neg\neg P \iff P$	$\bar{\bar{A}} = A$
2	$P \vee P \iff P$	$A \cup A = A$
3	$P \wedge P \iff P$	$A \cap A = A$
4	$P \vee Q \iff Q \vee P$	$A \cup B = B \cup A$
5	$P \wedge Q \iff Q \wedge P$	$A \cap B = B \cap A$
6	$(P \vee Q) \vee R \iff P \vee (Q \vee R)$	$(A \cup B) \cup C = A \cup (B \cup C)$
7	$(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$	$(A \cap B) \cap C = A \cap (B \cap C)$
8	$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9	$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
10	$\neg(P \vee Q) \iff \neg P \wedge \neg Q$	$\overline{A \cup B} = \bar{A} \cap \bar{B}$
11	$\neg(P \wedge Q) \iff \neg P \vee \neg Q$	$\overline{A \cap B} = \bar{A} \cap \bar{B}$
12	$P \vee (P \wedge Q) \iff P$	$A \cup (A \cap B) = A$
13	$P \wedge (P \vee Q) \iff P$	$A \cap (A \cup B) = A$
14	$(P \leftrightarrow Q) \iff (P \rightarrow Q) \wedge (Q \rightarrow P)$	$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
15	$P \vee T \iff T$	$A \cup U = U$
16	$P \wedge F \iff F$	$A \cap \emptyset = \emptyset$
17	$P \vee F \iff P$	$A \cup \emptyset = A$
18	$P \wedge T \iff P$	$A \cap U = A$

19	$P \vee \neg P \iff T$	$A \cup \bar{A} = U$
20	$P \wedge \neg P \iff F$	$A \cap \bar{A} = \emptyset$
21	$(P \to Q) \iff (\neg Q \to \neg P)$	if $A \subseteq B$, then $\bar{B} \subseteq \bar{A}$
22	$P \Longrightarrow P \vee Q$	$A \subseteq A \cup B$
23	$P \wedge Q \Longrightarrow P$	$A \cap B \subseteq A$
24	$(P \to Q) \wedge (Q \to R) \Longrightarrow (P \to R)$	if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
25	$(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Longrightarrow (P \leftrightarrow R)$	if $A = B$ and $B = C$, then $A = C$

Why are logical theorems and theorems in set theory so similar? Because predicate logic and set theory are the same discipline. For instance, it means the same that Aristotle belongs to the set of man and that Aristotle satisfies the predicate of man

Geometrical logic is constructed by the author. In geometrical logic there is no propositional calculus, predicate calculus is treated the same discipline with set theory. In geometrical logic there are five operators $\sim$, $\sqcap$, $\sqcup$, $\oplus$, $\otimes$ and seven connectives $\sqcap^{-1}$, $\sqcup^{-1}$, $\subseteq^{-1}$, $=^{-1}$, $\subset^{-1}$, X^{-1}, $\oplus^{-1}$. $\sim$ corresponds to $\neg$ in logic and $\bar{\ }$ in set theory, $\sqcap$ to $\wedge$ in logic and $\cap$ in set theory, $\sqcup$ to $\vee$ in logic and $\cup$ in set theory, $\subseteq^{-1}$ to $\to$ in logic and $\subseteq$ in set theory, $=^{-1}$ to $\leftrightarrow$ in logic and $=$ in set theory, $\subset^{-1}$ to $\to$ in logic and $\subset$ in set theory.

[1] FANG SHICHANG, ***Discrete Mathematics***, Press of Northwestern Communication Engineering Institute, 1987.

THE BULLETIN OF SYMBOLIC LOGIC
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THIRD SOUSLIN CONFERENCE

CO-SPONSORED BY THE ASSOCIATION FOR SYMBOLIC LOGIC

Saratov, Russia, July 21–27, 1994

The Third Souslin Conference, an international meeting dedicated to the 100th anniversary of the birth of M. Souslin, was held July 21–27, 1994, in Saratov, Russia, at Saratov Higher Education University. Sixty-two people participated in the conference

The conference was held in cooperation with the Association for Symbolic Logic and the Russian Academy of Sciences. It received financial support from the Russian Foundation of Fundamental Research and the Institute for Studies in Theoretical Physics and Mathematics (Tehran, Iran).

There were nine invited talks:

N. Bamber (England)
O. Verbitskii (Ternopol, Ukraina and MIAN, Moscow)
A. Vernitskii (Ural State University)
I. Kayumov (Kazan State University)
V. N. Krupskii (Moscow State University)
T. Obedkov (Kazan State University)
V. A. Uspensky (Moscow State University)
V. M. Tihomirov (Moscow State University)
F. Topsoe (University of Copenhagen, Denmark)

There were forty-two other talks The abstracts from talks given at the conference were published. More information about the conference proceedings may be obtained from the Organizer by sending an e-mail message to the address `vapr@scnit.saratov.su` with the subject line `attn: Molchanov`.

Organizing Chair: V. MOLCHANOV
Souslin Foundation, Michurina 92, Saratov 410071 Russia

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Volume 1, Number 3, Sept 1995

AUSTRALASIAN ASSOCIATION FOR LOGIC GEORGE HUGHES MEMORIAL CONFERENCE

CO-SPONSORED BY THE ASSOCIATION FOR SYMBOLIC LOGIC

Otago, Dunedin, New Zealand, August 26–29, 1994

The George Hughes Memorial Conference of the Australasian Association for Logic was held at Knox College of the University of Otago, Dunedin, New Zealand, on August 26–29, 1994. About forty people participated in the conference, which was organized by Pavel Tichy, the retiring AAL President.

The conference was held in cooperation with the Association for Symbolic Logic. Abstracts of contributed talks given (in person or by title) follow

For the ASL Committee on Logic in Australia
MARTIN BUNDER, CHAIR

Abstracts of contributed talks

► ROSS T. BRADY, *Gentzenization of relevant logics with distribution*
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I present Gentzenizations of the common sentential relevant logics, B, DW, DJ, TW, TJ, RW, T and R. We use Inversion Rules and t-rules, in addition to the customary structural and connective rules, in order to eliminate the Cut-rule. In interpreting the Gentzenizations back into the logics, we need to add classical negation '$-$', fusions 'o' and 'o_t' and the sentential constant 't'. Either or both fusions will be deleted for some logics We then show conservative extension results for each of the logics, with respect to these additions, so as to establish the Gentzenizations for the logics without these additions.

First, we show the conservative extension result for B, i.e., if A is provable in Bt^{-oo_t} then A is provable in B, for formulae A without occurrences of 't', '$-$', 'o', and 'o_t' The method uses a reduced Routley-Meyer semantics for B with the special postulate, $\mathrm{RT}bc \iff b = c$. The key step is the completeness argument for B, in which this postulate is ensured by adding two extra elements $\mathrm{B}'$and $\mathrm{B}^{*\prime}$ to the set of prime B-theories. Use is made of the work of Meyer, Slaney, Sylvan, Giambrone, Priest and Restall in establishing this for B and the other logics.

► MARTIN BUNDER AND NORMAN MEGILL, *Weaker D-complete logics.*
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To show that an implicational logic is D-complete it is enough to show that all substitution instances of its axioms are deducible in the corresponding logic from the condensed detachment rule D alone.

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It is well known that every logic with axioms from the set (I), (B), (B′), (C), (K) is D-incomplete. Meyer and Bunder (1988) showed that BB′IW is D-complete.

The main result of this paper is that there is an infinite sequence of independent D-complete systems that are all weaker than BB′IW. We also consider the question: Given a D-incomplete system, how many substitution instances of the axioms must be added to make the system D-complete? The answer to the question is a finite number (such as 1) in some cases, an infinite number in others and an unknown number in most.

▶ CRISTIAN CALUDE, *Godel's incompleteness theorem. An information-theoretic perspective.*

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One of the main applications of *Algorithmic Information Theory* [1, 3] concerns the incompleteness phenomenon discovered by Gödel. The major results are due to G. Chaitin [4, 5]; they point out information-theoretic limits of Chaitin's results. Using topological tools as described in [2], we show that the incompleteness in Chaitin's sense is pervasive

[1] C. Calude, ***Information and randomness: An algorithmic perspective***, Springer-Verlag, 1994 (in press)

[2] C Calude, H. Jürgensen, and M Zimand, *Is independence an exception?*, ***Appl. Math. Comput.*** (in press).

[3] G. J. Chaitin, ***Algebraic information theory***, Cambridge University Press, 1987. (third printing 1990)

[4] G J. Chaitin, ***Information-theoretic incompleteness***, World Scientific, 1992.

[5] G J. Chaitin, *The limits of mathematics. Course outline and software*, IBM Watson Center, December 12, 1993.

▶ PHILIP CATTON, *Problems with a deductivist image of scientific reasoning.*

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Deductivism as an approach to the characterisation of scientific inference has from the vantage point of several philosophers' concerns some significant merits. I briefly canvass these. Yet despite these merits, I believe the deductivist image of scientific reasoning is pie-in-the-sky Actual sciences have not the perfection of logical organisation required for the deductivist image to ring true The point moreover is not a merely psychologistic one, that actual scientists fail to reason logically. Rather it concerns how enormously difficult it is to achieve in a science the perfection of logical organisation required for the deductivist image to ring true. To argue this point, I look to where support for this point seems least likely. I look to the most exact and rigorous of sciences, mechanics. Just as it was historically extraordinary difficult so to formulate Euclidean geometry that proof procedures become discursive deductive logical, so the task of bringing to mechanics this degree of perfection of logical organisation is enormously challenging, so difficult, indeed, that to this day it has not been satisfactorily carried through This has consequences for the perspicuity of mechanics, and by exploring these consequences I explore how far even of the most exact and rigorous of sciences the deductivist image fails to ring true.

▶ BRIAN F. CHELLAS, *S1, close relatives, classical precursors.*

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The paper reports research carried out jointly with Krister Segerberg and presents a uniform approach to the modal logics S1, S1°, S0.9, and S0.9° by way of classical, "prenormal" logics. Building on work of Cresswell, we develop semantics for prenormal logics and prove determination and decidability. Then we show that S1, S1°, S0.9, and S0.9° are non-classical extensions ("Lewis versions") of certain prenormal logics, determination and decidability for which are immediate. We also use generalizations of a key schema and equivalent inference rule to characterize a class of "Cresswell logics". In light of these we ask some questions about C. I. Lewis's concept of strict implication in S1.

[1] M. J. Cresswell, *The completeness of S1 and some related problems*, ***Notre Dame Journal of Formal Logic***, vol. 13 (1972), pp. 485–496.

[2] M J Cresswell, *S1 is not so simple*, in ***Modality, morality and belief; essays in honor of Ruth Barcan Marcus***, (Walter Sinnot-Armstrong, Diana Raffman and Nicholas Asher, editors), Cambridge University Press, Cambridge, 1994.

[3] Robert Feys, ***Modal Logics***, edited with some complements by Joseph Dopp. Louvain· E Nauwelaerts, and Paris: Gauthiers-Villars, 1965.

[4] E. J. Lemmon, *New foundations for Lewis modal systems*, ***Journal of Symbolic Logic***, vol. 22 (1957), pp. 176–186.

[5] C. I. Lewis and C H. Langford, ***Symbolic Logic***, New York, 1932, second edition, Dover Publications, 1959.

► M. J CRESSWELL, *Incompleteness and the Barcan formula.*

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A (normal) system of propositional modal logic is said to be complete if and only if it is characterised by a class of (Kripke) frames. One of the simplest incomplete logics is K+

$$\mathrm{H} \qquad L(Lp \equiv p) \supset Lp$$

When we move to modal predicate logic the question of completeness can again be raised It is not hard to prove that if a predicate modal logic is complete then it is characterised by the class of all frames for the propositional logic on which it is based. Nor is it hard to prove that if a propositional modal logic is incomplete then so is the predicate logic based on it. But the interesting question is whether a complete propositional modal logic can have an incomplete extension. In 1967 Kripke announced the incompleteness of a predicate extension of S4 + $LMp \supset MLp$. It is now known that such incompleteness is common. Montagna in 1984 proved that the predicate extension of KW is one such, since the wff

$$\exists x M\phi x \,\&\, \forall x \exists y L(\phi x \supset M\phi y)$$

forces a transitive and irreflexive frame to have an infinite chain, yet its negation is not a theorem. It is perhaps not surprising that a non-compact logic has an incomplete predicate extension since a single wff of the lower predicate calculus (LPC) can often take the place of an infinite set of propositional wff. What is more surprising is that complete systems whose class of frames are definable by a single first-order sentence can have incomplete extensions. What is even more surprising is that it can make a difference whether these systems contain the *Barcan Formula*, which can be stated by the schema

$$\mathrm{BF} \qquad \forall x L\alpha \supset L\forall x\alpha.$$

The purpose of the present talk is to present several such systems. In the first group it is the systems *with* the Barcan Formula which are incomplete, while those without it are complete. In the second group it is those *without* the Barcan Formula which are incomplete, while those with the Barcan Formula are complete Yet the class of frames is definable in each case by a single first-order sentence. A point worth noting is that, unlike incomplete propositional logics, which often look as if they have been 'cooked up' simply to provide examples of incomplete logics, these incomplete predicate logics are based on propositional logics which have a history going back to the late fifties or early sixties.

The results I shall be presenting are not new, although the importance of the Barcan Formula has not always been appreciated. However the proofs already in the literature use complicated mathematical techniques, and one of the purposes of this paper is to give alternative proofs using only methods standard in modal logic.

► JOHN N. CROSSLEY, *Natural deduction and linear logic*
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We introduce a natural deduction system for linear logic equivalent to Girard's. The motivation is to be able to prove a strong normalization theorem. The proof will be outlined but there will also be a discussion of what is required of a natural deduction system and why the notion of substitution of proofs for proofs is so important—and difficult.

► RODERIC A. GIRLE, *The dynamics of belief revision.*
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Belief revision is the process of changing a set of beliefs by either adding beliefs, giving up beliefs, or doing both. In classical AGM belief revision theory the belief sets are logically consistent and closed under logical consequence The addition of beliefs consistent with a belief set gives the expansion of a belief set, and the giving up of beliefs gives the contraction of a belief set. The addition of beliefs which are inconsistent with the belief set require the giving up of those beliefs inconsistent with the new beliefs, a contraction, and the expansion of the set. Such a process is called revision. If the set of beliefs is seen as a state description, the changes can be seen as state changes Dynamic Logic provides a logic for reasoning about such processes of change. In the standard semantics for Dynamic Logic, the states are seen as possible worlds and the changes from state to state are seen as accessibility relations between worlds. In this paper we explore the extension of Dynamic Logic by adding to it a set of belief change processes to match the AGM processes of expansion, contraction and revision. We show that with the appropriate definitions of the three processes of belief revision we get a system equivalent to the AGM system.

► ROBERT GOLDBLATT, *Bisimulation and modal equivalence.*
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This talk will discuss the Hennessy-Milner modal logic of processes (finitary version) in relation to notions that have become standard in modal model theory (filtrations, bounded morphisms, modal saturation, preservation theorems)

A *labelled transition system* $(\mathcal{P}, \{ \xrightarrow{\alpha} \cdot \alpha \in \mathcal{A} \})$ comprises a collection of binary transition relations $\xrightarrow{\alpha}$ on a set $\mathcal{P}$ of *processes* (or *states*), indexed by a set $\mathcal{A}$ of *action labels* α. The associated modal language is the variable-free set of sentences A given by the abstract syntax

$$\mathrm{A} ::= \bot \mid \neg A \mid A_1 \wedge A_2 \mid [\alpha]A.$$

The satisfaction relation $P \models A$ between processes and states is defined in the usual way. Two states are *logically equivalent* if they satisfy the same modal sentences. A system has the *Hennessy-Milner Property* if this logical equivalence relation is a bisimulation (and hence is the maximal bisimulation on $\mathcal{P}$). Hennessy and Milner [1] showed that this always holds under the *image-finiteness* assumption that $\{ Q . P \xrightarrow{\alpha} Q \}$ is finite for all α and P

Associated with a given transition system are new definable transitions $\xRightarrow{\underline{\alpha}}$ having $P \xRightarrow{\underline{\alpha}} Q$

if and only if $P \models [\alpha]A$ implies $Q \models A$. The system is $\mathcal{H}$-closed if $P \overset{\alpha}{\Rightarrow} Q$ implies $P \xrightarrow{\alpha} R$ for some process R logically equivalent to Q.

We prove

- Every image-finite system is $\mathcal{H}$-closed (converse false).
- Every $\mathcal{H}$-closed system has the Hennessy-Milner Property (converse false).
- Every $\mathcal{H}$-closed system has exactly one filtration (quotient).
- A system is $\mathcal{H}$-closed if and only if
  (1) it has the Hennessy-Milner Property, and
  (2) it has exactly one filtration
- For any system S, the following are equivalent:
  (1) S satisfies the Hennessy-Milner Property;
  (2) logical equivalence between S and its least filtration is a bisimulation;
  (3) the quotient map from S to its least filtration is a bounded morphism.

[1] MATTHEW HENNESSY and ROBIN MILNER, *Algebraic laws for nondeterminism and concurrency*, ***Journal of the Association for Computing Machinery***, vol. 32 (1985), pp. 137–161.

[2] ROBERT GOLDBLATT, *Saturation and the Hennessy-Milner property*, in ***Modal logic and process algebra***, (Alban Ponse, Maarten de Rijke and Yde Venema, editors), CSLI Lecture Notes No. 53, CSLI Publications, Stanford (1995), pp. 107–129

► J. ROGER HINDLEY, *How many proofs can a formula have?*

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In the propositional logic of implication, formulae provable by the intuitionistic natural deduction rules correspond exactly to types in simple type theory that are inhabited by lambda-terms. And proofs and lambda-terms also correspond. Thus the well known decision-procedures for intuitionistic logic translate into lambda-calculus as algorithms to tell whether the set of inhabitants of a type is empty. This talk will describe an extension of these algorithms that will tell exactly how many irreducible members this set has. In particular, the algorithm will tell in finite time whether the number of such members is infinite. The algorithm is due to C. Ben-Yelles.

► DAVID LEWIS, *Humean supervenience debugged.*

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The thesis of 'Humean Supervenience' says that the whole truth about a world like ours supervenes on the spatiotemporal distribution of local qualities—for short, on history. Elsewhere I've argued that there was one 'big bad bug' for Humean Supervenience: chance, in other words single-case objective probability. But now, thanks to a suggestion of Michael Thau, I think I know how to get around the problem.

The problem: Suppose present chances supervene on the whole of history, future as well as present and past. (This would be so if present chances were given by actual frequencies throughout all time; but that's only one especially simple case) Suppose two different futures each have non-zero chance of coming about; and suppose these two different futures would complete two different chance-making patterns, which in turn would make different present chances. Then there is some chance that the present chances would be other than they actually are. Then we get inconsistent results if we apply the 'Principal Principle', which tells us what our subjective probabilities for future events should be, conditional on hypotheses about present chances.

The old part of the solution: We know that the PP doesn't apply if we conditionalize on hypotheses that give us direct information about future history. Call this *inadmissible* information. If Humean Supervenience is true, then information about present chances is, in part, information about chance-making patterns in future history. So the PP *never* applies. That solves the problem; but it seems to clobber the PP altogether, leaving us no way at all to relate chance to degrees of belief.

The rest of the solution, due to Thau: A hypothesis about present Humean-Supervenient chances is never perfectly admissible, but it may come close enough. Inadmissibility admits of degree For some uses of the PP, namely, the very ones we make in practice, the error introduced by not-quite-perfectly admissible hypotheses about chance can be utterly negligible.

It's an old story. Maybe nothing could *perfectly* deserve the name 'sensation' unless it were infallibly introspectible; or the name 'simultaneity' unless it were a frame-independent equivalence relation; or the name 'value' unless it were 'objectively prescriptive'; or the name 'chance' unless information about its present distribution were perfectly admissible. In each case, an imperfect candidate may deserve the name quite well enough.

▶ ED MARES, *A star-free semantics for relevant logic.*
Department of Philosophy, Victoria University, Wellington, New Zealand.

In this paper, I develop an intuitive semantics for R and some close neighbours which does not use the Routley star operator. This semantics is based on the Routley-Meyer semantics for negation-free R It uses a set of worlds and a ternary relation. Moreover, it distinguishes between possible and impossible worlds and a false constant 'f' is said to hold at all and only impossible worlds. Negation is defined implicationally, viz $\sim A =_{\text{def}} A \rightarrow f$ It is shown that by placing constraints on the ternary relation, a semantics can be constructed over which R is complete. In addition, I explore the semantics for some intuitionistic and intermediate relevant logics.

▶ ROBERT K. MEYER AND E. P. MARTIN, *S (for syllogism).*
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We discuss Logic Past, Logic Present, and Logic Yet to Come. Logic Past is Aristotelian logic; Logic Present is modern formal logic; Logic Yet to Come is, we claim, the system S, based on the prefixing and suffixing axioms. (These are B and B′, for combinatory logic fans.) Closely related is P-W (T-W, in some circles), which adds the I identity axiom $A \rightarrow A$. Martin in his Ph.D. thesis (ANU, 1978) showed that I is *totally* independent of the B and B′principles—that is, *no* instances of $A \rightarrow A$ are provable in S. We call such a formula *universally invalid*. By contrast, a formula is *potentially valid* if *some* instance of it is a theorem. If *every* formula of a system T is potentially valid, the system is *lax*. Most systems of implicational (or even positive) logic are in fact lax; classically or intuitionistically (or even in relevant logics like R and E), simply substituting $A \rightarrow A$ for all sentential variables p in an arbitrary positive formula produces a theorem. But, by Martin's result, $p \rightarrow p$ is universally invalid in S; as a corollary, we can find universally invalid formulas in P-W as well; for example, $p \rightarrow ((p \rightarrow q) \rightarrow q)$ is such a formula. In conclusion, S honours syllogistic principles, but nonetheless begs no questions; it is a *progressive* logic, validating only arguments that *go* somewhere.

▶ CHRIS NOWAK, *Logics of combinatorial possible worlds*
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In this paper we will try to discuss logics of combinatorial possible worlds. These worlds are built up from states of affairs associating particulars (individuals) with universals (properties). We will first review results obtained in applying combinatorial worlds to an analysis of necessity and possibility; these include such concepts as concept lattice, concept entailment, accessibility and similarity between combinatorial possible worlds, and systems of similarity spheres. Similarity spheres appear not only in Lewis' counterfactual reasoning, but also in the area of belief revision. Combinatorial possible worlds seem to form an interesting framework for knowledge representation issues, including ontological ones. The first step is to develop logics of combinatorial possible worlds, but many issues relating to belief revision, nonmonotonic reasoning, and knowledge representation in general might be clarified as well.

▶ GRAHAM PRIEST, *On an observation of Hilbert and Bennays*
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The paper is a discussion of a result in Hilbert and Bernays in their *Grundlagen der Mathematik*. Their interpretation of the result is similar to the standard interpretation of Tarski's Theorem. This and other interpretations of the result are discussed, and shown to be inadequate. Instead, it is argued, the result refutes certain versions of Meinongianism. In addition, it poses new problems for classical logic that are solved by dialetheism.

▶ KRISTER SEGERBERG, *Belief revision from the point of view of doxastic logic.*
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In 1985 Alchourrón, Gardenfors and Mackinson presented their now classic theory (AGM) of theory change (belief revision). In 1988 Adam Grove, generalizing David Lewis' theory of counterfactuals, presented a model theory suitable for the AGM theory Although AGM and Grove mentioned object languages, neither used them. But recently, Maarten de Rijke has shown how object languages can be brought into the picture. In the present paper we take de Rijke's idea further, addressing the question whether there is a particular doxastic or epistemic logic implicit in the AGM.

▶ PAVEL TICHY, *Cracking the natural language code.*
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Current grammatical theories either do not aspire to generate interpretations of well-formed English strings along with the strings themselves, or else attempt to do it indirectly by generating translations of those strings into some other language. In either case the coding system peculiar to the English language remains unexamined. To decipher the code of English, or any other natural language, one has to construct a generator of string-meaning pairs such that the second component of a pair is the actual denoting complex expressed by the first component The paper reports on progress made towards developing a grammar of this sort.

Abstracts presented by title

▶ B J. COPELAND, *Prior's role in the development of temporal logic and the possible worlds semantics.*
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Logics that form the basis of a number of currently much-studied software technologies were invented by the New Zealand philosopher A. N Prior. For the □ (necessity) and ◇ (possibility) operators of standard modal logic he substituted a variety of other operators which he called quasi modal operators. An example is the K operator (it is known that) of epistemic logic. Prior's quasi modal operators migrated from philosophy to computer science via various mathematical logicians who were associated with departments of computer science. To date the tense operators have proved to be the most useful of Prior's inventions. His tense logic has found numerous applications in computer science, for example logical deduction involving time-dependent data, natural language processing, knowledge representation, planning, database management, specification and verification of concurrent programs, and VLSI. Prior's discovery of the so-called possible worlds semantics, now regarded as a representational system of great generality, was a direct result of his work on tense logic. (Priority is commonly assigned to Kripke but Prior, in collaboration with the Irish logician Carew Meredith, arrived at the essentials of the semantics several years ahead of Kripke.) It is noteworthy that two of the major forces in the genesis of these various software technologies were a love of ancient and medieval logic and a philosopher's concern to make conceptual room for freedom of the human will.

► STAN J. SURMA, *On closure operators one-to-one associated with fixed object languages.*
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In its spirit, this paper is an exercise in algebraic logic and it aims at providing a unifying conceptual and terminological schema for generation of closure operators one-to-one associated with each reduct of an arbitrary but fixed object language The main result of Part One is the following theorem.

THEOREM 1 (Normal Form Theorem). *If* Cn *is a regular* (¬)*-closure operator over the set* S *of all sentences of an object language with negation* ¬ *and an extra* m*-ary connective* #*, then the conditions below are equivalent.*

(i) Cn *is* (#)*-normal,*
[*i e.,* $\mathrm{Cn}(X, \#A_1A_2\dots A_m) = \bigcap\{\mathrm{Cn}(X, A_1[f_j], A_2[f_j], \dots, A_m[f_j]) : j \leq 2^m$ and $f_j(\#A_1A_2\dots A_m) = 1\}$, *where* f_j *is a valuation function (a homomorphism) from* $(S, \#)$ *to* $(\{0,1\}, \underline{\#})$ *such that* $\underline{\#}$ *is an operation on* $\{0,1\}$ f_j*-correlated with* #*, and where* $A_i[f_j] = A_i$ *if and only if* $f_j(A_i) = 1$*; and* $A_i[f_j] = \neg A_i$ *if and only if* $f_j(A_i) = 0$ *for any* $A_1, A_2, \dots, A_m \in S$ *and any* $X \subseteq S$.]

(ii) Cn *is* (#)*-stratified*
[*i.e., the fact that* $\#A_1A_2\dots A_m \in X$ *is equivalent to the fact that* $\#^*(A_1 \in X)(A_2 \in X)\dots(A_m \in X)$*, where* $\#^*$ *is the meta language correlate of* #*, for any* $A_1, A_2, \dots, A_m \in S$ *and any* Cn*-maximal* $X \subseteq S$]

In Part Two a uniform way of weakening of the above concept of (#)-stratification is proposed so as to make a classical closure-theoretic set up a non-classical one. An intuitionistic, a Johansson-minimal and a modal closure-theoretic framework are given as an illustration.

The rest of the terminology used in this abstract is as follows. An operator Cn is a closure operator if and only if it is inclusive (i.e., $X \subseteq \mathrm{Cn}(X)$ for any $X \subseteq S$); monotonic (i.e., the fact that $X \subseteq Y$ implies that $\mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y)$ for any X, $Y \subseteq S$); and idempotent (i.e., $\mathrm{Cn}(\mathrm{Cn}(X)) \subseteq \mathrm{Cn}(X)$ for any $X \subseteq S$). For any $A \in S$ and $X \subseteq S$, X is (Cn, A)-maximal if and only if $A \notin \mathrm{Cn}(X)$ and $A \in \mathrm{Cn}(X, B)$ for any $B \notin X$ X is Cn-maximal if and only if X is (Cn, A)-maximal for some A. An operator Cn is regular if and only if, for any $A \in S$ and $X \subseteq S$, the fact that $A \notin \mathrm{Cn}(X)$ implies that there is $Y \in 2_X$ such that $A \notin \mathrm{Cn}(Y)$ and $A \in \mathrm{Cn}(Y, B)$ for any $B \notin Y$. The expression $Y \in 2_X$ means that $X \subseteq Y$ Cn is a (¬)-operator if and only if Cn is (¬)-compact (i.e, $\mathrm{Cn}(A, \neg A) = S$ for any $A \in S$) and Cn has the (¬)-cut property (i.e., $\mathrm{Cn}(X, A) \cap \mathrm{Cn}(X)$ for any $A \in S$ and any $X \subseteq S$).

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1994–1995 WINTER MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

San Francisco, California, January 6–7, 1995

The Winter Meeting of the Association was held in San Francisco, California, on January 6–7, 1995, in conjunction with the American Mathematical Society. The program committee consisted of Andreas Blass (chair), Julia Knight, Robert Soare, and W Hugh Woodin. The program included seven invited talks, twenty-five contributed talks, and six papers presented by title.

The following invited talks were presented at the meeting

John Steel (UCLA), *Recent results in core model theory*

Alexandra Shlapentokh (East Carolina University), *Algebraic and Turing inseparability of rings*

Valentina Harizanov (George Washington University), *Recursive model theory. Examining computability in the theory of theories.*

Tomek Bartoszyński (Boise State University), *Cardinal invariants and Borel sets.*

Alexander Kechris (Caltech), *A Glimm-Effros dichotomy for analytic equivalence relations and Ulm-type classifications.*

Ted Slaman (University of Chicago), *Recursion theory and second order arithmetic.*

Steffen Lempp (University of Wisconsin), *Coding and undecidability in computability theory.*

Abstracts of contributed talks given (in person or by title) by members of the Association for Symbolic Logic follow.

For the Program Committee
ANDREAS BLASS, CHAIR

Abstracts of contributed talks

► A. NAIT ABDALLAH, *Variational principles for reasoning about actions in the context of partial information.*
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The characteristic surface [1, 2] of a given syntactic path Φ representing an evolving physical system Π is defined by $\Sigma(\Phi) = \prod_{n \in I} (\Omega_{\min}(\models)(\Phi_n) \equiv_{\star}^{G(\cup_n \Phi_n)})$. The least action principle postulate that *"Of all the possible paths in $\Sigma(\Phi)$ that it might take to get from one state to another, the point representing a (physical) logic system takes a path (m_n) which is extremum, i.e , the smoothest possible on the characteristic surface representing the system."* This variational principle provides a solution to the frame problem in its temporal projection form.

Temporal explanation problems require a combination of the least action principle with

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abduction. An example is given by the Gun problem. *After performing an action, things normally remain the same. After a man is shot with a loaded gun, he dies Fred was initially alive. After shooting Fred, Fred was dead Was the gun originally loaded?* The Gun problem is not readily solved by the sole application of the least action principle. It requires a formalization of the following line of argument: *After shooting a person with a loaded gun, that person dies After Fred was shot with a gun, Fred died Therefore the gun must have been loaded.* To formalize this, one introduces the following inference rule, which expresses an *abduction principle*:

$$\frac{a' \ , \ C[b] \ , \ a \wedge a' \rightarrow b}{C[\star(a, a)]}$$

and uses it as part of a new variational principle. Let P be an ionic logic program specifying some physical system Π, and $H \subseteq P$ be the set of hard facts to be explained. The abductive variation δP of P with respect to H is the set of ions generated from H by means of the abduction principle. Program $P' = (P \backslash H) \cup \delta P$ is the "variational" logic program. Define syntactic path $D'_n = \{\varphi \in \mathit{lfp}(P') : \text{state-index}(\varphi) \leq n\}$, where $\mathit{lfp}(P')$ is the least fixpoint of P'. Let $\Sigma(D')$ be the characteristic surface of (D'_n) The abductive variational principle postulates that *"Among all possible paths that it might take to get from one state to another, the point representing system* Π *takes some extremum path* (m_n) *in* $\Sigma(D')$ *such that* $H \subseteq \Omega(\models_{\mathit{soft}})(\lim_n m_n)$*."* In other words the limit of the path taken satisfies in a soft sense every hard fact $\in H$. In summary: one introduces a variation, applies the least action principle, and then removes the variation; the final result obtained is the expected evolution path.

For the Gun problem, the characteristic surface of the "nearby" syntactic path D' has two minimal semantic paths, one of which converges towards a model where Fred is dead after the shooting (this is the "dangling semantic path" removal condition.) When the evolution of the system follows that path, the gun was originally loaded. In the other minimal semantic path, Fred is still alive after the shooting, and the gun was originally unloaded. Another example is the Murder Mystery problem: *After performing an action, things normally remain the same. After a man is shot with a loaded gun, he dies. Fred was initially alive. After shooting Fred and waiting, Fred was dead. When did Fred die? Was the gun originally loaded?* The characteristic surface of the "nearby" syntactic path (D'_n) here has two extremum semantic paths, one of which converges toward a model where Fred was dead after the waiting period (this is the "dangling semantic path" removal condition), where he died during the shooting, and where the gun was originally loaded. In the other minimal semantic path, Fred is still alive after the shooting, and the gun was originally unloaded.

[1] M. A Nait Abdallah, *The dynamics of logic systems and its applications*, ***Logic Colloquium 94***, 1994.

[2] ———, ***The logic of partial information***, Springer Verlag, 1994.

► JEREMY AVIGAD, *Two effective conservation results involving subsystems of second-order arithmetic.*

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First we give an effective proof of the Harrington-Simpson-Brown result that $\mathit{WKL}+_0$ is conservative over RCA_0 for Π^1_1 sentences, and show that the increase in the length of proofs is polynomially bounded. In doing so we give general methods by which iterated-forcing arguments involving subsystems of second-order arithmetic can be formalized in the base theory. We also give an effective proof that ATR_0 is conservative over $\widehat{\mathit{ID}}_{<\omega}$ for arithmetic sentences, by showing that ATR_0's main axiom is equivalent ro a second-order version of $\widehat{\mathit{ID}}_{<\omega}$'s inductive definition scheme. In this case, however, there is a significant increase in the lengths of proofs, since ATR_0 has short proofs of the consistency of superexponentially-many inductive definitions.

▶ MATTHIAS BAAZ AND RICHARD ZACH, *Generalizing theorems in real closed fields.*

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Krajíček recently [2, Problem 23] asked the question: Is there a generalization result in the theory of real closed fields of the form· If $A(1 + \cdots + 1)$($n$occurrences of 1) is provable in length k for all $n \in \omega$, then $(\forall x)A(x)$ is provable? Questions and results about generalizations of proofs in mathematical theories depend substantially on the presentation of the theory. We investigate four distinct formalizations with respect to their generalization behavior. We show that there is a positive answer to Krajíček's question for (1) the usual axiom system *RCF* of Artin-Schreier with Gentzen's sequent calculus **LK** as underlying logical calculus, (2) *RCF* with the variant LK_B of **LK** allowing introduction of several quantifiers of the same kind in one step, and (3) LK_B and the first-order schemata corresponding to Dedekind cuts and the supremum principle. A negative answer is given for (4) any system containing the schema of extensionality.

The results (1) and (2) build on earlier work by Parikh and Krajíček-Pudlák, showing that the logical complexity of the formulas occurring in a proof in **LK** can be bounded in the length of the proof. Krajíček and Pudlák [3] use *unification* to prove this. For the axiom system investigated in (1) and (2), it follows that everything that can be proved in k steps in **LK** can already be proved in a finite subtheory of *RCF*. (Note that the logical complexity of the axiom asserting the existence of a root for all polynomials of degree $2n + 1$ grows as n gets larger.) Finite theories admit *finite term bases*, i.e.. For every $A(x)$ and k, there are $t_1, \dots, t_m$ s.t. if $A(t)$ is provable in k steps, then t is a substitution instance of t_i for some i. From this the generalization result follows. A similar argument establishes (2), but a more careful analysis is needed

Result (3) follows from an analysis of the main theorem of [1], where a similar generalization result is proved for arithmetic with the least number principle for $\exists_1$-formulas.

Using a variant of Yukami's Trick [4], we give conditions under which an axiomatization of the theory of real closed fields containing the schema of extensionality does not admit generalization of theorems.

[1] M. BAAZ and P. PUDL'AK, *Kreisel's conjecture for $L\exists_1$*, ***Arithmetic, proof theory and computational complexity*** (P. Clote and J. Krajíček, editors), Oxford University Press, Oxford, 1993, pp. 29–59

[2] P CLOTE and J. KRAJÍČEK, *Open problems*, ***Arithmetic, proof theory and computational complexity***, Oxford University Press, Oxford, 1993, pp. 1–19.

[3] J. KRAJÍČEK and P. PUDLÁK, *The number of proof lines and the size of proofs in first order logic*, ***Arch. Math. Logic***, vol 27 (1988), pp. 69–84.

[4] T. YUKAMI, *Some results on speed-up*, ***Ann. Japan Assoc. Philos. Sci.***, vol. 6 (1984), pp. 195–205

▶ ROY BENTON, *A simple incomplete modal extension of KT which is the logical union of two logics with the finite model property.*

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I present an incomplete modal extension of KT ($= K\{Lp \rightarrow p\}$) which has two points of interest: (1) it is simpler than incomplete KT-extensions which have appeared before (2) it can be realized as the union of two logics which have the finite model property (F.M.P.) and hence are themselves complete.

The resulting incomplete logic, with its three simple axioms, is simpler than van Benthem's classic examples ("Two incomplete logics", Theoria 44, 25–37). Van Benthem used two measures of simplicity of a logic, both of them involving the complexity of its characteristic axioms (i.e., axioms beyond those needed for the minimal normal modal logic K)—namely, modal depth and number of sentence letters One of his logics extends KT, and its four axioms have maximum modal depth 2 and use only two sentence letters. My incomplete logic actually discards one of these four axioms—the only one with two sentence letters—so that the result uses only one sentence letter, and of course maintains modal depth at 2. Since van Benthem's other "simple" logic uses only one sentence letter, but requires a modal depth of 3 (and, incidentally, does not extend KT and cannot be easily modified to do so), my logic is strictly simpler on both counts. (More recently, other simple incomplete logics have appeared.)

I have previously showed that the class of normal logics with the F.M.P. do not form a lattice, by presenting another such failure of logical union. However, the resulting incomplete logic, while quite simple, did not extend KT.

My two "component" logics are $K\{MLp \rightarrow LMp\}$, previously shown to have the F.M.P.; and $K\{Lp \rightarrow p, Mp \,\&\, L(p \rightarrow Lp) \rightarrow p\}$, which I show to have F.M.P. using a technique similar to normal "forms."

▶ WILLIAM C. CALHOUN, *An untangled proof of the recursion theorem.*

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Kleene's Recursion Theorem has central importance in recursion theory. The usual proof of the theorem is very short, but rather cryptic. I will discuss a slightly longer "untangled" proof of the Recursion Theorem. I find the "untangled" proof more intuitive than the usual proof. It consists of three lemmas:

LEMMA 1. *If $\{\phi_i\}$ is an acceptable numbering of the partial recursive functions and the Recursion Theorem holds for $\{\phi_i\}$, then the Recursion Theorem holds for the standard numbering.*

LEMMA 2. *Define $\phi_i(x) = \phi_i(\langle i, x\rangle)$ Then $\{\phi_i\}$ is an acceptable numbering.*

LEMMA 3. *The Recursion Theorem holds for $\{\phi_i\}$.*

I will also discuss generalized versions of the Recursion Theorem.

▶ LUIZ CARLOS, P. D. PEREIRA, EDWARD HERMANN HAUSLER, MARíA CLAUDIA MERÉ, AND MARCELO F. FRIAS, *Bounded infinite parallel logic.*

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It is quite well known that the elimination of contraction and weakening gives rise to a splitting of the usual logical conjunction and disjunction into multiplicative and additive versions. It would be quite natural to think of a corresponding splitting with respect to quantifiers

The purpose of this paper is twofold. (1) to present an **infinitary propositional system for classical linear logic** with infinite tensor and infinite par which satisfies the Normalization

Theorem; and (2) to show how to use this infinitary system in order to define multiplicative universal and existential quantifiers.

Infinitary propositional systems for classical and intuitionistic logic have been studied in several works [2, 3]. The natural infinitary generalization of the tensor operator does not introduce any significant additional difficulty. On the other hand the natural infinitary generalization of the par operator does not seem to allow for a natural infinitary generalization of the par operator does not seem to allow for a natural concept of $\square_\infty$-reduction. In a certain sense (as it was remarked by Blass [1]) our concept of sequent and our concept of derivation as a well-founded tree are not adequate for the definition of $\square_\infty$-reduction. In order to solve this well-foundness problem we introduce the concept of *bounded parallel inference*

Instead of applying one rule at a time, we now have a unique rule "R-Sim" whose effect is the simultaneous application of a (possibly infinite) sequence of rules. We can show that· (1) If there is a finite bound for the length of the derivations of the minor premises of $\square_\infty$-elimination and (2) If there is a finite bound for the degree of the maximum formulas occurring in a (possibly infinite) list of formulas, then we can both define a $\square_\infty$-reduction and use it in order to prove normalization.

When we think of multiplicative quantifiers as infinite tensors and pars we see that the restrictions imposed on derivations in our system of parallel inference are perfectly adequate for the treatment of multiplicative quantification.

[1] A. BLASS, ***Game semantics for affine logic***, Marseille, 1990.

[2] P. MARTIN-LÖF, *Infinite terms and a system of natural deduction*, ***Composito Mathematica***, vol. 2 (1972), fasc. 1, pp. 92–103.

[3] L. C. P. D. PEREIRA, *On the estimation of normal derivations*, PhD thesis (Filosofiska institutionen), Stockholm, 1982.

▶ GREGORY CHERLIN AND NIANDONG SHI, *Graphs omitting a finite set of cycles*.

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We say that a class of graphs has a universal element if every graph in this class can be embedded in the universal one as a subgraph. We prove that the class of graphs omitting a finite set $\mathfrak{C}$ of cycles has a universal element if and only if the set $\mathfrak{C}$ consists precisely of all the odd cycles of length less than some specified bound. The sufficiency of this condition was proved by P. Komjath, A Mekler and J. Pach.

▶ ALI ENAYAT, *A new transfer theorem in the constructible universe*.

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Recall that given two cardinals κ and λ, the transfer relation $\kappa \longrightarrow \lambda$ asserts that any theory (with a distinguished linear order) which has a κ-like model must have a λ-like model.

THEOREM A. *If κ and λ are inaccessible cardinals and λ is* ***not*** *Mahlo, then $\kappa \longrightarrow \lambda$.*

Let μ be the function defined on inaccessible cardinals as follows:

$$\mu(\kappa) = \begin{cases} \omega, \text{ if } \kappa \text{ is } n\text{-Mahlo for each } n, \\ \text{The largest } n \in \omega \text{ such that } \kappa \text{ is } n\text{-Mahlo, } \textit{otherwise}. \end{cases}$$

THEOREM B ($V = L$). *Assume κ and λ are inaccessible cardinals The following are equivalent·*

(i) $\kappa \longrightarrow \lambda$.
(ii) $\mu(\kappa) \geq \mu(\lambda)$.

▶ DON THOMAS FALLIS, *A defense of a probabilistic method of establishing mathematical truths.*

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One of the primary goals of mathematicians is to establish new mathematical truths. Toward this end, mathematicians are almost invariably theorem provers. However, there are several methods other than writing down a proof which seem to achieve this epistemic goal of establishing mathematical truths. For instance in [1], Michael Rabin described a probabilistic test for primality which establishes to an arbitrarily high degree of certainty that a number is prime. Nevertheless, the vast majority of mathematicians are unwilling to employ such probabilistic methods in their attempts to establish new mathematical truths. I will argue that mathematicians do not have good grounds for this rejection of probabilistic methods.

[1] M. O. RABIN, *Probabilistic algorithm for testing primality*, ***Journal of Number Theory***, vol. 12 (1980), pp. 128–138.

▶ MARCELO F. FRIAS, ARMANDO M. HAEBERER, AND PAULO A. S. VELOSO, *Metalogical properties of fork algebras.*

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In this abstract we present some results on the boundary between algebra and logic, which show the suitability of abstract fork algebras as a framework for doing logic in an algebraic setting.

DEFINITION (proper fork algebras). Consider a set $\mathcal{U}$ with an injective operation $\star : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ (constructing pairs $y \star z$). This induces an operation on subsets of $\mathcal{U} \times \mathcal{U}$:

$$r \lhd s = \{\langle x, y \star z\rangle \in \mathcal{U} \times \mathcal{U} : \langle x, y\rangle \in r \wedge \langle x, z\rangle \in s \wedge \langle y, z\rangle \in V\}.$$

A *proper fork algebra* (PFA) is a structure

$$\mathcal{F} = \langle F, \cup, \cap, |, \lhd, \sim, \imath, \emptyset, \Delta, V\rangle$$

such that F is closed under $\lhd$ and the reduct $\langle F, \cup, \cap, |, \sim, \imath, \emptyset, \Delta, V\rangle$ is a complete and atomic proper relation algebra.

DEFINITION (abstract fork algebras). An *abstract fork algebra* (AFA) is a structure

$$\mathcal{A} = \langle A, +, \bullet, ;, \nabla, -, \breve{}, 0, 1, \infty\rangle$$

such that $\langle A, +, \bullet, ,, -, \breve{}, 0, 1, \infty\rangle$ is a complete and atomic relation algebra [1], and the following axioms hold·

(1) $r \nabla s = (r;\pi) \bullet (s;p)$,
(2) $\pi \nabla \rho \preceq 1$,
(3) $(r \nabla s), (p \nabla q)\breve{} = (r; p\breve{}) \bullet (s; q\breve{})$,
(4) $0 \neq t \preceq \infty \otimes \infty \to (\exists v)(\exists w)(0 \neq v \otimes w \preceq t)$

where *projections* and *cross* product are defined by

$$\pi = (1 \nabla \infty)\breve{}, \rho = (\infty \nabla 1)\breve{} \text{ and } r \otimes s = (\pi;\imath) \nabla (\rho;s).$$

These projections satisfy the properties of quasi-projections [1], but are rather special In fact, every atom of an atomic AFA is functional.

NOTATION. We will denote by $\mathcal{F}$ the set of sentences of classical first-order logic, and by $\mathcal{T}$ the set of fork-terms In order to state the main results, we will use the following result which is proved in [3, 2].

THEOREM. *There exists a recursive function* $I : \mathcal{F} \rightarrow \mathcal{T}$ *such that* $\models \phi$ *iff* $\vdash_{=,\mathrm{AFA}^-} I(\phi) \neq 0$ (*Where* $\vdash_{=,\mathrm{AFA}^-}$ *means entailment using only equational reasoning, and the set of equational* AFA *axioms* (1), (2) *and* (3).)

THEOREM 1 (Strong Completeness). *Let* Γ *be a set of first-order sentences, and let us denote by* $I(\Gamma) \neq 0$ *the set* $\{I(\gamma) \neq 0 / \gamma \in \Gamma\}$. *Then*

$$I(\Gamma) \neq 0 \models_{\mathrm{PFA}} I(\psi) \neq 0 \text{ iff } I(\Gamma) \neq 0 \vdash_{=,\mathrm{AFA}^-} I(\psi) \neq 0$$

The previous theorems allow the translation of the machinery of classical first-order logic into the framework of fork algebras, in such a way that proofs can be performed equationally. Thus fork algebras algebraize first-order logic, much as Boolean algebras do for propositional logic Moreover, we have the analogous of the theorems of *compactness* and *Lowenheim-Skolem*

DEFINITION. By a *(proper) fork model* of a set Σ of terms we mean a (proper) fork algebra $\mathcal{A}$ such that $\mathcal{A} \models t \neq 0$ for every $t \in \Sigma$

Consider a (possibly) infinite set Γ of first-order sentences

THEOREM 3 (Compactness). *If every finite subset of* $I(\Gamma)$ *has a proper fork model, then so does* Γ.

THEOREM 4 (Löwenheim-Skolem). *Suppose that the set* $I(\Gamma) \neq 0$ *has a* fork model *whose underlying set* $\mathcal{U}$ *is infinite Then for each infinite cardinality,* α, $I(\Gamma)$ *has a proper fork model with underlying set of cardinality* α.

Notice that in order to prove the theorems it is not enough to use their versions for first-order logic since in that case the equationality in proofs is lost.

[1] A TARSKI and S. A. GIVANT, *A formalization of set theory without variables*, ***American Mathematical Society Colloquium Publications***, vol. 41 (1987).

[2] P. A S. VELSOS and A. M. HAEBERER, *A finitary relational algebra for classical first-order logic*, ***Bulletin of the Section of Logic, Polish Academy of Sciences***, vol. 20 (1991), no. 2, pp. 52–62.

[3] P. A. S. VELOSO, A. M. HAEBERER, and M. F. FRIAS, *Fork algebras as algebras of logic*, ***Proceedings of the logic colloquium '94***, July 1994, p. 127, to appear also in: ***Journal of Symbolic Logic***.

► E. I. GORDON AND A. A BUSHUEV, *On some extensions of IST and the predicates of relative standardness*

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We consider one finite axiomatics for nonstandard set theory, which is based on the notion of class. This axiomatics is in the same relation to IST as NBG to ZFC. We add to NBG one unary predicate—St(X)—"class X is standard" and some axioms for this predicate. For example we require that classes which are constructed from standard classes by NBG axioms of classes existence, are standard themselves. The separation and substitution principles are formulated only for standard classes. The transfer, idealization and standardization principles are here axioms but not the schemes of axioms. So we have here for classes a possibility to be not the sets but subclasses of the sets. It is proved that a set does not contain such proper subclass iff its cardinality is standard natural number.

The internal classes are defined in such a way that all sets are internal and the proper classes are internal iff they are sections of standard classes—relations. This definition of internal classes is connected with definitions of the predicates of relative standardness, which were introduced by the first author earlier (Siberian Journal of Mathematics—1989, N2). Using these predicates we prove that standardization principle and right-to-left implication in idealization principle are independent from other axioms of IST.

We consider also some consistent extensions of described theory, in which Alternative Set Theory can be interpreted.

► JOEL DAVID HAMKINS, *Lifting and extending measures through forcing.*

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Measures in V can relate to measures in a forcing extension $V[G]$ in two subtly different but connected ways Namely, a measure μ in V might *extend* to a measure ν in $V[G]$ when $\mu \subset \nu$. Alternatively, μ might *lift* to ν when the corresponding ultrapower embedding j_μ lifts to j_ν I will show how these similar notions subtly differ. In particular, a normal measure can extend to normal measures to which it does not lift, and a normal measure can extend to many measures, none normal, and hence not lift at all. I will also discuss how lifts and extensions respect or disrespect the Rudin-Kiesler and Mitchell orders

► JEFFRY L. HIRST AND STEFFEN LEMPP, *Infinite versions of some NP-complete problems*

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Some connections between NP-complete problems for finite objects and the complexity of their analogues for infinite objects have appeared recently. In particular, the problem of deciding which finite graphs have a Hamilton path is known to be NP-complete [3], while Harel [2] has shown that the problem of deciding which infinite recursive graphs have a Hamilton path is Σ^1_1-complete. Within the framework of reverse mathematics, Harel's construction can be used to prove that the existence of a "deciding function" for Hamiltonicity is equivalent to Π^1_1 comprehension [1].

We have considered some other infinite versions of NP-complete problems, particularly those related to graph theory, in terms of their complexity and proof-theoretic strength. The infinite analogues of NP-complete problems can behave in a variety of unexpected ways, as exemplified by the results below.

THEOREM 1A. *The following are provably equivalent over* $\mathbf{RCA}_0$*:*

(1) $\Pi^1_1 - \mathbf{CA}_0$.

(2) *For any sequence of ordered pairs of graphs,* $\langle G^L_i, G^R_i \rangle_{i\in\mathbf{N}}$, *there is a function* $f : \mathbf{N} \to 2$ *such that* $f(i) = 1$ *if and only if* G^L_i *is isomorphic to a subgraph of* G^R_i

(3) *For any sequence of graphs,* $\langle G_i \rangle_{i\in\mathbf{N}}$, *there is a function* $f : \mathbf{N} \to 2$ *such that* $f(i) = 1$ *if and only if* G_i *contains an infinite clique.*

(4) *For any sequence of graphs,* $\langle G_i \rangle_{i\in\mathbf{N}}$, *there is a function* $f : \mathbf{N} \to 2$ *such that* $f(i) = 1$ *if and only if* G_i *has a co-infinite vertex cover.*

THEOREM 1B. *The index sets for the recursive graphs with the properties of* (2), (3), *or* (4) *above are all* Σ^1_1*-complete.*

By contrast, other NP-complete problems naturally lead to arithmetical problems in the infinite case.

THEOREM 2A. *The following are provably equivalent over* $\mathbf{RCA}_0$*:*

(1) $\mathbf{ACA}_0$
(2) *For any sequence of graphs,* $\langle G_i \rangle_{i \in \mathbf{N}}$*, there is a function* $f : \mathbf{N} \to 2$ *such that* $f(i) = 1$ *if and only if* G_i *can be 3-colored*
(3) *For any sequence of graphs,* $\langle G_i \rangle_{i \in \mathbf{N}}$*, there is a function* $f : \mathbf{N} \to 2$ *such that* $f(i) = 1$ *if and only if each connected component of* G_i *can be colored*

THEOREM 2B. *The index set for the recursive graphs with the property in* (2) *is* Π^0_1*-complete, while the index set for the recursive graphs with the property in* (3) *is* Π^0_3*-complete*

[1] W. GASARCH and J. HIRST, ***Reverse mathematics and recursive graph theory***, (in preparation).

[2] D. HAREL, *Hamiltonian paths in infinite graphs*, ***Israel J. Math.***, vol. 76 (1991), pp. 317–336.

[3] R. M. KARP, *Reducibility among combinatorial problems*, ***Complexity of computer computations*** (R. Miller and J Thatcher, editors), Plenum Press, New York, 1972, pp. 85–103.

▶ GREG HJORTH, Π^1_1 *sets universal for being non-Borel*

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It is independent of ZFC whether there exists a Π^1_1 set in the plane whose sections correspond exactly to the Π^1_1 non-Borel sets in the line. This answers a question raised by Mauldin.

▶ HASEO KI, *Kechris-Woodin rank and Zalcwasser rank.*

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Zalcwasser introduced a rank that measures the uniform convergence of sequences of continuous functions on the unit interval. We apply the Zalcwasser rank to the Fourier series of a continuous function on the unit circle. In it is shown that on C (the set of series of a continuous functions, on the unit circle, with convergent Fourier series), the Zalcwasser rank is a Π^1_1-norm which is unbounded below ω_1, i.e., functions in C are arbitrarily bad in terms of this rank. Kechris and Woodin defined a rank that measures the uniform continuity of the derivative of a differentiable function. We shall refer to this rank as the Kechris-Woodin rank In fact, they have shown that on the set of all differentiable functions, the Kechris-Woodin rank is a Π^1_1-norm which is unbounded below ω_1.

Ajtai and Kechris conjectured that the Kechris-Woodin rank is finer than the Zalcwasser rank, meaning that for any function f, the Zalcwasser rank is less than or equal to the Kechris-Woodin rank. There is a fair amount of evidence supporting this conjecture. For example, the Zalcwasser rank is 1, i.e., the smallest possible number, for all differentiable functions f, whose derivative f', is bounded. On the other hand, on the set of all differentiable functions with bounded derivatives, the Kechris-Woodin rank is unbounded below ω_1. Our main result is an affirmative answer to this conjecture of Atjai and Kechris.

Also Kechris and Louveau asked if Kechris-Woodin rank can be approximated by Zalcwasser rank. Recently, we have shown that it is true.

▶ CLAUDE LAFLAMME, *A few ordinal ultrafilters.*

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We discuss ordinal ultrafilters especially from the point of view of the Rudin-Keisler ordering. In particular, answering a question of J. Baumgartner, we show that consistently there is a downward closed ω^ω ultrafilter, i.e., an ultrafilter all of whose Rudin-Keisler predecessors are also ω^ω ultrafilters. We also show that there cannot be a downward closed $\omega^{\omega+\omega}$ ultrafilter but the question whether every $\omega^{\omega+\omega}$ ultrafilter has a ω^ω ultrafilter below it remains open.

▶ GEOFFREY LAFORTE, *Some phenomenon in the n-r.e. and n-REA degrees.*

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A set $A \subseteq \omega$ is said to be 1-r.e. if it is recursively enumerable, and $n+1$-r.e. if $A = C - B$ where C is r.e. and B is n-r.e. (Since 2-r.e. sets are differences of r.e. sets, we call them d.r.e.) If $W \subseteq \omega$, then A is said to be W-REA if A is recursively enumerable in W and $W \leq_T A$. Analogously to the n-r.e. sets, a set is 1-REA if it is recursively enumerable, and $n+1$-REA if it is W-REA for some n-REA set W. We examine some phenomena relating $n+1$-REA to $n+1$-r.e., n-REA, and n-r.e. sets.

By a result of Soare and Stob, if W is nonrecursive, then some W-REA set fails to be of r.e. degree. We show that this phenomenon fails in the d.r.e. degrees: there is a nonrecursive r.e. set W such that every W-REA set has d.r.e. degree. We conjecture that this result can be extended to $n > 2$ to construct, for example, an r.e. set W such that every W-REA set has 3-r.e. degree and some W-REA set is not of d.r.e. degree. An obstacle here is the lack of any direct method of constructing a 2-REA set with properly (that is, non-d.r.e.) 3-r.e. degree. Indeed, we show that any 3-r.e. set which is simultaneously 2-REA *as a set* must have d.r.e. degree. The existence of some 2-REA set with properly 3-r.e. degree, however, does not follow from some results of Cooper and Yi on isolated degrees.

If S is a class of subsets of ω, a set A is said to be *isolated in the S-degrees* by a set W if $W <_T A$ but there is no $C \in S$ with $W <_T C <_T A$. Cooper and Yi have shown that there is a d.r.e. set that is isolated in the r.e. degrees by an r.e. set below it. Turning to such isolation phenomena, we show that d.r.e. sets isolated in the r.e. degrees by an r.e. set below them occur in any nontrivial interval in the r.e. degrees.

[1] S. B. Cooper, S. Lempp, and P. Watson, *Weak density and cupping in the d.r.e. degrees*, ***Israel J. Math.***, vol 67(1989).

[2] S. B. Cooper and X. Yi, *Isolated d.r.e. degrees*, ***University of Leeds, Dept. of Pure Math., 1993 Preprint Series***, no. 17.

[3] R. I. Soare and M. Stob, *Relative recursive enumerability*, ***Proceeding of the Herbrand Symposium: Logic Colloquium '81*** (J. Stern, editor), North-Holland, 1982

▶ ROBERT LUBARSKY, *V is probably L, intuitionistically.*

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We present a very simple construction, under very mild set-theoretic hypotheses, that, starting from a Kripke model of IZF built upon most any partial order, will put *any* set into L

▶ GRZEGORZ MICHALSKI, *Relatively r.e. vs relatively Σ_1 in models of Peano arithmetic.*

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The usual setting for recursion theory is the standard model for second order arithmetic; i.e., the structure $(\mathbf{N}, \mathcal{P}(\omega))$, where $\mathbf{N}$ is the standard model for first order arithmetic, and $\mathcal{P}(\omega)$ is the family of all subsets of ω—the set of natural numbers. In recent years people (M.J. Groszek, P. Hájek, K. Kontostathis, A. Kučera, M. E. Mytilinaios, P. Pudlák, T. A. Slaman, W. H. Woodin, Y. Yang—see references) have considered recursion theory in other second order structures of the form $(\mathcal{M}, \mathcal{S})$, where $\mathcal{M}$ is a nonstandard model of Peano Arithmetic (*PA*) or some fragment of *PA*, and $\mathcal{S}$ is a family of subsets of $\mathcal{M}$. The purpose of these investigations in general has been to develop a better understanding of the properties actually needed in proving various results from recursion theory.

In the present paper, we consider the following elementary result:

Proposition 1. For any sets X and Y, the following are equivalent

(1) X is r.e. relative to Y
(2) X is definable by a formula which is $\Sigma_1(Y)$; i.e., Σ_1 with a predicate for Y.

In structures of the form $(\mathcal{M}, \mathcal{S})$ where $\mathcal{M}$ is a model of (a sufficient fragment of) *PA*, (1) implies (2), and if $\mathcal{S}$ consists of amenable sets, then (2) implies (1). Here we show that without amenability, (2) need not imply (1). More precisely, we prove that for any countable model $\mathcal{N}$ of *PA*, there is a conservative extension $\mathcal{M}$, with a subset Y such that a certain $\Sigma_1(Y)$-formula defines a set which is not r.e. relative to Y.

The model $\mathcal{M}$ is produced using methods of Specker-MacDowell and Gaifman, and the set Y is produced by forcing, using $\mathcal{N}$-finite forcing conditions.

[1] M. J. Groszek and M. E. Mytilinaios, *Σ_2-induction and the construction of a high degree*, ***Recursion Theory Week, Proceedings Oberwolfach 1989*** (K. Ambros-Spies et al., editors), Lecture Notes in Mathematics, vol. 1432 (1990), Springer-Verlag, Berlin, Heidelburg, New York, Tokyo, pp. 205–223.

[2] M. J. Groszek and T. A. Slaman, *On Turing reducibility*, preprint.

[3] P. Hájek and A. Kučera, *On recursion theory in $I\Sigma_1$*, ***Journal of Symbolic Logic***, vol 534 (1989), pp. 576–589.

[4] K. Kontostathis, *The combinatorics of the Friedburg-Muchnick theorem*, preprint.

[5] M. .E. Mytilinaios, *Finite injury and Σ_1-induction*, ***Journal of Symbolic Logic***, vol. 54 (1989), pp. 38–49.

[6] M. .E. Mytilinaios and T. A. Slaman, *Σ_2-collection and the infinite injury priority method*, ***Journal of Symbolic Logic***, vol. 53 (1988), pp. 212–221.

[7] T. A. Slaman and W. H. Woodin, *Σ_1-collection and the finite injury priority method*, ***Mathematical Logic and applications*** (K. Ambros-Spies et al., editors), Lecture Notes in Mathematics, vol. 1388 (1989), Springer-Verlag, Berlin, Heidelburg, New York, Tokyo, pp. 178–188.

[8] Y. Yang, *The existence of high incomplete r.e. set does not imply $I\Sigma_2$*, preprint.

▶ G. MINTS AND S. TUPAILO, *Epsilon-substitution method for ramified analysis.*

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We extend to Ramified Analysis (RA) the definition and termination proof of Hilbert's ε-substitution method [1, 2, 3]. Second order variables of RA are assigned ordinals as levels. *Critical formulas* are closed formulas

$$F[t] \to F[\varepsilon x F[x]] \quad (\phi_0) \qquad F[T] \to F[\varepsilon X F[X]] \quad (\phi_1)$$

with a suitable restriction for levels. *ε-substitution* assigns numerals to first order ε-terms and abstracts of lower level to second order ε-terms. The existence of values of closed terms under given substitution S is proved by induction on level.

The ε-substitution process is defined for a fixed system E of critical formulas as follows. Initial substitution assigns default values: $S_0(\varepsilon xA) = 0$, $S_0(\varepsilon YB) = \lambda x(x = 0)$. If $S_0, \ldots, S_i$ are already generated, and S_i is not yet a solving substitution (some critical formula in E is false), then S_{i+1} is found as follows. Take a false critical formula If it is of the form (ϕ_0), set $S_{i+1}(\varepsilon xF) :=$ the least n such that $S_i(F[n]) =$ **true**, and set all terms of higher complexity to default value 0. In the case of (ϕ_1) set $S_{i+1}(\varepsilon XF[X]) = S_i(T)$.

THEOREM 1. *The sequence $S_0, \ldots, S_n, \ldots$ always terminates*

[1] D. HILBERT and P. BERNAYS, ***Grundlagen der Mathematik***, Bd. 2, Springer, 1970.

[2] G. MINTS,*Substitution method and Gentzen-type systems*, ***Proceedings of 9-th International Congress of Logic, Method. and Philos. of Sci.*** (Uppsala, Sweden), August, 1991 (to appear).

[3] G. MINTS and S. TUPAILO, *Epsilon substitution method for elementary analysis*, ***Report No. CSLI-93-175, CSLI***, Stanford University, February 1993, submitted to Archive for mathematical logic.

▶ DANIEL E TALAYCO, *Applications of cohomology to set theory Todorčević trees.*

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Recent developments have lead to the description and investigation of certain set theoretic phenomena such as Hausdorff gaps and Aronszajn trees using (co)homological terminology and machinery. An example of this will be explored in this talk and a new class of Aronszajn trees, called Todorčević trees, that has arisen through this research will be described.

▶ ALAN D. VLACH, *Hyperarithmetical expansions of recursive structures: A new result and an application.*

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Let $\mathfrak{U}$ be a recursive $\mathcal{L}$-structure, P a relation symbol not in $\mathcal{L}$, ψ a recursive infinitary Π_2 sentence in the language $\mathcal{L} \cup \{P\}$, and let $0 < \alpha < \omega_1^{\mathrm{CK}}$. In [5] we found sufficient conditions under which we can construct a recursive $\mathcal{L}$-structure $\mathfrak{B} \simeq \mathfrak{U}$ such that $\langle \mathfrak{B}, P_{\mathfrak{B}} \rangle \not\models \psi$ for every Δ_α^0 relation $P_{\mathfrak{B}}$ on $\mathfrak{B}$. Here we sharpen the result, providing necessary and sufficient conditions which involve a notion of rank. The argument is a nested priority construction, making use of the metatheorems of C. Ash found in [1, 2, 3]

We also give an example of a structure $\mathfrak{U}$ of arbitrarily large recursive rank to which our new result applies The structure $\mathfrak{U}$ will be a tree, isomorphic to a subtree of $\omega^{<\omega}$, with added unary predicates, for which ψ will say of the new unary predicate P that is a path.

[1] C J. ASH, *Stability of recursive structures in arithmetical degrees*, ***Annals of Pure and Applied Logic***, vol 32 (1986), pp. 113–135.

[2] C. J. ASH, *Recursive labelling systems and stability of recursive structures in hyperarithmetical degrees*, ***Transactions of the American Mathematical Society***, vol. 298 (1986), pp. 497–514. Errata, vol. 310 (1988), p 851.

[3] C. J. ASH, *Labelling systems and r.e. structures*, ***Annals of Pure and Applied Logic***, vol. 47 (1990), pp. 99–119.

[4] HARTLEY ROGERS, JR., ***Theory of recursive functions and effective computability***, The MIT Press, Cambridge, Massachusetts, 1987. Reprint (with corrections) of the 1967 McGraw-Hill edition.

[5] ALAN D. VLACH, *Hyperarithmetical relations in expansions of recursive structures*, ***Annals of Pure and Applied Logic***, vol. 60 (1990), pp. 163–196.

► BENJAMIN WELLS, *Simple recursively-based pseudorecursive varieties of semigroups.*

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The equational theory of a variety may be decidable (recursive), as in the case of commutative semigroups, or undecidable, as in the case of relation algebras. In some sense, the *pseudorecursive* variety, announced elliptically in [1], is an intermediate notion: its equational theory is not recursive, but for every $n \in \omega$ the set of equations in the theory with n variables *is* recursive. (For relation algebras, even the variable-free equations in the theory form a nonrecursive set.)

Since 1983, six authors in four papers have constructed simple infinite recursive bases for pseudorecursive varieties of squarezero semigroups with additional operations. All have been inspired directly or indirectly by [2], the first published specification of a pseudorecursive variety. The author examines this growing cottage industry, reviews recent products, and reveals the latest models.

[1] A. I. MAL'CEV, ***The metamathematics of algebraic systems—Collected Papers: 1936–1967***, North-Holland, Amsterdam, 1971. Translator's notes by BENJAMIN WELLS, pp. 395, 473.

[2] BENJAMIN WELLS, *A simple pseudorecursive variety of infinite type*, ***Abstracts of Amer. Math. Soc.***, vol. 3 (1982), p. 592.

► DAN E. WILLARD, *Alternate versions of self-verifying axiom systems.*

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Define an axiom system A to be **self-verifying** iff

(i) one of the theorems implied by A is the statement of its own consistency, and
(ii) the axiom system is in fact consistent.

Kleene's Fixed Point Theorem [1] implies that every recursively enumerable axiom system A can be easily expanded into a broader system A^* which satisfies condition (i). The catch is that A^* can be inconsistent even while it asserts its own consistency, hence violating condition (ii). In the prior articles [3, 4], we showed how a self-verifying system, satisfying both conditions (i) and (ii), could prove all the Π_1 theorems of Peano Arithmetic without violating the constraints imposed by Gödel's Incompleteness Theorem. In this Abstract, we will describe two alternate systems which have additional nice properties.

Let d denote any cut-free rule of inference, such as Gentzen's cut-free sequent calculus, semantic tableaux or resolution. Let A denote any consistent axiom system. Define $f(x_1, x_2, \ldots, x_n)$ to be a **non-growth function** iff it satisfies $f(x_1, x_2, \ldots, x_n) \leq \max(x_1, x_2, \ldots, x_n)$. Consider a self-verifying system, denoted as $\mathrm{IS}_d(A)$, that will consist of the five groups of axioms described below:

(1) An initial axiom group defining all the non-growth functions of A, as well as the growth functions of Addition and Scalar Multiplication. (Integer multiplication will not be a function under $\mathrm{IS}_d(A)$, but it can be implicitly represented as a relation through the non-growth function of Non-zero Division.)
(2) For any Π_1 sentence Φ, the sentence $\Phi \vee \neg\Phi$ will be an axiom.
(3) For any Δ_0 formula Ψ and constant k, the sentence $\forall x \Psi(x) \supset \Psi(k)$ will be an axiom.
(4) Let $\mathrm{Prf}_A(\ulcorner\Phi\urcorner, x)$ be a formula asserting that x is the Gödel number of a proof of Φ from the axiom system A. Then for each Π_1 sentence Φ, $\mathrm{IS}_d(A)$ will have an axiom of the form $\exists \mathrm{Prf}_A(\ulcorner\Phi\urcorner, x) \supset \Phi$.
(5) $\mathrm{IS}_d(A)$'s Group-5 axiom will be a single sentence asserting that the entire $\mathrm{IS}_d(A)$

system (including this sentence looking at itself) is unable to support a proof of $0 = 1$.

Our first main theorem will state that if A is consistent then so must $\mathrm{IS}_d(A)$ be consistent. This theorem is stronger than our prior work [3, 4] because the latter did not include the Group-2 and Group-3 axioms. (It is a non-trivial generalization because Solovay [2] has shown essentially that the *cut-free* system $\mathrm{IS}_d(A)$ becomes inconsistent when its Group-2 axioms are extended to include for each sentence Φ, the axiom $\Phi \vee \neg\Phi$.)

It is also possible to develop versions of $\mathrm{IS}(A)$ that employ a substantially stronger notion of self-consistency than the Group-5 sentence above. Let Ψ denote an arbitrary sentence with n unbounded universal quantifiers, and $\Psi_k(p)$ denote a formula which differs from Ψ by requiring all Ψ's universal quantifiers to be bounded by k, and all its existential quantifiers to be bounded by the quantity pk. Let $\mathrm{IS}_d^R(A)$ be a modified version of $\mathrm{IS}_d(A)$ whose group-5 Class includes one axiom of the form $\forall x\{\mathrm{Prf}_{\mathrm{IS}_d^R(A)}(\ulcorner\Psi\urcorner, x) \supset \Psi_k(x)\}$, for each constant k and sentence Ψ. Then a stronger version of our theorem would state that if A is consistent then $\mathrm{IS}_d^R(A)$ must be consistent. Indeed, it is possible to develop a yet even slightly stronger system which asserts the preceding Group-5 reflection principle is valid in a fully uniform sense.

[1] S C. Kleene, *On the notation of ordinal numbers*, ***Journal of Symbolic Logic***, vol 3 (1938), pp. 150–156.

[2] R. Solovay, private communication, in a somewhat different notation, demonstrating how variants of $\mathrm{IS}_d(A)$(Peano) are inconsistent.

[3] D. Willard, *Self-verifying axiom systems*, ***Proceedings of the Third Kurt Gödel Symposium*** (1993), pp. 325–336, published in vol 713, Springer-Verlag Lecture Notes in Computer Science.

[4] D. Willard, *Self-verifying axiom systems and the Incompleteness Theorem*, ***SUNY-Albany Computer Science TR 93-10*** (a 54 page manuscript, submitted for publication, whose results were orally presented at the 1994 Spring ASL conference in Gainesville, Florida).

► JINDŘICH ZAPLETAL AND SAHARON SHELAH, *Small forcings and Cohen reals.*

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Hebrew University of Jerusalem, Jerusalem, Israel.

Consistently all forcings of density $\aleph_1$ embed "$\aleph_1$ Cohen reals." We also derive this statement or its approximations from various forcing axioms

Abstracts presented by title

► VLADIMIR KANOVEI, *On external Scott algebras and external definability in nonstandard models of Peano arithmetic*

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D. Scott gave in [1] a necessary and sufficient condition for a countable set $A \subseteq \mathcal{P}(\omega)$ to be equal to SA^M, the *Scott algebra* of a **PA** model M, that is, the family of all sets $Z \subseteq \omega$ definable in M by a parameter-free **PA** formula.

It is evident that SA_M is exactly the collection of all arithmetical $Z \subseteq \omega$ provided M is an elementary extension of ω. One may, however, extend the *PA* language $\mathcal{L}$ by the unary predicate of standardness st interpreted, in all nonstandard models of **PA**, as being a member of ω Let ESA^M, the *external Scott algebra* of M, denote the family of all subsets of ω definable in M by a parameter-free formula of the extended language $\mathcal{L}^{st}$. We give a necessary and sufficient condition for a countable family of subsets of ω to be an external Scott algebra for an elementary extension of ω.

THEOREM 1. *Let $\mathcal{Z} \subseteq \mathcal{P}(\omega)$ be countable. Conjunction of the following two conditions is necessary and sufficient for there to exist a countable $M \models$ **PA**, $M \succ \omega$, such that $ESA_M = \mathcal{Z}$.*

(1) *$\mathcal{Z}$ is arithmetically closed*

(2) *$\mathcal{Z}$ contains $0^{(\omega)}$, the set of all (Gödel numbers of) true arithmetical sentences.*

The necessity is quite easy, the hard part is the sufficiency. To make every $Z \in \mathcal{Z}$ definable in M by a formula of the extended language, we use a coding system, which also gives a tool for the following definability theorem:

THEOREM 2. *Let $Z \subseteq \omega$, $n \in \omega$ There exists a countable model $M \models$ **PA**, $M \succ \omega$, such that Z is Σ^{st}_{n+2} in M but every $Y \subseteq \omega$, $Y \in \Sigma^{st}_{n+2}$ in M, is $\Sigma_{n+1}[0^{(\omega)})]$.*

By Σ^{st}_n we denote the class of all $\mathcal{L}^{st}$ formulas having Σ_n prefix, followed by a formula composed from recursive formulas by &,$\vee$, and quantifiers $\exists^{st}$ and $\forall^{st}$ $0^{(\omega)}$) in square brackets means that $0^{(\omega)}$) can participate as an extra parameter.

Notice that $0^{(\omega)}$) itself is Δ^{st}_1 in M; moreover, any $Y \subseteq \omega$, $Y \in \Sigma_{n+1}[0^{(\omega)})]$ is Σ^{st}_{n+1} in any **PA** model $M \succ \omega$. Thus Theorem 2 tells us that a set Z of natural numbers can be made Σ^{st}_{n+2} but not Σ^{st}_{n+1} in an appropriate **PA** model $M \succ \omega$ unless it is essentially simple (belongs to $\Sigma_{n+1}[0^{(\omega)})]$.

The proof of both Theorem 1 and Theorem 2 is based on several forcing ideas, in particular, the notion of arithmetical forcing and generic ultrafilters.

[1] D. SCOTT, *Algebras of sets binumerable in complete extensions of arithmetic*, ***Recursive Function Theory, Proceedings of the Fifth Symposium in Pure Mathematics of the AMS***, 1962, pp 117–121.

► GUN WON LEE, *CS ideal for properties and a definition of finiteness.*
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$\aleph_0 = \{0, 1, 2, \dots\}$ and $\beth_0$ = the power set of $\aleph_0$ by definition. If $n \varepsilon \aleph_0$, n is said to be finite. Thus the *finiteness* has been defined. The poset of $\aleph_0$ is assigned for the properties of the elements of objects, whereas the elements of the set of natural numbers are assigned for the objects in question. However the properties are represented in terms of the equivalent sets of power set which are assigned to them. The equivalent sets of the power set in terms of their cardinalities are said to be the *ideal.*

There is an one-to-one mapping in between the ideal and the $\aleph_0$. Therefore the definition of the finiteness in the above can be an equivalent set of posets whose base is $\aleph_0$. This hypothesis assumes that the targets of measurements are properties, and the cardinalities of the objects are represented in terms of natural numbers. And the definitions of the finiteness can be fixed in this hypothesis at least. The operations admitted in this hypothesis are *closed* because of the infiniteness of the $\aleph_0$ set and also in the case of the $\beth_0$ set. This model can be posed as:

$$M = \{U, CS, LE, E, +\}.$$

▶ BILLY JOE LUCAS, *The logic of epistemic obligation, part 1.*

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Reasoning about epistemic obligation is hardcore philosophy, and goes back at least to Socrates, yet, oddly enough, reflection on the underlying principles behind such reasoning is quite rare The basic motivation for this paper is to introduce a variety of alternative concepts of epistemic obligation and to study the logical structure of each such concept as well as to investigate how these concepts interact with each other. Two particularly important distinct families of such concepts are: *deontic* epistemic obligations, and *alethic* epistemic obligations.

The results in Part 1 presuppose as base systems a minimal deontic epistemic logic (MDE) and a minimal logic of alethic epistemic obligation (MAE). Let L be a language with $\sim$, $\vee$, $\bigcirc$ (it ought to be true that), and sK (s knows it is true that), as the only primitive operators. An MDE frame is a triple $\langle X, R, S\rangle$, with R serial and S reflexive. An MDE model is the usual sort of thing, with $\bigvee(\bigcirc p, x) = T$ iff $\forall y(xRy \supset \bigvee(p, y) = T)$ and $\bigvee(sKp, x) = T$ iff $\forall y(xSy \supset \bigvee(p, y) = T)$. Let L' be a language with these primitive operators $\sim$, $\vee$, sOK (s has an alethic epistemic obligation to know it is true that). An MAE frame is a couple $\langle X, T\rangle$, with T reflexive, and an MAE model has $\bigvee(sOKp, x) = T$ iff $\forall y(xTy \supset \bigvee(p, y) = T)$. For languages that include both L and L', we use MADE frames (and their extensions): $\langle X, R, S, T\rangle$, with R, S, T, and truth clauses for $\bigcirc$, sK, and sOK as above. In this paper I'm concerned with what we can learn about those concepts of epistemic obligation which can be articulated on MDE (MAE) frames and extensions of them formed in just two ways· adding first-order definable conditions (separately) on any of R, S, and T; adding first-order definable principles that link at least two of R, S, and T. Subsequent parts of this study will focus on the epistemic obligation for additional extensions of MDE (MAE) frames, as well as for systems in which various principles sound on MDE (MAE) frames fail

At least since Aqvist [1], deontic logics that extend *KD* (including *KD* itself) have been suspect as candidates for being the appropriate deontic systems to be used in combination with epistemic logics, and since not long after Hintikka [3], epistemic logics that extend *KT* (including *KT* itself) have been suspect as candidates for anything. MDE models make all of these suspect assumptions, and it turns out that study of such systems can substantially enrich our understanding of the logic of epistemic obligation. For example, Aqvist's paradox of epistemic obligation can be avoided without mutilating the basic deontic logic in the ways said to be required in Goble [2], and a philosophically *useful* interpretation of 'sK' can be found relative to which a normal modal logic for knowledge has some plausibility.

[1] Lennart Aqvist, *Good Samaritans, contrary-to-duty imperatives, and epistemic obligations*, ***Nous***, vol. 1 (1967), pp. 361–379.

[2] Lou Goble, *A logic of good, should, and would*, ***Journal of Philosophical Logic***, vol. 19 (1990), pp. 169–199.

[3] Jaakko Hintikka, ***Knowledge and belief***, Cornell University Press, 1962.

▶ MARÍA CLAUDIA MERÉ AND PAULO AUGUSTO VELOSO, *Definability in extensions by sorts.*

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The process of implementing a formal specification on another one is very important in formal software development and it can be described in terms of simple basic logical concepts. Formal specifications are presentations of theories in many-sorted logic, and an implementation of a formal specification A on another formal specification C amounts to an interpretation of A into a conservative extension B of C.

An extension consists of the addition of new symbols and axioms. The new symbols to be introduced correspond to sorts, functions, or predicates. When performing an extension one expects an increase in both deductive and expressive powers. On the other hand, one often does not wish to disturb the smaller theory. When the extension only adds properties of the new symbols, without disturbing the old ones, one has a **conservative extension**, which does not increase deductive power for the properties of the old symbols. Similarly, the increase in expressive power may be inessential, in which case our extension has the **property of eliminability**.

Extensions by the addition of function and predicate symbols are well studied in the literature. In particular, **extensions by definitions** of function and predicate symbols have some nice properties Consider such an extension of theory T over language L to T' over L' We then have conservativeness and eliminability. Consequently, we have the following properties:

- **Interpretability**. T' can be interpreted back into L.
- On the model-theoretic side, we have **Invariance under elementary equivalence**. Two models of T' are elementarily equivalent whenever their reducts to L are so.
- **Definability**. A relation is definable in a model of T' iff it is definable in its reduct to L.

The many-sorted case has not been deeply studied because of the existence of the well known reduction to unsorted logic. Our purpose here is to analyse conservative extensions that introduce new sorts. This is of importance because it occurs often in implementing formal specifications, when the new sorts are "constructed" from the concrete ones. We specify and analyse some well-known sort introduction constructs akin to those found in programming languages ([1]), namely cartesian product, discriminated union, subsorting and quotient

In a previous work [2] we have axiomatised the specifications and established their adequacy in the sense of exhibiting the intended behaviour. In this work we centre our attention in establishing that these extensions are like extensions by definition of a new sort, even though we have proved that we have only unique expandibility up to isomorphism.

In the case of introduction of a sort, we have new variables over the new sort, and new universes realising the new sort. To cope with these, we use two ideas

- **connection**: a local translation of variables ranging over the new sort to the old sorts.
- **internalisation**: a view of relations involving the new sorts.

With these new ideas we will be able to generalise the above properties of extensions by definitions of predicate or function symbols to our sort introducing constructs. Actually, we will prove that there are no new definable relations, up to internalisation, which corroborates the intuitive feeling that these amount to extensions by definition of new sorts.

[1] C.A.R. HOARE, *Notes on data structuring*, ***Structured programming*** (O. J Dahl, E. W. Dijkstra, and C.A.R. Hoare, editors), Academic Press, 1974, pp. 83–174.

[2] M. C. MERÉ and P. A. S. VELOSO, *On extensions by sorts*, ***Monografias em Ciência da Computação, No. 38/92 D.I., PUC-Rio***, 1992.

▶ ALBERT A MULLIN, *Applications of recursive number theory to literature,* dedicated to the memory of Prof. S. C. Kleene (23 83,2 · 997).

506 Seaborn Drive, Huntsville Alabama, USA

Recall that the author has shown that *popular* Japanese poetry (e.g., Haiku and Tanka) is Eastern poetry in prime numbers with its own additive problems and recursive results for prime-number theory. We show that more *private* Japanese poetry (e.g., Sedōka and Dodoitsu) is Eastern poetry in bi-composites; e.g., with 38 and 26 syllables, respectively. Here recursive number-theoretic results on the number of essentially different representations of bi-composites provide estimates on the number of essentially different kinds of Sedōka and Dodoitsu. The methodology used is analogous to the author's representation theory for Haiku and Tanka which depends on use of primality results rather than bi-compositeness results. Poetry in *odd* bi-composites provides yet another motif for the lyrics of Japanese folk songs.

Problem 1 Give an example of a (validromic) verse in one natural language which backwards is prose in a different natural language

Problem 2. Give an example of a WFF in one formal language which backwards is a WFF in a different formal language

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1994–1995 ANNUAL MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

Irvine, California March 30–April 2, 1995

The Annual Meeting of the Association was held at the University of California, Irvine, March 30–April 2, 1995 The program included eight invited hour talks and two special sessions, in model theory and recursion theory respectively, each consisting of six half-hour talks. Leo Harrington gave the sixth annual Godel lecture, entitled *Gödel, Heidegger and direct perception (or, why I am a recursion theorist).* In addition, there were twenty-eight contributed talks and four papers presented by title. The Program Committee consisted of C. A. Anderson, S. Buss, M Foreman, C. Laskowski (chair), and M Lerman. The local organizers were P. Eklof (chair), M. Foreman, P. Maddy, and A. Nesin.

The following invited talks were presented:

George Boolos (MIT), *The recent rehabilitation* (?) *of logicism.*

Barry Cooper (University of Leeds), *Turing definability and automorphisms.*

Andras Hajnal (DIMACS, Rutgers University), *Partition relations revisited.*

H. Jerome Keisler (University of Wisconsin), *Quantifier elimination in probability theory.*

Christos Papadimitriou (University of California San Diego), *Logic, complexity and artificial intelligence.*

William Reinhardt (University of Colorado), *On sets, set theory and higher type objects.*

Marion Scheepers (Boise State University), *Games that involve set theory or topology.*

Alasdair Urquhart (University of Toronto), *Russell and the foundations of logic, 1903–05*

The following talks were presented in a Special Session on Model Theory·

Bradd Hart (McMaster University), *Stable varieties· A survey of results and open questions.*

Renling Jin (University of Illinois), *An easy way to use the isomorphism property.*

James Schmerl (University of Connecticut), *The reals in non-standard models.*

Chris Miller (University of Illinois-Chicago), *Polynomially bounded, o-minimal expansions of the field of real numbers.*

Alexander Kechris (Caltech), *Countable Borel equivalence relations and the classification problem for rank 2 torsion-free groups*

Ambar Chowdhury (University of Connecticut), *The small index property and smoothly approximable structures.*

The following talks were presented in a Special Session on Recursion Theory:

Doug Cenzer (University of Florida), *Index sets for* Π^0_1 *classes.*

Tamara Hummel (Dartmouth University), *Effective versions of Ramsey's theorem.*

Carl Jockusch (University of Illinois) [with Rod Downey and Michael Stob], *Array nonrecursive degrees and genericity.*

William Calhoun (Kalamazoo College), *Structured programming for priority arguments*

Andre Nies (University of Heidelberg), *Relativizations of structures arising from computability theory.*

Klaus Ambos-Spies (University of Heidelberg), *Resource bounded genericity concepts.*

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Abstracts of three invited talks, and all contributed talks given (in person or by title) by members of the Association for Symbolic Logic follow.

For the Program Committee
CHRIS LASKOWSKI, CHAIR

Abstracts of invited talks in the Special Session in Model Theory

► RENLING JIN, *An easy way to use the isomorphism property.*
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Let V denote an ω_1-saturated nonstandard universe. V is said to be satisfy the isomorphism property if for any finite language $\mathcal{L}$, for any two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ which are internal in V, $\mathcal{A}$ is elementarily equivalent to $\mathcal{B}$ implies $\mathcal{A}$ is isomorphic to $\mathcal{B}$. The isomorphism property was suggested by C. W. Henson in early 70's for studying nonstandard hull of Banach spaces. Recently, Shelah and I discovered an equivalent form of the isomorphism property in terms of the realizability of some second-order types. This equivalent form makes the use of the isomorphism property much easier than before. In the talk I am going to give three examples to illustrate the idea.

I will show that the isomorphism property implies:

(1) the hyperreal line is not Scott complete,
(2) every unbounded Loeb measure space has a measurable subset with infinite outer measure and zero inner measure,
(3) every hyperfinite time line has a bad cut and a good type two cut.

► CHRIS MILLER, *Polynomially bounded, o-minimal expansions of the field of real numbers.*
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An expansion of $(\mathbb{R}, +, \cdot)$ is said to be *o-minimal* if every subset $\mathbf{R}$ (parametrically) definable in the structure is a finite union of points and open intervals, and is said to be *polynomially bounded* if for every definable $f : \mathbf{R} \to \mathbb{R}$ there is an $n \in \mathbf{N}$ such that $f(t) = O(t^n)$ as $t \to +\infty$. Sets and functions definable in polynomially bounded o-minimal expansions of $(\mathbf{R}, +, \cdot)$ have many of the nice analytic, metric and geometric properties of real semialgebraic sets and functions. Applications of this fact to particular polynomially bounded o-minimal expansions of $(\mathbf{R}, +, \;)$ yield new results in real analytic geometry.

► JAMES H. SCHMERL, *The reals in non-standard models*
Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA.

There are constructions of nonstandard models of Peano Arithmetic which yield models whose corresponding reals have certain desirable properties. We will discuss a method for extending these constructions to other theories which are very weak versions of set theory and second-order number theory.

Abstracts of contributed talks

► ALEXANDRU BALTAG, *Modal logic and topology: Multidimensional frames.*
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The project is to relate topology with modal logic, by giving a modal system, weak enough to be decidable, but with enough expressive power to account for elementary topological reasoning.

First, a modal logic with indexed modalities is proposed, having two natural interpretations: "n-dimensional Kripke frames" and n-dimensional topological spaces (a generalization of McKinsey's semantics for S4). I have a completeness and a decidability result for $n = 2$.

Second, generalizing a paper of R Parikh and L. Moss, I am proposing a language with two types of indexed modalities (one for **knowledge** and one for **effort**), with topological and knowledge-theoretical applications.

▶ DOUGLAS BURKE, *Generic ultrafilters with well-founded ultrapowers.*

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A tower of height δ is a sequence $\langle F_x \mid X \in V_\delta \rangle$ where F_X is a normal filter on $\mathcal{P}(X)$ and $X \subseteq Y$ implies F_Y projects to F_X. A tower is based on stationary sets if each F_X is the club filter restricted to a stationary set. A tower is precipitous if the generic direct limit ultrapower is well-founded—forcing with the positive sets from the tower.

THEOREM 1 (Woodin). *Assume κ is regular and $\delta > \kappa$ is Woodin. Then there is a forcing of size δ such that it is forced that there is an elementary $j : V \to M$ with c.p $(j) = \kappa$, $j(k) = \delta$, and $M^{<\delta} \subseteq M$.*

The forcing in this theorem comes from the tower of height δ where F_X is the club filter restricted to $\{a \subseteq X \mid |a| < \kappa\}$. We generalize this theorem as follows:

THEOREM 2. *Assume T is a tower of height δ, δ supercompact, and T is based on stationary sets. Then T is precipitous.*

We do not know if the large cardinal hypothesis can be weakened (down to a Woodin) but there are examples that show that the assumption "based on stationary sets" is necessary.

We use Theorem 1 to prove the following: Assume F is a countable complete filter and $\kappa > |F|$ is measurable or Woodin. Then there is a forcing P of size κ and a P name $\dot{G}$ such that it is forced that $\dot{G}$ is a V-ultrafilter extending F and

a. If κ is Woodin then (it is forced that) $\mathrm{Ult}(V, \dot{G})$ is well-founded.
b. If κ is measurable then (it is forced that) $j_G(\kappa)$ is in the well-founded part of $\mathrm{Ult}(V, \dot{G})$.

Sometimes this lemma can be used instead of assuming F is precipitous. For example we use this lemma to answer a question of Abe ([1]):

THEOREM 3. *Assume there is a weakly normal (see* [1] *for the definition) filter on $\mathcal{P}_\kappa\lambda$ where $\kappa < cof(\lambda) < \lambda$ (and there is a measurable cardinal $> \lambda$). Then $\lambda^{<\kappa} = 2^{<\kappa}\ \lambda$*

[1] YOSHIHIRO ABE, *Weakly normal ideals on $\mathcal{P}_\kappa\lambda$ and the singular cardinal hypothesis*, ***Fundamenta Mathematicae***, vol. 143 (1993), pp. 97–106

[2] H. WOODIN, *Supercompact cardinals, sets of reals and weakly homogeneous trees*, ***Proc. Nat. Acad. Sci. U.S.A.***, vol. 85 (1988), pp. 6587–6591.

▶ JAMSHID DERAKHSHAN, *The Fitting subgroup of MC-groups and locally nilpotent substable groups.*

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This paper is concerned with the structure of groups with the chain condition on centralizers (denoted as MC-groups), and the structural group theoretical properties of substable groups. The following are proved. The Fitting subgroup of an MC-group is nilpotent, (this solves problems in Wagner (1992)). Any locally nilpotent substable group is hypercentral. Any uniformly locally nilpotent MC-group is nilpotent, (this generalizes a theorem of Poizat-Wagner (1993) on substable groups and a theorem of Bryant (1979) on MC-groups of finite exponent). Some consequences of these results are obtained which solve some other problems on the algebraic structure of stable groups. In this regard, among other results, an Engel theory is developed, and applications to skew linear groups are obtained.

[1] R. M. Bryant, *Groups with the minimal condition on centralisers*, ***The Journal of Algebra***, vol. 60 (1979), pp. 371–383.

[2] B. Poizat and F. O. Wagner, ***Journal of Symbolic Logic***, 1993.

[3] F. O. Wagner, *The Fitting subgroup of a stable group*, ***The Journal of Algebra***, to appear, 1992.

▶ DERRICK DUBOSE, *Determinacy and the sharp function on all sets.*

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We characterize in terms of determinacy, the existence of indiscernibles for the least inner model of the sharp function on all sets.

Let $\#^k$ be the partial sharp function on objects of type k. The following result has been known for some time:

(i) [2, 3] The existence of indiscernibles for $L[\#^k]$ is equivalent to the determinacy of $(\Pi^0_k)^*_+$. (See the last paragraph for the definition of $(\Gamma)^*_+$.)

Let $\#^\infty$ be the partial sharp function on all sets. The following analogous result is now known:

(ii) The existence of indiscernibles for $L[\#^\infty]$ is equivalent to the determinacy of $(\Pi^1_1)^*_+$. Around 1986–1987, Martin showed the $\Leftarrow$ direction of (ii). Its proof is similar to that of the $\Leftarrow$ direction of (i), which in turn is similar to Martin's proof of the converse of (iii) below. "Recently" Martin and I independently showed the $\Rightarrow$ direction of (ii). The proof combines quasi-Borel Determinacy [4, 5] (formulated and proven by Martin) with the method of integrating strategies with respect to ordinal auxiliary moves, as done in Martin's proof of the following:

(iii) [Martin] $\beta - \Pi^1_1$ games are determined for $\beta < \omega^2$ if $0^\#$ exists (see [5] or [1]).

The classes $(\Pi^0_k)^*_+$ and $(\Pi^1_1)^*_+$ lie strictly between $\bigcup_{\beta<\omega^2}(\beta - \Pi^1_1)$ and $\Delta(\omega^2 - \Pi^1_1)$. The definition of $(\Gamma)^*_+$ is given in what follows. Suppose that the sequence $\vec{A} = \langle A_\alpha \,|\, \alpha < \omega^2 \rangle$ of Π^1_1 sets witnesses that A is $\omega^2 - \Pi^1_1$. Let us say that "x falls out of $\vec{A}$ for player I by stage n" if

$$\mu\hat{\alpha}(x \notin A_{\hat{\alpha}} \text{ or } \hat{\alpha} = \omega \cdot n) \text{ is odd.}$$

For $B \subseteq {}^\omega\omega \times \omega$, define $N^B(x)$ to be the least n such that $B(x, n)$; it is undefined if no such n exists. For $\vec{D}$ witnessing that some set is $\beta - \Pi^1_1$ for some $\beta < \omega^2$, let

$$x \in (B)^*(\vec{A}; \vec{D}) \text{ iff either (a) or (b) below holds.}$$

(a) $N^B(x)$ exists and x falls out of $\vec{A}$ for player I by stage $N^B(x)$.

(b) $N^B(x)$ is undefined and x falls out of $\vec{D}$ for player I.

Define $(\Gamma)^*_+$ to be the collection of all $(B)^*(\vec{A}; \vec{D})$ such that $B \in \Gamma$, $\vec{A}$ witnesses that some set is $\omega^2 - \Pi^1_1$, and $\vec{D}$ witnesses that some set is $\beta - \Pi^1_1$ for some $\beta < \omega^2$.

[1] D. A. DuBose, *The equivalence of determinacy and iterated sharps*, ***Journal of Symbolic Logic***, vol. 55 (1990), pp. 502–525.

[2] D. A. DuBose, *Determinacy and the sharp function on the reals*, ***Annals of Pure and Applied Logic***, vol. 55 (1992), pp 237–263

[3] D. A. DuBose, *Determinacy and the sharp function on the objects of type k*, ***Journal of Symbolic Logic***, to appear

[4] D. A. Martin, *An extension of Borel determinacy*, ***Annals of Pure and Applied Logic***, vol. 49 (1990), pp. 279–293

[5] D. A. Martin, his forthcoming book entitled "Borel and Projective Games."

► LEONA F. FASS, *Modeling perfect behavior in an imperfect world.*

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We have taken a formal, foundational approach to problems of reasoning about knowledge, with particular attention to determining a mathematical structure for a specified body of knowledge (and its identifiable components). Considering the knowledge to-be-acquired as a "perfect behavior" within an imperfect world, we need distinguish that which is the desired behavior from that which is not.

From our perspective (originally, in theoretical computer science) knowledge acquisition or learning is determination of a "device" that models a desired behavior perfectly. With suitable constraints on the process: guidance of the learning procedure, selection of illustrative information; and description of the goal "device" also can be made precise or, well-defined.

During the earliest development of formal theories for computer science there was much interest in designing such "abstract devices" and in reasoning about their "states". Work by E. F. Moore, Myhill, Nerode, Rabin and Scott, Buchi, J. W. Thatcher and J. Wright investigated the areas of: relationships between component-state structure and device behavior; necessity or elimination of device states; choice or certainty in state-to-state transitions; and achievement of goals through accessing of "final states". A typical Moore thought-experiment on an abstract device could involve determining what state it might or must be in, based on observations of how it has behaved. Much research was devoted to defining realizable functions and "behaviors", determining their processing devices and specifying their states. If a device could be shown to produce a precisely specified behavior, then it realized its "function", behaved "perfectly", and was verifiably correct.

As time progressed the focus of formal theory in computer science shifted from logical and algebraic-machine "foundations" to mathematics-of-the-feasible and time-and-space efficiency. However, there has been renewed interest in the foundational approach, in connection with artificial intelligence research into such issues as "human-like" reasoning, logical/philosophical epistemics, and computational learning theory. We ourselves have defined learning in terms of *modeling* a *behavior*, based on reasoning about informational observations to determine the model's *function, structure, choices,* and *states.*

Here we describe the evolution of our approach to reasoning and learning as we have sought to apply our theory to a variety of behavioral domains. We consider the case where a perfect behavioral model is *known* to exist and can be found effectively; where it is *suspected* that a perfect model exists; where it is suspected that *no* perfect behavioral model exists, so we accept a result that is approximately (probably) correct; and finally, the fuzzy domains where perfection, approximation and probability are replaced by possibility

[1] L. F. Fass, *Perfect learning (more or less)*, presented at the ***1992 Annual Meeting of the Society for Exact Philosophy***, University of Southwestern Louisiana, Lafayette, LA, May, 1992.

[2] ———, *Modeling perfect behavior· A goal-driven learning analysis*, ***Notes of the AAAI—Stanford 1994 Spring Symposium on Goal-Driven Learning***, SS 94-02, AAAI Press, pp. 125–127.

▶ RAYMOND D. GUMB, *A free theory of types.*

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Partial functions can be formalized in a natural manner in free logics. The aim of the present work is to explicate the positive free theory of types underlying higher-order functional programs employing both strict and lazy evaluation. The logic is positive and intensional because, for example, the following sentences are simultaneously satisfiable: $\neg\exists X(X = \text{error})$, $\text{zero}(\text{error}) = 0$, $1 \div 0 = \text{error}$, and $\forall X(f(X) = g(X)) \wedge f \neq g$. The free theory is an extension of the standard theory, rather than an alternative to it.

▶ GEOFFREY HELLMAN, *Constructivization problems in space-time physics Singularity theorems*

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To what extent is constructive mathematics (á la Bishop, et al.) really able to recover mathematics of importance in the physical and other sciences? In previous work, we have already elaborated on some significant obstacles in connection both with quantum mechanics (Gleason's theorem, some important classes of unbounded linear operators acting in Hilbert space) [1] and with cases of ordinary reasoning in which the use of constructive logical connectives and quantifiers results in systematic distortions of intended meaning [2]. This paper calls attention to some related problems in connection with the singularity theorems of general relativity (Hawking, Penrose, et al.) and their applications to our universe.

The main points are the following:

1. The basic Robertson-Walker singularity theorem (Proposition 12.15, p 348 of reference [3]) *can* be given a constructive proof. (This is the highly idealized case of completely isotropic "dust" models.) The key point is that a comparative order on the reals rather than a total linear order suffices

2. Important parts of Lorentz manifold theory, including the main tools used to prove the more powerful and general Hawking (cosmological) singularity theorems, however, are essentially non-constructive. For example, (a) the "construction" of "quasi-limit" curves—limits of sequences of continuous causal curves—is not constructively legitimate (depending on the König Infinity Lemma); (b) the existence of length-maximizing geodesics relies essentially on sequential compactness. (Examples of non-constructive theorems are Theorems 9.4.4 and 9.4 5, pp. 236–237, of reference [4], which assert that the maximum value of an upper semi-continuous function on a compact domain is assumed.) Similar problems confront efforts to constructivize related singularity theorems of Penrose and of Hawking and Penrose.

3. Ordinary reasoning used in applying, says, the cosmological theorems, including even the constructive Robertson-Walker theorem, to our cosmos appear to involve non-constructive existence claims and rules in an ineliminable way. The data constrain the mathematical models only very approximately (e.g., the range of a positive Hubble constant, nearly vanishing radiation pressure "sufficiently after" the initial singularity, etc.), but we still conclude that, with reasonable probability, the universe had a "big bang" initial point at some time in the past (e.g., prior to recent, controversial recalculations, 1.5×10^{10} years ago give or take 5×10^{9} years!). Yet we have no method (and quite conceivably could never have any) for narrowing this to the point at which we could be said to instantiate this existential quantifier (as the generalized disjunction property of constructive "∃' requires). Appeal to "interval-valued" or "fuzzy" magnitudes—however useful in more mundane contexts, such as measuring length, mass, frequency, etc.—can hardly help here.

[1] Geoffrey Hellman, *Gleason's theorem is not constructively provable*, ***Journal of Philosophical Logic***, vol. 22 (1993), pp. 193–203, and *Constructive mathematics of quantum mechanics: Unbounded operators and the spectral theorem*, ***Journal of Philosophical Logic***, vol. 22 (1993), pp. 221–248.

[2] ———, *Never say 'never'! On the communication problem between intuitionism and classicism*, ***Philosophical Topics***, vol. 17, no. 2, pp. 47–67.

[3] Barrett O'Neill, ***Semi-Riemannian geometry***, Academic Press, 1983.

[4] Roberst Wald, ***General relativity***, University of Chicago Press, 1984.

► GREG HJORTH, *On uncountably many minimal models.*

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Under $ZFC + \neg CH$, it is not decidable whether a countable complete theory can have exactly $\aleph_1$ many minimal models.

► RENLING JIN AND SAHARON SHELAH, *Can a small forcing create Kurepa trees?.*

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On one side, we prove that if one chooses a Levy's model (a model obtained by forcing with a countable support Levy collapsing order which collapses all cardinals between ω_1 and a strongly inaccessible cardinal) as his ground model, then it is extremely hard to create Kurepa trees by a cardinal-preserving forcing notion of size at most ω_1 (if it is ever possible). On the other side, we prove that by assuming the consistency of one inaccessible cardinal there exists a model of CH plus no Kurepa trees, in which there is an ω-distributive Aronszajn tree such that forcing with that tree creates a Kurepa tree in the generic extension.

► DARRYL JUNG, *The vicious-circle principle.*

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Prompted by Henri Poincaré, Bertrand Russell put forward his celebrated **vicious-circle principle** (vcp) as the solution to the modern paradoxes. (The logical theory of Russell's 1910 *Principia Mathematica* may be understood as the result of modifying the logical theory implicit in his 1903 *Principles of Mathematics* in order to abide by the vcp.) Gödel and Quine, among others, have raised two salient objections against Russell's vcp. First, Gödel has claimed that Russell's various renderings of the vcp really express distinct principles and, thus, distinct solutions to the paradoxes, a claim that gainsays one of Russell's positions on the nature of the solution to the paradoxes, *viz.*, that such a solution be uniform. Secondly, Gödel and Quine have protested that the vcp's proscription against impredicative specification is incompatible with a realistic conception of the domain of quantification, a conception that Russell certainly held. I examine Russell's vcp and defend it against these objections.

► THOMAS E. LEATHRUM, *More results concerning off-branch families.*

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In recent work, the author has defined two new "small cardinal characteristics," using variations on almost disjoint families. A set of nodes in the tree ${}^{<\omega}\omega$ (ordered by inclusion) is *off-branch* if it is disjoint from every branch in the tree; an *off-branch family* is an almost disjoint family of off-branch sets;

$$\mathfrak{o} = \min\{|\mathcal{O}| : \mathcal{O} \text{ is a maximal off-branch family}\}.$$

Previous results concerning $\mathfrak{o}$ include: (in ZFC) $\mathfrak{a} \leq \mathfrak{o}$, and (consistent with ZFC) $\mathfrak{o}$ is not equal to any of the standard small cardinal invariants $\mathfrak{b}$, $\mathfrak{a}$,$\mathfrak{s}$,$\mathfrak{d}$, or $\mathfrak{c} = 2^\omega$. (Most of these consistency results use standard forcing notions.) Several interesting consistency questions remained open, however—in particular, Con($\mathfrak{d} < \mathfrak{o}$) was answered positively (add random reals over a model of CH), but Con($\mathfrak{o} < \mathfrak{d}$) remained open. Results will be presented in this direction, based on Shelah's proof of Con($\mathfrak{b} < \mathfrak{s}$) [1].

In this model (in its purest form), Shelah showed that $\mathfrak{a} = \mathfrak{b}$. The current results investigate adapting that proof to show $\mathfrak{o} = \mathfrak{b}$ in the same model.

[1] SAHARON SHELAH, *On cardinal invariants of the continuum*, ***Contemporary Mathematics***, vol 31, American Mathematical Society, New York, 1984, pp. 183–207.

▶ ADAM H LEWENBERG, *Embedding partial groups in groups.*
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A **partial group** is a triple (X, p, i) where X is a set, $p : U \to X$ and $i : V \to X$, with $U \subseteq X \times X$ and $V \subseteq X$, are functions, such that, writing ab for $p(a, b$ and a^{-1} for $i(a)$, the following equations are true for all a, b, and c in X (when both sides of the equation are defined):

$$(ab)b^{-1} = a, a^{-1}(ab) = b \text{ and } a(bc) = (ab)c.$$

Using a nonstandard-analysis argument, I show that when a partial group is a group "almost everywhere," then the partial group can be embedded in a group of only slightly larger size.

▶ ISABEL LUENGO, *Symbols and icons in geometry proofs.*
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Diagrams used in geometry proofs are finite combinations of three kinds of objects: iconic objects, symbolic objects and labels. Iconic objects represent objects of geometry like points, lines, angles, polygons, etc. Symbolic objects are conventional devices used to represent properties and relations between geometry objects such as the property of being a right angle or the congruence relation among segments. Labels establish the link between the diagrams and the sentences at the semantic level. In this paper I am just going to consider strictly diagrammatic systems where no language is involved, so I am going to focus on the features of the systems arising from the distinction between iconic and symbolic objects.

In particular, any diagrammatic inference system has rules to transform the diagrams into new diagrams. The transformation rules of the system can be divided into two exhaustive classes: *construction rules* that add new iconic objects to the diagram, and *inference rules* that do not add new iconic objects to the diagram. That is, the latter only add symbolic and labels or they may erase information. Whereas the traditional notion of logical consequence is sufficient to account for the validity of inference rules we need a new notion of consequence that I call *geometric consequence* to account for the validity of construction rules.

▶ ALEX. M. MCALLISTER, *Coding sets for the recursive subtrees of* $2^{<\omega}$.
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A consequence of the lemma of Lachlan and Soare in [1] is: if $\mathcal{X}_{0'}$ is a path through the recursive subtree of $2^{<\omega}$, then one can "code" a set using the paths through the subtree. In particular, there exists a recursive set B such that for all $h \in 2^B$, there exists a path f through the recursive subtree with $h \subseteq f$. The proof of this lemma requires the Recursion Theorem and is dependent on a specific definition of $\mathbf{0}'$.

However, there exists a different "coding strategy" that does not involve the Recursion Theorem and that enables one to construct other sets which provide a slightly stronger form of coding.

THEOREM 1. *There exists $C \subset \omega$ such that if T is a recursive subtree of $2^{<\omega}$ and C is a path through T, then there exists $\sigma \in T$ such that $\sigma \subset \mathcal{X}_C$ and for all $\tau \supseteq \sigma$, $\tau \in T$*

Such a set C is called a coding set for the recursive subtrees of $2^{<\omega}$ Further, this coding strategy can be combined with a permitting argument to give us the following result.

THEOREM 2. *If $Y \leq_T \emptyset'$, then there exists $C \leq_T Y$ such that C is a coding set for the recursive subtrees of $2^{<\omega}$*

One consequence of this theorem is that, surprisingly enough, there exist $\aleph_0$ many different coding sets for the recursive subtrees of $2^{<\omega}$.

[1] A. H. LACHLAN AND R. I. SOARE, *Models of arithmetic and upper bounds of arithmetic sets*, ***Journal of Symbolic Logic***, vol. 59, no. 3 (1994), pp. 977–983.

▶ MONICA MCARTHUR, *Thresholds for 0-1 laws for $L^k_{\infty,\omega}$ on random graphs.*
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Using general theorems which characterize the existence of an $L^k_{\infty,\omega}$ 0-1 law (due to Kolaitis and Vardi) and an $L^k_{\infty,\omega}$ convergence law on a class of finite models with arbitrary measure, we establish some results about both 0-1 laws and nonconvergence for $L^k_{\infty,\omega}$ on $G_{n^{-\alpha}}$, the random graph with edge probability $p(n) = n^{-\alpha}$, where α is rational and $0 < \alpha \leq 1$. Shelah and Spencer showed that for such $\alpha < 1$ these classes $G_{n^{-\alpha}}$ do not have first-order convergence laws, and thus they do not have $L^\omega_{\infty,\omega}$ convergence laws. However, when we consider $L^k_{\infty,\omega}$ instead, we can obtain some bounds on k in terms of α which guarantee the existence of $L^k_{\infty,\omega}$ 0-1 laws; we also obtain bounds on how large k must be to guarantee nonconvergence for $L^k_{\infty,\omega}$ on $G_{n^{-\alpha}}$. For $\alpha = 1/m$, m a positive integer, we obtain a complete characterization of the asymptotic of $L^k_{\infty,\omega}$ on $G_{n^{-\alpha}}$. if $k \leq m$, there is an ∞, ω^k 0-1 law on $G_{n^{-\alpha}}$; otherwise there is a $L^k_{\infty,\omega}$ nonconvergence. Thus we establish a single threshold for k such that there is $L^k_{\infty,\omega}$ convergence on $G_{n^{-1/m}}$. For $G_{n^{-\alpha}}$, $\alpha = a/b$, a, b relatively prime positive integers with $a < b$, we establish bounds on the region in which such thresholds exist, but do no obtain a complete characterization.

These results fill in much of the remaining gaps in the knowledge of asymptotic behavior of $L^k_{\infty,\omega}$ on $G_{n^{-\alpha}}$: Lynch has shown that $G_{n^{-\alpha}}$ has an $L^\omega_{\infty,\omega}$ 0-1 law for $\alpha > 1$, and Lynch and Tyszkiewicz have announced a 0-1 law for $L^\omega_{\infty,\omega}$ on $G_{n^{-\alpha}}$, α irrational, $0 < \alpha < 1$.

▶ GARVIN MELLES, *Set Turing machines.*
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Classification by set recursive functions and Shelah's classification theory are intimately related. For example a class of $\aleph_1$ models of a superstable theory T has a set recursive classification relative to the reals and the cardinality function provable in ZFC if and only if T has $NDOP$. However, the notion of set recursion that is used was originally defined in a somewhat ad hoc way. Here we outline the proof that the class of set recursive functions and the more naturally defined class of set computable functions are equivalent.

▶ GRZEGORZ MICHALSKI, *A metetheorem for finite-injury priority arguments*
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The paper presents a theorem which seems to capture essential combinatorial features common to all known finite-injury priority arguments in recursion theory and recursive model theory. In this sense the theorem can serve as a metatheorem for all such arguments, providing a general framework for them.

There are similar metatheorems of Ash and Knight (cf [1], [2], [3]) designed to do complicated α-nested priority arguments – they do not however apply to the classical Friedberg-Muchnik theorem. The statement of our metatheorem is as follows

THEOREM 1. *Suppose T is an r.e. tree, $I \subseteq T$ is an r.e. relation and $E : T \to \mathcal{P}^{<\omega}(\omega)$ is a partial recursive function. Suppose also that*

(1) *For every node $l \in T$ there is a node $l \in T'$ with $E(l) \subseteq E(l')$ which is either an immediate successor of l or satisfies $I(l, l')$*

(2) *There is no node $l \in T$ with a recursive sequence $(l_k)_{k\in\omega}$ of immediate successors of l such that for all $k \in \omega I(l_k, l_{k+1})$ and $E(l_k) \subseteq E(l_{k+1})$.*

Then, there is a Δ^0_2 path p through T, such that $E(p) := \cup_{n\in\omega} E(p \restriction n)$ is r.e.

To show how this theorem works, we consider a sample of finite-injury priority arguments; including the Friedberg-Muchnik theorem, two more theorems from recursion theory, and the following theorem from recursive model theory (cf [4], [5]):

THEOREM 2 (Goncharon, Peretuat'kin). *Let $\mathbf{A}$ be a countable homogeneous structure and let $\mathcal{E}$ be an enumeration of the complete types realized in $\mathbf{A}$. Let f be a function such that if $\mathcal{E}_n = \mathcal{E}(\bar{U})$, x is the first variable not in $\bar{u}$ and $\exists x\varphi(\bar{u}, x) \in \mathcal{E}_n(\bar{u})$, then $\mathcal{E}_{f(n,\varphi)}(\bar{u}, x) \supseteq \mathcal{E}_n(\bar{u}) \cup \{\varphi(\bar{u}, x)\}$. If $\mathcal{E}$ and f are recursive, then there is a decidable $\mathbf{B} \cong \mathbf{A}$.*

[1] C. J. ASH, *Labeling systems and r.e. structures*, ***Annals of Pure and Applied Logic***, vol 47 (1990), pp. 99–119.

[2] C. J. ASH and J. F KNIGHT, *Mixed systems*, ***Journal of Symbolic Logic***, to appear.

[3] C. J. ASH and J F. KNIGHT, *Mixed ramified systems*, ***Annals of Pure and Applied Logic***, to appear

[4] S. S. GONCHAROV, *Strong constructivizability of homogeneous models*, ***Algebra i Logika***, vol. 17 (1978)

[5] M. G PERETYAT'KIN, *Criterion for strong constructivizability of a homogeneous model*, ***Algebra i Logika***, vol 17 (1978).

▶ CHARLES MORGAN, *Historicized forcing and pseudo-morasses.*

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In this talk I shall reconsider historicized forcing and explore its connections with pseudo-morasses. Some applications in general topology and combinatorics will be given.

▶ ITAY NEEMAN, *Woodin cardinals and determinacy.*

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There are essentially two different techniques used in proofs of determinacy from Woodin cardinals. The first, due to Martin-Steel, uses iteration trees and by itself suffices to show that the existence of n Woodin cardinals and a measurable cardinal above implies Π^1_{n+1}-determinacy. The other, due to Woodin, uses among other things the technique of stationary tower forcing (developed by Foreman, Magidor, Shelah and Woodin) to obtain such results as $L[R^*] \models AD$ assuming that δ is the sup of ω Woodin cardinals, and R^* represents the reals in the symmetric collapse up to δ. (More general results can also be obtained.)

I shall show how the Martin-Steel method can be adapted to yield some of Woodin's results.

► KRISTER SEGERBERG, *AGM as a dynamic, doxastic logic.*

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Dynamic doxastic logic is the result of allowing the agent of Hintikka type doxastic logic three types of doxastic action: expanding by a proposition, revising by a proposition and contracting by a proposition. Within this framework, theories of belief revision can be represented. In particular, one may represent the famous theory of Alchourrón, Gärdenfors and Makinson (AGM) in this way.

► SLAWOMIR SOLECKI, *A characterization of the meager ideal.*

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This is an account of a joint work with A. S. Kechris. We find all σ-ideals on a Polish space which are generated by closed sets and fulfill the countable chain condition. We derive from it the following characterization of the meager ideal: given a Polish group G, the meager ideal is the unique translation invariant σ-ideal on G which is generated by closed sets and fulfills the countable chain condition. This gives a partial solution to a problem of Kunen. Balcerzak and Rogowska found another application of the main result solving an old problem of Mauldin about the Baire order of families of functions whose sets of discontinuity points belong to a given σ-ideal.

► OTMAR SPINAS, *Partitioning products of $\mathcal{P}(\omega)/fin$*

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We generalize the cardinal invariant a to products of $\mathcal{P}(\omega)/\mathrm{fin}$ and then sharpen Solomon's inequality $b \leq a$ by proving $b \leq a(\lambda)$ for every $\lambda \leq \omega$. Here $a(n)$, for $0 < n < \omega$, is the least size of an infinite partition of $(\mathcal{P}(\omega)/\mathrm{fin})^n$, $a(\omega)$ is the least size of an uncountable partition of $(\mathcal{P}(\omega)/\mathrm{fin})^\omega$ (there are always countably infinite partitions of $(\mathcal{P}(\omega)/\mathrm{fin})^\omega$)), and b is the least size of an unbounded family of functions from ω to ω ordered by eventual dominance. We also prove the consistency of $b < a(n)$ for every $n < \omega$.

► KATHERINE ST. JOHN, *Iteration in classes of finite structures.*

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On the standard model of the natural numbers, iteration and recursion are equally expressive (due to the Kleene Normal Form Theorem, see [1]). However, on classes of finite structures, iteration can be less expressive than recursion. Tiuryn [2], as the culmination of several results of others, for which references can be found in his paper, showed this for the class of complete binary trees. The class, $\mathcal{K}_T$, of complete binary trees is all of the finite Herbrand interpretations of the signature $\sigma = \{\text{root}, \text{left}, \text{right}, =\}$, where *root* is a constant, *left* and *right* are unary functions, and $=$ is the standard equality on elements. His elegant proof showed that iterative functions could only "see" exponentially (i.e., all) elements in the structure (see [3]). We extend this result to any class of Herbrand interpretations of a finite functional signature.

THEOREM 1. *For any finite functional signature, for every $k \in N$, and for every iterative function S with k variables, there exists a polynomial, $p(n)$ such that*

$$|Val(S, \mathcal{A}_n, \vec{a})| < p(n)$$

holds for all $n \in N$ and all $\vec{a} \in (\mathcal{A}_n)^k$, where $\mathcal{A}_n$ is the finite Herbrand interpretation of height n, $|S|$ is the length of S, and $Val(S, \mathcal{A}_n, a)$ are the values "seen" by $S(a)$

A corollary to these results gives polynomial bounds on the closure ordinals of iterative functions. Due to the work of Urzyczyn [3], this theorem is sharp.

[1] S. C Kleene, ***Introduction to metamathematics***, D. Van Nostrand Company, Inc., Princeton, NJ, 1952.

[2] Jerzy Tiuryn, *Simplified proof of $DDL < DL$*, ***Information and Computation***, vol. 81 (1989), pp. 1–12.

[3] Pawel Urzyczyn, *A necessary and sufficient condition in order that a Herbrand interpretation be expressive relative to recursive programs*, ***Information and Control***, vol. 56 (1993), pp. 212–219.

► JINDRICH ZAPLETAL, THOMAS JECH, AND BOHUSLAV BALCAR, *Approximations to Cohen algebras.*

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We define several properties of complete Boolean algebras which are weakenings of the property "being a Cohen algebra" and investigate their relationships We prove for example that the completion of a measure algebra from the ground model can be a Cohen algebra in an $\aleph_1$-preserving generic extension.

Abstracts presented by title

► MATIAS FRANCISCO DIAS, *Self-reference and diagonalization.*

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I give a new effective way of constructing true undecidable formulas of elementary arithmetic (PA), and remark that they are neither self-referential nor diagonalized. Let W_x be any r.e. set. fx is the recursive function whose range is W_x and $D_x(y, z)$ is the formula of PA that represents f_x. $g(u)$ is a recursive function that gives the Gödel number of a formula $A_u(u)$ which is the result of substituting in $A_u(Y)$ the numeral u for all free occurrences of the variable y. Let $G(x, y)$ represent in PA the function g. Ψ is a productive partial recursive function for the set $V = \{x\ .\ A_x$ is true in the standard model of PA$\}$, such that for a r.e. set $W_x \# \emptyset$ and $W_x C V$, $\Psi(x0$ gives the Gödel number of $\forall z \forall w \neg (D_x(z, w) \vee G(p, w))$ where p is the Gödel number of $\forall z \forall w \neg (D_x(z, w) \vee G(y, w))$. So $\Psi(x) = g(p)$. Of course the formulas obtained in this way are diagonalized and self-referential. However my next step is to construct true undecidable formulas that are neither diagonalized nor self-referential. Given W_x as above let us construct a r.e. set W_y in the following way: first define the sets $W_x \bigcup \{\Psi(x)\} = W_{x_1}$, $W_x \bigcup \{\Psi(x_1)\} = W_{x_2}$ and so on. Finally define $W_y = \{\Psi(x), y(x_1), \Psi(x_2), \dots\}$. $W_y C V$. Now let h be a recursive function whose range is W_y and let h be represented in PA by $H(z, w)$. Then consider the formula $\forall w [\exists t \exists y (H(y, t) \wedge G(w, t) \Rightarrow \forall y \forall z \neg (^{D_x}(y, z) \wedge G(w, z))]$. We should note that such formulas are neither self-referential nor diagonalized, although they are true and undecidable. Once more diagonalization and self-reference do not seem to be a good criterion for undecidability in PA.

[1] Matias Francisco Dias, *On elementary arithmetic*, ***Journal of Symbolic Logic***, vol. 59, no. 1 (1994), pp. 716–717, annual meeting of the ASL abstracts of papers: Notre Dame, 1993.

► V. KANOVEI AND M. REEKEN, *Internal approach to external sets and universes.*

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It has been considered as a problem for nonstandard set theories how external sets can be conveniently incorporated in the internal universe. To solve this problem, we use *Bounded set theory* **BST**, a nonstandard set theory which differs from **IST** of Nelson so that the "standard" variable in Idealization is restricted by a standard set as the domain, and the axiom which postulates that every set belongs to a standard set is added. **BST** is a conservative and equiconsistent extension of **ZFC**.

Let **HST** be the external theory NS_1 of Hrbaček, plus Choice in the case when the choice function has a set of standard size as the domain. (A *set of standard size* is an image of a set of the form $\{x \in X : \mathrm{st}x\}$, where X is standard.) In particular **HST** contains Separation and Replacement for all st-$\in$-formulas, and Saturation for families (of internal sets) of standard size.

THEOREM 1. *HST has an interpretation in BST such that the class of all internal sets in the sense of the interpretation coincides with the ground BST universe*

In other words, a **BST** spectator can legitimately consider things so that his universe is embedded, as the class of all internal sets, in a universe of **HST**, a theory convenient enough to deal with external sets.

► GUNWON LEE, *Silk ball orders.*

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E-mail: `gwl@krsnuccl.bitnet`.

We pursue the orders which can be verifiable in our every day life such as the four dimensional coordinates in use. The points can be ordered as enumerations of the coordinations in the form of random collections. The extensions of the diagonal methods in the form of a silk ball are the finite subsets of the spatio-temporal continuum of Aristotle-type. The connections inbetween silk ball orders and the continuum are uncleared.

► ZHOU XUNWEI, *Brief introduction to geometrical logic.*

Department of Computer and Automation, College of Automation Engineering, Beijing Union University, Beijing 100009, P.R. China

Geometrical logic is constructed by the author. It is a geometrical approach to logic. Geometrical logic deems that human knowledge is divided into four levels: terms, fact, empirical connection and logical connection, which can be graphically represented by four coordinate spaces which constitute a hierarchical coordinate system

Human knowledge is denoted by prepositions, geometrical figures are made by the locus of moving points. In geometrical logic, one-to-one correspondence is established between propositions and geometrical figures through the hierarchical coordinate system, so that human knowledge is described by locus of moving points.

THE BULLETIN OF SYMBOLIC LOGIC
Volume 1, Number 3, Sept 1995

NOTICES

- THE 1995–96 ASL WINTER MEETING will be held January 12–13, 1996, in conjunction with the Annual Meeting of the American Mathematical Society, in Orlando, Florida. The dates of the AMS meeting are January 10–13. The Program Committee for the ASL meeting consists of J Larson, W. Mitchell, T. Slaman, and C. Wood. Abstracts of contributed papers from ASL members (300 words, one page limit) should be sent by the deadline of October 20, 1995, to the Program Chairman: William J. Mitchell, Department of Mathematics, University of Florida, Gainesville, Florida 32611; email: `mitchell@math.ufl.edu`.
- THE 1995–96 ASL ANNUAL MEETING will be held March 9–12, 1996, at the University of Wisconsin in Madison, Wisconsin. The Program Committee consists of S. Buss, P. Cholak, H. J. Keisler, Chair, D. A. Martin, and M. Resnik. Chair of the local organizing committee is Steffen Lempp. Abstracts of contributed papers from ASL members (300 words, one page limit) should be sent by the deadline of January 12, 1996, to: H. J. Keisler, Mathematics Department, University of Wisconsin, Madison, Wisconsin 53706; email. `keisler@math.wisc.edu`.
- THE 1996 ASL EUROPEAN SUMMER MEETING (LOGIC COLLOQUIUM '96) will be held July 9–15, 1996, in Donostia, San Sebastián, Spain. (Note the change in dates from earlier announcements) The Program Committee consists of K. Ambos Spies, J. L. Balcazar, J. E. Fenstad, D. Israel, H. Kamp, R. Kaye, J M. Larrazabal, D. Lascar, Chair, A. Marcja, G. Mints, M Otero, S. Ronchi della Rocca, K. Segerberg, and L Vega. The Organizing Committee consists of X. Arrazola, A. Arrieta, R. Beneyto, B. Carrascal, K. Korta, J M. Larrazabal, Chair, J. C. Martinez, J. M. Mendez, F. Migura, and J. Perez Topics of the main lectures include Model Theory, Proof Theory, Recursion and Complexity, Models of Arithmetic, Logic for Artificial Intelligence, Formal Semantics of Natural Language, and Philosophy of Contemporary Logic. Contributed papers are invited from all areas of Logic. Abstracts (300 words, one page limit) should be submitted by the deadline of April 1, 1996, to the address below The registration fee is 28,000 ptas. (35,000 ptas. after April 1). Students and accompanying persons may register for 14,000 ptas. (18,000 ptas. after April 1). Further information may be obtained from: Jesus M. Larrazabal, Logic Colloquium '96, ILCLI, University of the Basque Country, Apdo 220, 20080 Donostia, San Sebastián, Spain; telephone: +34 43 320940; Fax: +34 43 293677; email: `ilcli@sf.ehu.es`.
- THE 1996–97 ASL WINTER MEETING will be held in conjunction with the Annual Meeting of the American Mathematical Society during January 8–11, 1997, in San Diego, California.
- THE 1996–97 ASL ANNUAL MEETING will be held March 22–25, 1997, at the Massachusetts Institute of Technology in Cambridge, Massachusetts Chair of the local organizing committee is Sy Friedman.
- THE 1997 ASL EUROPEAN SUMMER MEETING (LOGIC COLLOQUIUM '97) will be held in early July, 1997, in Leeds, England. The local organizing committee consists of B. Cooper, J. Derrick, F. Drake, D. Macpherson, A. Slomson, J Truss, and S. Wainer
- A MEETING ON THE LOGICAL FOUNDATIONS OF MATHEMATICS, COMPUTER SCIENCE, AND PHYSICS – KURT GÖDEL'S LEGACY (GÖDEL '96) will be held August 25–29, 1996, in the city of Gödel's

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birth, Brno, Czech Republic. The conference is organized by Masaryk University in cooperation with the international Kurt Gödel Society, which is based in Vienna. Chair of the Organizing Committee is J. Zlatuska. The Program Committee consists of Z. Adamovicz, J. Bicak, L. Bukovsky, D. de Jongh, J. Grygar, E. Koehler, J. Krajicek, P. Hájek, Chair, A. Leitsch, D. Mundici, G. Mueller, J. Paris, and C. Parsons. Submissions of full draft papers should be sent by the deadline of January 14, 1996, to the Program Chair: Petr Hájek, Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod vodarenskou vezi 2, 182 07 Prague, Czech Republic; email: `goedel96-program@uivt.cas.cz`; telephone: +42-2-66414244, +42-2-66413418; Fax: +42-2-8585789. Further information may be obtained by contacting: Jiri Zlatuska, Faculty of Informatics, Masaryk University, Buresova 20, 602 00 Brno, Czech Republic; email: `goedel96@informatics.muni.cz`; telephone: +42-5-41213125, +42-5-41211646, +42-5-41213219; Fax: +42-5-41212747; WWW: `http://www.fi.muni.cz/~zlatuska/goedel96.html`. This meeting is co-sponsored by the Association for Symbolic Logic.

- THE TENTH AMSTERDAM COLLOQUIUM will be held December 18–21, 1995, at the University of Amsterdam. This biennial meeting is intended to foster interdisciplinary research among logicians, philosophers, linguists, and computer scientists. The invited speakers include R. Carpenter, G. Chierchia, J. Halpern, H. Kamp, A. Kratzer, F. Landman, B. Partee, K. Segerberg, and A. Szabolsci For further information contact: Organizing Committee, Tenth Amsterdam Colloquium, ILLC/Department of Philosophy, University of Amsterdam, Nieuwe Doelenstraat 15, 1012 CP Amsterdam, The Netherlands; telephone: +31 20 5254541; Fax: +31 20 5254503; email: `AcTen@illc.uva.nl`, WWW: `http://www.fwi.uva.nl/fwi/research/vg2/illc/acten.html`.

- THE 5TH INTERNATIONAL WORKSHOP ON EXTENSIONS OF LOGIC PROGRAMMING (ELP'96) will be held March 28–30, 1996, in Leipzig, Germany. The program co-chairs are R. Dyckhoff (St. Andrews), H. Herre (Leipzig), and P Schroeder-Heister (Tübingen). For further information write to ELP'96, Heinrich Herre, Institut für Informatik, Universität Leipzig, Augustusplatz 10–11, D–04109 Leipzig, Germany; telephone: +49 341 973 2201; Fax: +49 341 973 2209; email: `elp96@informatik.uni-leipzig.de`; WWW: `http://www-theory.dcs.st-and.ac.uk/elp96.html`

- LUKASIEWISZ IN DUBLIN will be held July 8–10, 1996, at University College Dublin, Ireland, immediately following the Joint Session of the Aristotelian Society and the Mind Association. This meeting commemorates the 40th anniversary of the death of the eminent logician and historian of logic, Jan Lukasiewicz, who spent his last years in Dublin. The meeting is being organized by the Department of Philosophy of University College Dublin and the European Society for Analytic Philosophy. Invited speakers include C. Lejewski, G. Malinowski, W. Marciszewski, G. Priest, P. Simons, T. Smiley, A. Weir, and J. Wolenski. Abstracts of 30–40 minute contributed papers (1 or 2 sides of A4 paper double spaced) should be submitted by November 30, 1995. For further information contact: Dr. Maria Baghramian, Department of Philosophy, University College Dublin, Dublin 4, Ireland; telephone: +353-1-7068125; Fax: +352-1-2693469; email: `Baghram@macollamh.ucd.ie`.

- THE SECOND CONFERENCE ON INFORMATION-THEORETIC APPROACHES TO LOGIC, LANGUAGE, AND COMPUTATION will be held July 18–21, 1996, at Regent's College, London. The Program Committee consists of J. Ginzburg, R. Koons, L. Moss, Chair, H. Nakashima, and M. de Rijke. Authors are invited to submit detailed abstracts of a full paper by the deadline of January 1, 1996, by email using the subject line `ITALLC96 Submission`, to the address `ITALLC96@cs.indiana.edu`; abstracts should be no more than 10 pages, and the cover page should include title, authors, and the coordinates of the corresponding author. Submissions in LaTeX are encouraged. For further information on the World Wide Web: `http://www.etl.go.jp:8080/etl/captain/ITALLC96`.

- The European Summer School in Logic, Language and Information (ESSLLI'96) will be held August 12–23, 1996, in Prague. Proposals for introductory and advanced courses to be offered at this summer school are sought by the program committee, which consists of. Language and Logic: Patrick Blackburn, Chair (Saarbrücken), `patrick@coli.uni-sb.de`; Language· Anne Abeille (Paris), `Anne.Abeille@linguist.jussieu.fr`; Language and Computation. Ted Briscoe (Cambridge), `ejb@linc.cis.upenn.edu`; Computation: Luis Damas (Porto), `luis@ncc.up.pt`; Logic and Computation: Dov Gabbay (London), `dg@doc.ic.ac.uk`; and Logic: Petr Hájek (Prague), `hajek@uivt.cas.cz`.

- A Program in Algebraic Model Theory will be held from August, 1996, to June, 1997, at the Fields Institute for Research in Mathematical Sciences, Toronto, Canada. The organizing committee consists of B. Hart, A. Lachlan, A. Macintyre, M. Makkai, R. McKenzie, and M Valeriote. During the year workshops are planned in Geometric Model Theory (January 13–17, 1997) and The Model Theory of Analytic Functions (March 17–21, 1997), and there will be a general workshop during May 26–June 9, 1997. Participants include G. Cherlin, L. van den Dries, R. Freese, E. Hrushovski, D Marker, A Pillay, and R. Willard. Some support for participation by graduate students and postdoctoral fellows will be available. Further information may be obtained by writing to: Algebraic Model Theory Program, The Fields Institute, 185 Columbia Street W., Waterloo, Ontario, Canada N2L 5Z5; Fax: 519-725-0704; email: `model@fields.uwaterloo.ca`.

- A Master of Logic Program has been established at the Institute for Logic, Language and Computation (ILLC) of the University of Amsterdam. Courses and areas of specialization cover a very broad range of topics in logic. All courses are taught in English. For an informational brochure contact: ILLC, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands; email: `illc@fwi.uva.nl`; Fax: +31-20-525-5206.

- Volume 8 of the Cahiers du Centre de Logique entitled *The Curry–Howard Isomorphism*, and edited by Ph. de Groote, is available from the Université Catholique de Louvain, Academia, Louvain-la-Neuve, Belgium The volume is devoted to the Formulæ–as–Types correspondence, also widely known as the Curry–Howard isomorphism, which has become of interest to theoretical computer scientists, through the program–as–proof correspondence The ISBN number of this volume is 2-87209-363-X (1995) and it may be ordered from the address above for US $50.

- The Institute for Studies in Theoretical Physics and Mathematics (IPM) in Tehran, Iran, has started publishing volumes of possible interest to logicians. The first volume (234 pages) is the *Proceedings of the First Logic Congress in Iran (May 1990)* with articles by I. Hacking, W D. Hart, J Higginbotham, J. Stern, M. Boffa, R. Chuaqui, D R. Kurepa, K. Badie, A. Edalat, M. B. Smyth, and C. Lucas. The second volume (152 pages) is *A Course on Foundations of Nonstandard Analysis*, by V. Kanovei. The cost of each volume (including the postage fee) is $20 for hardcover, and $15 for softcover. For more information, contact IPM, P.O. Box 19395-1795, Tehran, Iran; email: `IPMPUB@IREARN.BITNET`.

THE BULLETIN OF SYMBOLIC LOGIC
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INNER MODELS AND LARGE CARDINALS

RONALD JENSEN

In this paper, we sketch the development of two important themes of modern set theory,[1] both of which can be regarded as growing out of work of Kurt Gödel. We begin with a review of some basic concepts and conventions of set theory.

§0. The ordinal numbers were Georg Cantor's deepest contribution to mathematics. After the natural numbers $0, 1, \dots, n, \dots$ comes the first infinite ordinal number ω, followed by $\omega + 1, \omega + 2, \dots, \omega + \omega, \dots$ and so forth. ω is the first *limit ordinal* as it is neither 0 nor a successor ordinal. We follow the von Neumann convention, according to which each ordinal number α is identified with the set $\{\nu \mid \nu < \alpha\}$ of its predecessors. The $\in$ relation on ordinals thus coincides with $<$. We have $0 = \emptyset$ and $\alpha + 1 = \alpha \cup \{\alpha\}$. According to the usual set-theoretic conventions, ω is identified with the first infinite cardinal $\aleph_0$, similarly for the first uncountable ordinal number ω_1 and the first uncountable cardinal number $\aleph_1$, etc.[2] We thus arrive at the following picture:

$$\begin{array}{cccccccc} 0,1,\dots, & \omega, & \omega+1,\dots,\omega+\omega,\dots, & \omega_1, & \dots, & \omega_2, & \dots \\ & \| & & \| & & \| & \\ & \aleph_0 & & \aleph_1 & & \aleph_2 & \end{array}$$

The *von Neumann hierarchy*[3] divides the class V of all sets into a hierarchy of sets V_α indexed by the ordinal numbers. The recursive definition reads: $V_0 = \emptyset$; $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ (where $\mathcal{P}(x) = \{z \mid z \subseteq x\}$ is the *power set* of x);

This article is based on the Gödel Lecture given at the meeting of the Association for Symbolic Logic at Berkeley in January, 1990. A German version of this paper, "Innere Modelle und Große Kardinalzahlen," was published in Jahresbericht der Deutschen Mathematiker-Vereinigung, Jubiläumstagung 100 Jahre DMV (B. G. Teubner, Stuttgart, 1992), pp. 265–280. This English translation was prepared by Andreas Blass. It is published by permission of B. G. Teubner.

[1] This paper and the lecture on which it is based are not intended as historical surveys. In particular, no attempt has been made to give proper credit for many of the early contributions to the subject.

[2] ω_ν is the νth infinite ordinal number of cardinality greater than all its predecessors and is identified with the corresponding cardinal number $\aleph_\nu$.

[3] This hierarchy was introduced independently by Zermelo.

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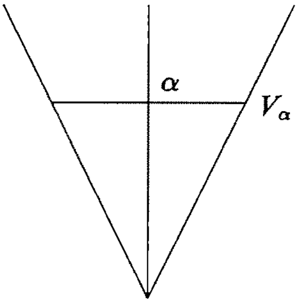
$V_\lambda = \bigcup_{\nu<\lambda} V_\nu$ for limit ordinals λ.[4] We can represent this hierarchy by the following picture.



The V_α increase in width as α increases. The ordinal numbers, here drawn as a straight line, grow linearly, since in the transition from α to $\alpha+1=\alpha\cup\{\alpha\}$ exactly one new element is added. The *axiom of foundation* in set theory asserts that $V=\bigcup_\alpha V_\alpha$.

This structure has two complementary motivations. The iteration can be understood as a generalization of analysis, since the real numbers can be represented in $\mathscr{P}(\omega)$, subsets and functions on $\mathbb{R}$ in $\mathscr{P}\mathscr{P}(\omega)$, etc. On the other hand, set theory can be regarded as a relativization of finite combinatorics, where the concepts "finite" and "infinite" are replaced by "set" and "proper class." From this point of view it is particularly easy to motivate the usual axioms of set theory,[5] for example the axiom of separation which says that a subclass of a set is a set or the axiom of replacement which says that the image of a set under a function is a set. It is perhaps not a coincidence that the young Cantor was influenced by Kronecker as well as by Weierstraß.

§1. The most famous question in set theory is the continuum problem. The *continuum hypothesis* (CH) asserts that the cardinal number Card($\mathbb{R}$) of $\mathbb{R}$ has the smallest possible value ω_1. An equivalent formulation is $\mathrm{Card}(\mathscr{P}(\omega))=\omega_1$. The *generalized continuum hypothesis* (GCH) asserts that $\mathrm{Card}(\mathscr{P}(\omega_\nu))=\omega_{\nu+1}$ for all ν. GCH implies the axiom of choice (AC). Despite great efforts, Cantor did not succeed in deciding whether CH is true, and neither did his successors. In this century, the suspicion grew that CH might be independent of the axioms of set theory, i.e., that these axioms do not suffice to prove either the continuum hypothesis or its negation. This suspicion was based on the apparent insufficiency of the axioms to adequately characterize the power set operation. It proved to be correct, and the two research directions described in this article can be regarded as attempts to understand power sets better.

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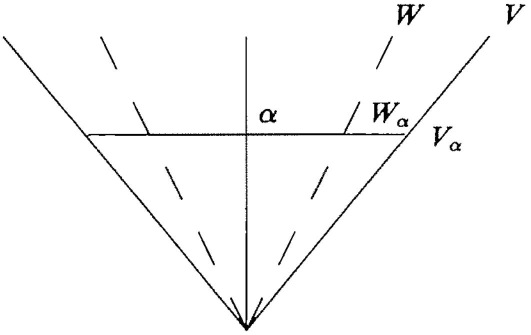
[4] *Urelements* or non-sets are not needed in mathematics. If we want to have them, we must modify the definition so that they occur in V_0.

[5] By this I mean the axioms of Zermelo-Fraenkel (ZF), possibly extended by the axiom of choice (ZFC).

The first progress came in 1939, when Gödel showed that GCH is consistent with the axioms. Then the unprovability of CH was shown by Cohen in 1963. Thus the independence was proved. Gödel's proof was the first application of the concept of *inner models*. This concept arises from an attempt to circumscribe the amorphous power set operation: one replaces the V_α-hierarchy with a new hierarchy W_α, defined just like V_α except that, in the transition from W_α to $W_{\alpha+1}$, not all subsets of W_α are adjoined but only those that are obtained by means of certain "well-understood" and clearly specified operations. $W = \bigcup_\alpha W_\alpha$ is then a "slice" of V, as indicated in the following picture.



We can hope that, for a suitable choice of these operations, W will *formally* satisfy all the axioms of set theory. In this case we call W an inner model.[6] Gödel defined the inner model $L = \bigcup_\alpha L_\alpha$ of *constructible* sets. $L_{\alpha+1}$ is the collection of subsets of L_α that can be obtained from the members of $L_\alpha \cup \{L_\alpha\}$ by *constructive* set operations. A pleasant property of this concept is its invariance with respect to different notions of constructivity. (For constructivists disagree as to what "constructive" means.) The ordinal numbers form a non-constructive ingredient in the definition of constructibility, and so we arrive at the same class L by using any "reasonable" notion of constructivity. It follows that many versions of the hierarchy L_α are possible. One frequently used definition sets $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$ = the set of subsets of L_α definable in first-order logic over the structure $\langle L_\alpha, \in\rangle$ with the use of arbitrary parameters from L_α.

Gödel showed that L is an inner model. He also showed that the definition of L is *absolute* in every inner model W in the following sense. If we interpret the definition of L in W rather than in V, then we obtain a class $L_W \subseteq W$. Absoluteness means that $L_W = L$. In particular, it follows that $L_L = L$.

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[6] The language of set theory is a first-order language with the predicates $=$ and $\in$. We interpret this language in W by restricting all quantifiers to range over W while the atomic formulas retain the same meaning in W that they have in V.

But then the statement $V = L$ holds in L, for this means just that $V_L = L_L$, where $L_L = L$ and $V_L = \{x \mid x = x\}_L = L$. So the axioms of set theory remain consistent if we adjoin the additional axiom $V = L$, called the *axiom of constructibility*. Gödel deduced GCH from $V = L$. So GCH is consistent relative to the other axioms. From GCH follows AC, which asserts that every set can be well-ordered. Gödel went further and showed, as a consequence of $V = L$, that the real continuum $\mathbb{R}$ has a *definable* well-ordering, and in fact one whose definition can be formulated in second-order number theory.[7] (The precise complexity of this definition is specified by its level, Σ^1_2, in the projective hierarchy.)

Gödel's discovery came shortly before the start of World War II and constituted a coda for the development of naive set theory in the early decades of this century. In the next twenty years, set theory was quiescent. In the sixties, after Cohen's discovery had revived interest in set theory, we discovered a series of strong and applicable combinatorial principles that follow from the axiom $V = L$. These principles, which can be regarded as natural strengthenings of GCH, are used in the infinitary parts of algebra, topology, and combinatorics. (The best known of these principles, called $\diamondsuit$, permits constructions in which the $\geq \omega_2$ subsets of ω_1 are diagonalized in ω_1 steps.)

§2. According to Gödel's theorem, the negation of CH is not provable. It remained to show that CH is not provable. But this could not be shown with the method of inner models, since precisely the success of this method precluded such an application: By absoluteness, we have $L = L_W \subseteq W$ for every inner model W. So $V = L$ says that V is the only inner model. As $V = L$ is consistent and implies CH, we cannot prove the existence of an inner model in which CH is false. To negate CH, we must apparently find an "outer model" that extends V. But how is this possible if V already contains all sets? The solution to this puzzle was given by Cohen's *forcing method*. Although forcing is somewhat outside the scope of this paper, I shall try to give a sketch of it (which the reader may skip without loss of continuity).[8] Instead of working with sets, one can consider their characteristic functions. This suggests an alternative to the V_α hierarchy: We define a hierarchy V^2_α of partial characteristic functions. As before, we set $V^2_0 = \emptyset$ and $V^2_\lambda = \bigcup_{\nu<\lambda} V^2_\nu$ for limit ordinals λ. At successor ordinals, we set $V^2_{\alpha+1}$ = the set of all $f : V^2_\alpha \to 2$. Finally, we define $V^2 = \bigcup_\alpha V^2_\alpha$. Then V^2 is "essentially" isomorphic to V, which can be made precise as follows. Each $f \in V^2$ is defined on a subset V^2_α of V^2. Extend f to a total function f^* on V^2 by setting $f^*(x) = 0$ wherever $f(x)$ is undefined. Define an *equality relation* I on V^2 by letting fIg iff $f^* = g^*$, and define a pseudo-$\in$ relation E by fEg iff $g^*(f) = 1$. Then I is a congruence relation on $\langle V^2, E\rangle$, and the quotient structure $\langle V^2, E\rangle/I$ is isomorphic to V.

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[7] Second-order number theory is the theory of the structure $\langle \mathcal{P}(\omega), \omega, +, \cdot, \in \rangle$. (Real numbers can be identified with elements of $\mathcal{P}(\omega)$.)

[8] Cohen's original version looks somewhat different from the version presented here, which is based on a later analysis by Solovay.

This analysis suggests a strategy for extending V^2. $2 = \{0, 1\}$ is the smallest Boolean algebra. If we carry out the same construction with another complete Boolean algebra $\mathbb{B}$ in place of 2, we obtain a hierarchy $V^{\mathbb{B}}_\alpha$. ($V^{\mathbb{B}}_{\alpha+1}$ is then defined as the set of all $f : V^{\mathbb{B}}_\alpha \to \mathbb{B}$.) We set $V^{\mathbb{B}} = \bigcup_\alpha V^{\mathbb{B}}_\alpha$. Then we have $V^2 \subseteq V^{\mathbb{B}}$ because $2 \subseteq \mathbb{B}$. Somewhat more complicated definitions than before give an "equality relation" I and a "pseudo-$\in$ relation" E. These are, however, not relations in the usual sense but "$\mathbb{B}$-valued relations," i.e., binary functions whose values lie in $\mathbb{B}$. In other words, we now assign to each statement of the form $x = y$ or $x \in y$ (with $x, y \in V^{\mathbb{B}}$) a *truth value* in $\mathbb{B}$. Then we can recursively define, for each statement φ of the language of set theory, a truth value $\|\varphi\| \in \mathbb{B}$. [9] ($\|\varphi\|$ may lie strictly between 0 and 1.) It turns out that all the axioms of set theory have truth value 1 (including AC if we assume AC in V) and the property of having truth value 1 is preserved by logical deduction. However, a suitable choice of $\mathbb{B}$ makes $\|\mathrm{CH}\| = 0$. This shows that CH does not follow from the axioms. (The consistency of CH with the axioms could also have been established by the forcing method, for one can choose $\mathbb{B}$ so that $\|\mathrm{CH}\| = 1$.)

Forcing also led to the discovery of useful combinatorial principles. The best known of these is Martin's Axiom (MA). The strength of MA depends on the size of the continuum. MA is a trivial consequence of CH. If CH is false, then MA has far-reaching consequences (but remains consistent relative to the axioms). MA is often a useful complement to the combinatorial principles that hold in L. For example, the independence of various statements can be proved by showing that one direction follows from $\diamondsuit$ and the other from MA and $\neg$CH.

§3. The forcing method proved to be an extraordinarily powerful tool for obtaining independence results in set theory and led to a renaissance of this field. It was an exciting time for the young mathematicians who flocked into this field. Major classical problems were solved with gratifying regularity. In the long run, though, it is somewhat unsatisfying to prove only that statements are unprovable. Increasingly, people asked whether the weakness of the set-theoretic axioms could be alleviated by accepting new axioms. An obvious candidate would be Gödel's axiom of constructibility, $V = L$. This

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[9] For example, we set $\|\neg\varphi\| = \neg\|\varphi\|$ =the complement of $\|\varphi\|$ in $\mathbb{B}$; $\|\varphi \wedge \psi\| = \|\varphi\| \cap \|\psi\|$; $\|\forall v\, \varphi(v)\| = \bigcap_{x \in V^{\mathbb{B}}} \|\varphi(x)\|$. Once this definition is given, the $\mathbb{B}$-valued relations are uniquely characterized by the requirements that the axioms of equality logic and the axiom of extensionality have truth value 1 and that $\|x \in y\| \subseteq \bigcup_{z \in \mathrm{dom}(y)} \|x = z\|$ for $x, y \in V^{\mathbb{B}}$.

axiom makes a clear statement about the nature of the mathematical universe. It is mathematically fruitful in that it solves many problems and leads to interesting new concepts and theories. It is philosophically attractive for adherents of "Ockham's razor," which says that one should avoid superfluous existence assumptions. I personally find it a very attractive axiom. Nevertheless, it has been rejected by the majority of set theorists, beginning with Gödel himself. One could perhaps summarize Gödel's objection as follows. The universe of sets is too large and varied to be fully grasped by the human spirit. Over time, however, mathematical intuition improves and with it our ability to describe this universe. The steps of this approximation process are marked by the axioms that we recognize as true. The axiom of constructibility presents, modulo the ordinal numbers, a complete description of sets and is thus a limiting principle that would halt the approximation process. Instead, one should look for additional existence axioms, called strong axioms of infinity. A typical axiom of this sort asserts the existence of a large cardinal number. As an example, Gödel suggests the inaccessible cardinals. As the universe satisfies all axioms of set theory, it should be so big that even some initial segment V_κ also satisfies the axioms. If V_κ satisfies the axioms (in the sense of second order logic[10]), then we call κ *inaccessible*. But if an inaccessible cardinal exists, then this statement also should be true in an initial segment V_τ, i.e., there should be two inaccessible cardinals κ and τ. By continuing this argument, we see that the inaccessible cardinals are unbounded in the ordinal numbers. But then also this assertion should hold in an initial segment, i.e., there is an inaccessible limit of inaccessible cardinals, etc.

Inaccessible cardinals are, however, not an alternative to the axiom of constructibility, for they are consistent with this axiom.[11] For this we need a principle saying that the universe is not only "long" but also "wide." The first large cardinal axiom proved to imply $V \neq L$ asserts the existence of a *measurable* cardinal. As the name implies, these cardinals originated from questions in measure theory. κ is measurable if and only if there is a κ-complete, non-trivial, 0,1-valued measure defined on all subsets of κ. (Or, equivalently, there is a κ-complete non-principal ultrafilter on κ.) The set-theoretic content of this concept becomes clearer in the following definition.

DEFINITION. κ is *measurable* if and only if there exist M and π such that

(1) M is an inner model.
(2) $\pi : V \to M$ is an elementary embedding.[12]
(3) κ is the critical point of π (i.e., $\pi \restriction \kappa = \mathrm{id}$, $\pi(\kappa) > \kappa$).

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[10] This means that V_κ is used as the domain of sets and $V_{\kappa+1}$ as the domain of classes

[11] If κ is inaccessible in V, then also in L.

[12] That is, $\varphi(x_1, \ldots, x_n)$ is true in V if and only if $\varphi(\pi(x_1), \ldots, \pi(x_n))$ is true in M

In the late fifties, Dana Scott showed that the existence of a measurable cardinal implies $V \neq L$. Later, it was shown that these cardinals have interesting consequences for the real continuum, e.g., that every Σ_2^1 set is Lebesgue measurable and has the Baire property. (Consequently, there can be no Σ_2^1 well-ordering of $\mathbb{R}$.)

We can generate a hierarchy of even stronger large cardinal principles by imposing additional conditions on the model M in the definition above. For example, we call κ *α-strong* iff $V_\alpha \subseteq M$. We call κ *strong* if it is α-strong for all α. κ is *superstrong* iff $V_{\pi(\kappa)} \subseteq M$. (This axiom is stronger than the previous one in the sense of *relative consistency* , i.e., the existence of superstrong cardinals implies the consistency of strong cardinals, but not conversely.)

These cardinals are the first stages of a hierarchy that has been intensively investigated in the last 20 years. Among the later stages are the compact and huge cardinals. The strongest principle of this sort was proposed by Reinhardt. We call κ a *Reinhardt cardinal* if $M = V$ in the definition above. At first, Reinhardt cardinals were greeted by many set theorists, including the author of this article, as the embodiment of an intuitively especially satisfactory existence principle. But after a few years Kunen proved that these cardinals do not exist. This set an upper limit for the hierarchy.

§4. Strong axioms of infinity say that the universe of sets is very large, and, as we have seen, they have important consequences for the real continuum. But one can also formulate axioms saying very directly that the continuum is large. The most successful principles of this sort are the so-called *axioms of determinacy*.

Let ω^ω be the set of all infinite sequences of natural numbers. To every set $A \subseteq \omega^\omega$ we associate a game as follows: Player I chooses a number $a_0 \in \omega$; Player II then chooses $a_1 \in \omega$; Player I chooses a_2; etc. After ω moves, the players have defined a sequence $a = \langle a_i | i < \omega \rangle$. Player I wins if $a \in A$; otherwise II wins. We say that A is *determined* if one of the two players has a *winning strategy*. A winning strategy for I, for example, is a function f, from finite sequences of natural numbers to natural numbers, such that I always wins if he plays $a_{2i} = f(a_1, a_3, \ldots, a_{2i-1})$ for all i. Strategies can be coded by real numbers, so the existence of a strategy for A is an existence postulate about $\mathbb{R}$. The simplest form of the axiom of determinacy (AD) asserts that every $A \subseteq \omega^\omega$ is determined. But AD implies that $\mathbb{R}$ cannot be well-ordered, so it cannot be accepted as an axiom. There are, however, various axioms of *definable* determinacy, saying that a certain class of "nicely definable" $A \subset \omega^\omega$ are determined. These seem to be consistent with the axiom of choice (AC) and have been intensively studied in the last twenty years. The axiom of *projective determinacy* (PD) says that the projective, i.e., Σ_ω^1, sets are determined. Naturally, Σ^1_n-AD means that the Σ^1_n sets are determined.[13] An especially attractive axiom is AD in $L[\mathbb{R}]$, which says that AD holds in the inner model $L[\mathbb{R}]$ that is defined exactly like L except that $L_0[\mathbb{R}] = \mathbb{R}$. This axiom is equivalent to the statement that every $A \in \mathcal{P}(\omega^\omega) \cap L[\mathbb{R}]$ is determined. It turns out that under this assumption the model $L[\mathbb{R}]$ becomes a paradise for analysts: Every set of real numbers is Lebesgue measurable and has the Baire property. AC is false there, but if we assume AC in V then $L[\mathbb{R}]$ still satisfies the weaker forms of AC—such as the axiom of dependent choice—that are needed by analysts.

It was soon discovered that there is a connection between determinacy and large cardinals. For example, AD implies the existence of an inner model with a measurable cardinal. (In fact, this cardinal is ω_1 in V.) In the other direction, Σ^1_1-AD follows from the existence of a measurable cardinal. The great hope of researchers in both fields was to deduce definable determinacy from the existence of sufficiently large cardinals. Until recently, however, this hope remained unfulfilled, and the three main areas of set theory—inner models, large cardinals, and determinacy—went their separate ways. Large cardinals probably attracted the most attention and were certainly the subject of very deep investigations. Nevertheless, this area remained in a certain sense the least satisfactory: It lacked—until recently—both the deep structure theory and the occasional dramatic shifts of perspective of the other two areas.

§5. The axiom of constructibility on the one hand and large cardinals and determinacy on the other embody two radically different conceptions of the universe of sets. How can these conceptions be justified? Most proponents of $V = L$ and similar axioms support their belief with a mild version of Ockham's razor. L is adequate for all of mathematics; it gives clear answers to deep questions; it leads to interesting mathematics. Why should one assume more? The proponents of strong axioms of infinity usually call themselves Platonists and use this to support their belief. I do not understand, however, why a belief in the objective existence of sets obligates one to seek ever stronger existence postulates. Why isn't Platonism compatible with the mild form of Ockham's razor cited above? Something else must be going on. I would like to propose a—necessarily somewhat speculative—hypothesis.[14] Could it be that the duality in modern set theory is nothing but a new manifestation of an ancient conflict between two points of view—I almost want to say two emotional states—which have always existed in mathematics? I call them the arithmetical and geometrical points of view. I also call the first one the Pythagorean point of view, for Pythagoras expressed it in its purest form: Everything consists of numbers. In other words, every mathematical structure can be interpreted in the natural structure of the positive integers. This idea is naturally very attractive; it gives to all of mathematics the intuitive clarity of the natural numbers. I conjecture that, if it could really be carried out, it would still be the dominant point of view today. In reality, however, the geometric point of view has been dominant since the rise of analysis. Thus I also call it the Newtonian point of view.[15] The Newtonian directs his gaze to the real rather than the natural numbers. He is less impressed by their clarity than by their boundless multiplicity. The real numbers constitute a gigantic, unfathomable sea. For every principle that generates real numbers, there must be a number not attainable by that principle. This excludes the possibility of an interpretation of the real numbers within the natural numbers.

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[13] Σ^1_0 consists of the Borel sets. Σ^1_{n+1} consists of projections of complements of Σ^1_n sets. $\Sigma^1_\omega = \bigcup_n \Sigma^1_n$.

[14] The reader can skip the rest of §5 without loss of continuity.

If one accepts set theory, then there is no doubt that Cantor has refuted Pythagoreanism in the strongest terms by showing that there are more real than natural numbers. But Cantor also introduced the ordinal numbers, which are in every sense the transfinite continuation of the natural numbers. They share much of the intuitive clarity of the natural numbers. Thus Cantor, who refuted the old Pythagoreanism, made possible a new Pythagoreanism in which the ordinal numbers take over the role of the natural numbers. In this sense, Gödel's axiom of constructibility seems to me to embody an entirely coherent Pythagorean picture of the world. And this picture cannot be refuted, for Gödel showed that $V = L$ is consistent if the other axioms are. But this axiom provides—modulo the ordinal numbers—a complete description of all sets and is therefore unacceptable for a Newtonian. For him, there must be a real number not generated by constructible processes. Since one cannot prove in set theory that such a number exists, one must seek new axioms. Thus, the ancient conflict is fought in a new arena.

Whether or not one accepts this analysis, there is no doubt that the quest for new axioms of infinity has led to very interesting mathematics. In addition, the proponents of these axioms have, through hard technical work, assembled a series of impressive plausibility arguments.[16] Nevertheless, I doubt that one could, with the sort of evidence that we have, convert the mathematical world to one or the other point of view.[17] Deeply rooted differences in mathematical taste are too strong and would persist.[18]

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[15] Of course one could also call it Leibnizian But perhaps Newton thought more geometrically.

[16] See §7, §8.

[17] The author confesses to being emotionally a Pythagorean.

[18] In the course of time I have had numerous opportunities to discuss the alternatives in set theory with mathematicians from other disciplines. I am always surprised how rapidly and

§6. Strong axioms of infinity should prevent the universe from being "constructed from below" like L. But do they really prevent this? It turns out that we can have a measurable cardinal in a very L-like model. The measurability of κ means that there is a normal ultrafilter U in $\mathcal{P}(\kappa)$.[19] Let the hierarchy L^U_α be defined like L_α except that $L^U_{\alpha+1} = \mathrm{Def}(\langle L^U_\alpha, \varepsilon, U \cap L^U_\alpha\rangle)$, and set $L^U = \bigcup_\alpha L^U_\alpha$. Then L^U is an inner model. In addition, κ is measurable in L^U, for $U \cap L^U \in L^U$ appears in L^U to be a normal ultrafilter. Silver and others have shown that L^U is thoroughly similar to L. It satisfies GCH. It has the combinatorial properties of L. (In fact, until now, no purely combinatorial difference between L and L^U has been discovered.) It has a well-ordering of the reals definable in second-order arithmetic (in this case Σ^1_3 instead of Σ^1_2). Even the absoluteness of L has an analog: L^U has a definition (using κ as a parameter) that is absolute in every inner model M with $L^U \subseteq M$.

But our definition of L^U does not by any means show that this model can be "built from below by well-understood operations," for the object U is explicitly used in the construction. That a construction from below is nevertheless possible to a certain extent was proved only later in the so-called *core model theory*. This theory is somewhat complicated, but I shall try to give a rough sketch of it. Let us assume not only that $V \neq L$ but also that there is a non-trivial elementary embedding $\pi : L \to L$.[20] It turns out that this assumption is equivalent to the existence of a certain set, denoted $0^\sharp$. In a very concrete way—which unfortunately cannot be described here—$0^\sharp$ encodes a complete description of the structure of L together with a canonical elementary embedding $\pi : L \to L$. The definition of $0^\sharp$ is absolute in every inner model that contains $0^\sharp$. Although it unfortunately cannot be justified here, there is no doubt that $0^\sharp$ is a "well-understood" mathematical object. If $0^\sharp$ does not exist, then the global structure of V does not deviate too much from that of L. This is the main content of the so-called "covering theorem": If $0^\sharp$ does not exist then every uncountable set X of ordinal numbers has a "covering set" $Y \supseteq X$ with $Y \in L$ and $\mathrm{Card}(Y) = \mathrm{Card}(X)$. (From this follows, for example, that every singular cardinal number has the same cardinal successor in L as in V.) But if $0^\sharp$ exists, then the structure of L-cardinals is "wiped out" in V. (For example, every uncountable cardinal looks inaccessible in L.) So $0^\sharp$ may be regarded as the "next larger construction step" after L. Then we form the inner model $L[0^\sharp]$, the constructible closure of

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with what certainty they express an opinion. Their opinions are divided and reflect, in my view, a pre-existing orientation.

[19] U is *normal* if for every regressive function $f : X \to \kappa$ (where "regressive" means $f(\xi) < \xi$ for all $\xi \in X$) with $X \in U$ there is an $\eta < \kappa$ with $\{\xi \mid f(\xi) = \eta\} \in U$. Normality implies κ-completeness.

[20] Non-trivial means $\pi \neq \mathrm{id}$.

$0^\sharp$.[21] Now let us assume that $L[0^\sharp]$ is non-trivially elementarily embeddable in itself by $\pi : L[0^\sharp] \to L[0^\sharp]$. This leads to a set $0^{\sharp\sharp}$ encoding $L[0^\sharp]$ and a canonical embedding π. Then we form $L[0^{\sharp\sharp}]$ and so on. After ω steps, we have a sequence $0^\sharp, 0^{\sharp\sharp}, \ldots, 0^{(n)}, \ldots$. We let $0^{(\omega)}$ encode this sequence and form $L[0^{(\omega)}]$. An embedding $\pi : L[0^{(\omega)}] \to L[0^{(\omega)}]$ yields $0^{(\omega+1)}$, and so on. When we have $0^{(\alpha)}$ for all ordinal numbers α, we form the constructible closure $L^\sharp$ of all the $0^{(\alpha)}$.[22] A non-trivial elementary embedding then yields a set $0^{(\infty)}$, and so forth.[23] We arrive at a sequence: $0^\sharp, 0^{\sharp\sharp}, \ldots, 0^{(\alpha)}, \ldots, 0^{(\infty)}, 0^{(\infty+1)}, \ldots$. The terms in this sequence are called *mice*. With some effort, one can define the class of all mice along with its natural ordering. (This ordering can be much longer than ∞.) But then we can build the constructible closure of all mice, which we denote by K.[24] K is called the *core model*. K is an inner model. The definition is absolute in the sense that $K_M = K$ for every inner model $M \supseteq K$. K is L-like in the same sense as L^U. What happens if there is a non-trivial elementary embedding $\pi : K \to K$? A new mouse cannot result, as all mice are already present in K. What we get is an inner model L^U, where $U \in L^U$ witnesses the measurability in L^U of some ordinal κ. We have $\mathcal{P}(\kappa) \cap K = \mathcal{P}(\kappa) \cap L^U$. U can be obtained in a fairly direct way from the embedding π. So we have, after all, obtained L^U "from below" via the long march through the mice.

The model L^U is uniquely determined by the number κ. Let us now choose κ as small as possible. What happens if there is a non-trivial elementary embedding $\pi : L^U \to L^U$? We obtain a set $0^\dagger$ coding L^U and a canonical embedding π. So, after all, $0^\dagger$ is "the next mouse." It differs from the previous mice in that, to obtain it, we needed an intermediate step marked not with a mouse but with an inner model L^U. This suggests the possibility of a continuing process, producing inner models for stronger and stronger axioms of infinity. How far can we go? Could it be that the directions mentioned in §5, Newtonianism and Pythagoreanism, are compatible after all, with Newtonianism indicating possibilities that then turn out to be realizable in a Pythagorean universe? Progress so far has been relatively modest. The next major step after a measurable cardinal is a strong cardinal. A few years ago, Dodd and the author set themselves the goal of defining a core model for a strong cardinal. Mitchell worked in the same direction but with a more modest goal. The technical difficulties turned out to be very great. Dodd and I barely got past the fundamental fine structure theory. Mitchell got closer to his goal. Recently, Mitchell—building on a suggestion of Baldwin—proposed a new definition of mice, which made immense simplifications possible.

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[21] $0^\sharp$ can be regarded as a set of ordinal numbers (even as a subset of ω). Then $L[0^\sharp]$ is defined from $0^\sharp$ as L^U was defined from U.

[22] $L^\sharp = \bigcup_\alpha L[0^{(\alpha)}]$. (Remark: $0^{(\nu)} \in L[0^{(\alpha)}]$ for $\nu \leq \alpha$.)

[23] ∞ is the class of all ordinal numbers and is also regarded as the "largest ordinal number."

[24] K is the union of all $L[m]$ such that m is a mouse.

The theory is still very long, and not all the details have been written down yet. Nevertheless, I believe I can claim that we now have an extended core model K in which a strong cardinal can exist. How big K is depends, naturally, on what there is. If there is no inner model with a measurable cardinal, then the new K coincides with the old one. If there is such an inner model but $0^{\dagger}$ does not exist, then $K = L^{U}$, etc. We again have $K_M = K$ if M is an inner model with $K \subseteq M$. K is again thoroughly L-like. The existence of a strong cardinal in V ensures the existence of one in K. If there is no elementary embedding $\pi : K \rightarrow K$, then again much of the global structure of K is preserved in V.[25] But if $\pi : K \rightarrow K$ exists, then we obtain the next mouse. So we can go further. But it turned out that one cannot go much further without seriously modifying the program.

§7. A few years ago, Hugh Woodin, in the wake of a deep result of Foreman, Magidor, and Shelah, proved the following theorem: If there is a superstrong cardinal, then there is no well-ordering of $\mathbb{R}$ in $L[\mathbb{R}]$. In particular, $\mathbb{R}$ has no well-ordering definable in second-order arithmetic. This result was surprising, because the concept of a superstrong cardinal had previously been regarded as a rather modest extension of the concept of measurability. The theorem set off an avalanche of results that have fundamentally altered the landscape of set theory. It also set a sharp limit for the conventional core model theory, because the methods that we had been using for the construction of these models always yielded a Σ^1_3 well-ordering of $\mathbb{R}$. Even more serious for this theory was another consequence of Woodin's proof: In a possible modified core model theory, one would have to give up the absoluteness theorem; if K contains a superstrong cardinal, then one cannot exclude the possibility that $K_M \neq K$ for some inner model $M \supseteq K$.[26]

A rapid sequence of discoveries followed. Martin and Steel produced the long-desired proof that determinacy axioms follow from strong axioms of infinity. A typical result—due to Martin, Steel, and Woodin—is that the existence of a superstrong cardinal implies AD in $L[\mathbb{R}]$. An even weaker cardinal—the so-called Woodin cardinal—played a key role in the new developments. Call a cardinal τ *strong with respect to a class A* if for each

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[25] But this can no longer be expressed by so strong and simple a statement as the covering theorem

[26] Using forcing, Woodin showed that, given an inner model M with a superstrong cardinal, one can consistently assume the existence of a strictly larger model M' and an elementary embedding $\pi : M \rightarrow M'$ Now let $M = K$. Then $V = K$ holds in M and hence also in M'. Absoluteness would imply $M' = K = M$, a contradiction.

β there exist an inner model M, a class B, and an elementary embedding $\pi : \langle V, A\rangle \to \langle M, B\rangle$ such that τ is the critical point, $V_\beta \subseteq M$, and $B \cap V_\beta = A \cap V_\beta$. If κ is inaccessible, $A \subseteq V_\kappa$, and $\tau < \kappa$, then we can relativize this concept to V_κ. We call κ a *Woodin cardinal* if for each $A \subseteq V_\kappa$ there exists a $\tau < \kappa$ that is strong with respect to A in V_κ. Martin and Steel showed that Σ^1_{n+1}-determinacy follows from the existence of n Woodin cardinals and a measurable cardinal above them. Then Woodin showed that the existence of ω Woodin cardinals and a measurable cardinal above them implies AD in $L[\mathbb{R}]$. Woodin also has interesting equiconsistency results: Σ^1_2-determinacy is equiconsistent with the existence of a Woodin cardinal. AD in $L[\mathbb{R}]$ is equiconsistent with the existence of ω Woodin cardinals.

§8. The two areas, large cardinals and determinacy, had come together. What about inner models? At first glance, the core model program seemed to have reached a dead end. For two essential attributes of this theory would have to be sacrificed in any extension: a nicely definable well-ordering of $\mathbb{R}$ and the absoluteness theorem. The second sacrifice seemed especially serious to me. K is defined as the union of its construction steps, where a construction step (e.g., a mouse) should be a "well understood" object. For me, this requirement always meant that one can recognize in an arbitrary inner model M whether a given element of M is a construction step. That implies $K_M = K$ if $K \subseteq M$. But this conclusion is false if a superstrong (or just a Woodin) cardinal occurs in K.

Fortunately, Martin, Mitchell, and Steel did not let these considerations trouble them but set out to find a concept of mouse suitable for the continuation of the theory. The first step in this extended core model program was to realize a Woodin cardinal in a core model. Martin and Steel first defined a "model of type L^U." Assuming a Woodin cardinal κ, they defined an inner model W. The construction of W is not really "from below" but uses a set whose existence follows from the assumed property of κ. They showed that, in W, κ remains a Woodin cardinal, CH holds, and $\mathbb{R}$ has a Σ^1_3 well-ordering. (So Σ^1_2-determinacy is false in W.) But the internal structure of this model remained quite mysterious; for example we do not know whether GCH holds in it. Mitchell and Steel then created a revised W using the fine structure techniques of core model theory. Their W satisfies GCH and is quite generally L-like. Finally, Steel managed to construct a genuine core model K. But to carry out his construction, he needs third order set theory and a large cardinal property (namely measurability) for ∞ (the class of all ordinals).[27] When one has this, one finds a first order definition of K that is absolute in K. At the moment, only a rather weak analog of the covering theorem has been proved.[28]

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[27] Steel thinks that second order set theory will suffice and that the measurability assumption can be weakened.

The mice studied in this extended theory are still "well understood" objects, but the definition of mice is of such high logical complexity that the absoluteness requirement is no longer satisfied.[29] The possibility that there are models M with $K \subseteq M$ and $K_M \neq K$ turned out to be less troublesome than one had expected. The increasing complexity of the concept of mouse also explains why we lose the nicely definable well-orderings of $\mathbb{R}$. The existence of such a well-ordering depends on the fact that every mouse that constructs a new real number is itself a countable object and so can be coded by a real number. The earlier mice were so simple that one could define the set of all such codes—together with their natural well-ordering—in second order arithmetic. With increasing complexity, this is no longer possible.

Will it be possible to realize all reasonable large cardinals in "well understood" inner models? This goal is important not only for the reasons indicated in §6 but also because of a consideration connected with Gödel's justification for strong axioms of infinity. A common plausibility argument for these axioms is that they are—as far as we can see—linearly ordered by the relation of relative consistency.[30] Historically, the axioms of infinity have many different mathematical sources, and there is no *prima facie* reason to suppose that they are linearly ordered. The existence of such an order—insofar as we can confirm it—is surely a strong argument for regarding these axioms as successive approximations to a "final" universe. The axioms in the sequence mentioned earlier—measurable, strong, superstrong, etc.—arise from weaker and stronger variations of a single theme. So it is not surprising that they are linearly ordered. We can regard this sequence as a yardstick with which to compare other strong axioms of infinity. (It turns out that such comparison is sensible even for some assertions that seem to have nothing to do with large cardinals, for example the existence of a saturated ideal on ω_1.) Very many open questions remain. Perhaps the best known is whether compact and supercompact cardinals are equiconsistent. If we go "down" a bit and consider principles of infinity that are weaker than measurability yet strong enough to imply $V \neq L$, then the so-called Erdős cardinals form an analogous yardstick. Here the core model is available, and it frequently allows us to compare other principles with this yardstick.

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[28] By this I mean theorems saying that the global structure of K is preserved in V if K is rigid with respect to elementary embeddings

[29] That the absoluteness condition held previously is due to the fact that the concept of mouse was Δ_1 in the parameter ω_1.

[30] An axiom A is *consistent relative to B* in the axiom system of set theory (ZFC) if it is provable in first order arithmetic that the consistency of ZFC+A follows from the consistency of ZFC+B Modulo equiconsistency, this relation is a partial order.

(Typical examples are the Chang conjecture[31] and the existence of Jónsson cardinals.[32]) The only strategy we have for attaining comparable results for larger cardinals is a corresponding generalization of the inner model theory.

I would like to close with a general comment on the direction of research in set theory. In recent years I have occasionally heard other mathematicians complain that set theory has turned inward, i.e., that set theorists are occupied too much with their own structural questions and not enough with applications. This reproach does not by any means apply to all set theorists, but this paper does little to refute it. I hope, however, to have made it plausible that a subject should from time to time turn inward. The deep inner problems have become steadily clearer in recent years, and the tools needed to overcome them have been developed. There has been enormous progress recently. If set theory now finds itself in a rather introspective phase, I am convinced that it will emerge from this phase stronger than ever and of greater use for other areas of mathematics.

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[31] The Chang conjecture is the following model-theoretic assertion: If $\langle A, B, \ldots \rangle$ is a model for a countable language with $\mathrm{Card}(A) = \omega_2$ and $\mathrm{Card}(B) = \omega_1$, then there is an elementary submodel $\langle \bar{A}, \bar{B}, \ldots \rangle$ with $\mathrm{Card}(\bar{A}) = \omega_1$ and $\mathrm{Card}(\bar{B}) = \omega$.

[32] κ is a Jónsson cardinal if every algebra $A = \langle |A|, f_1, f_2, \ldots, f_n \rangle$ of cardinality κ has a proper subalgebra of the same cardinality.

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SINGULAR CARDINALS AND THE PCF THEORY

THOMAS JECH

§1. **Introduction.** Among the most remarkable discoveries in set theory in the last quarter century is the rich structure of the arithmetic of singular cardinals, and its deep relationship to large cardinals. The problem of finding a complete set of rules describing the behavior of the continuum function $2^{\aleph_\alpha}$ for singular $\aleph_\alpha$'s, known as the *Singular Cardinals Problem,* has been attacked by many different techniques, involving forcing, large cardinals, inner models, and various combinatorial methods. The work on the singular cardinals problem has led to many often surprising results, culminating in a beautiful theory of Saharon Shelah called the pcf theory ("pcf" stands for "possible cofinalities"). The most striking result to date states that if $2^{\aleph_n} < \aleph_\omega$ for every $n = 0, 1, 2, \ldots$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

In this paper we present a brief history of the singular cardinals problem, the present knowledge, and an introduction into Shelah's pcf theory. In Sections 2, 3 and 4 we introduce the reader to cardinal arithmetic and to the singular cardinals problems. Sections 5, 6, 7 and 8 describe the main results and methods of the last 25 years and explain the role of large cardinals in the singular cardinals problem. In Section 9 we present an outline of the pcf theory.

§2. **The arithmetic of cardinal numbers.** Cardinal numbers were introduced by Cantor in the late 19th century and problems arising from investigations of rules of arithmetic of cardinal numbers led to the birth of set theory. The operations of addition, multiplication and exponentiation of infinite cardinal numbers are a natural generalization of such operations on integers. Addition and multiplication of infinite cardinals turns out to be simple: when at least one of the numbers κ, λ is infinite then both $\kappa + \lambda$ and $\kappa \cdot \lambda$ are equal to $\max\{\kappa, \lambda\}$. In contrast with $+$ and $\cdot$, exponentiation presents fundamental problems. In the simplest nontrivial case, 2^κ represents the cardinal number of the power set $P(\kappa)$, the set of all subsets of κ. (Here we adopt the usual convention of set theory that the number κ is identified with a set of cardinality κ, namely the set of all ordinal numbers

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smaller than κ. In this representation, the cardinal number $\aleph_\alpha$ is the same as the ordinal number ω_α.) By a celebrated theorem of Cantor, $2^\kappa > \kappa$ holds for all cardinals κ, and therefore $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$ for every infinite cardinal $\aleph_\alpha$. In [6], Cantor conjectured that $2^{\aleph_0} = \aleph_1$, which became known as the *Continuum Hypothesis* (and the similar conjecture $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ as the Generalized Continuum Hypothesis or GCH).

It has soon become apparent that if GCH were true, one could completely describe rules for all of cardinal arithmetic, including infinite sums and products ($\sum_{i\in I} \kappa_i$ and $\prod_{i\in I} \kappa_i$) of cardinal numbers. Despite efforts of Cantor himself and others, the question whether GCH, or even CH, is true, remained unanswered until the emergence of methods of modern logic.

For a long time, the only source of inequalities in cardinal arithmetic was König's Theorem [37]. The theorem states that if $\{\kappa_i : i \in I\}$ and $\{\lambda_i : i \in I\}$ are two indexed families of cardinal numbers such that $\kappa_i < \lambda_i$ for all i, then

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Note that Cantor's Theorem itself is a special case of this: letting $\kappa_i = 1$ and $\lambda_i = 2$ for all $i < \kappa$, we get $\kappa = \sum_{i<\kappa} 1 < \prod_{i<\kappa} 2 = 2^\kappa$.

König's Theorem also provides important information about singular cardinals. Let us recall that the *cofinality* of an infinite cardinal $\aleph_\alpha$ is the least cardinal $\aleph_\gamma = \operatorname{cf} \aleph_\alpha$ such that $\aleph_\alpha$ is the supremum of an increasing $\aleph_\gamma$-sequence of ordinal numbers. A cardinal $\aleph_\alpha$ is *regular* if $\operatorname{cf} \aleph_\alpha = \aleph_\alpha$ and *singular* if $\operatorname{cf} \aleph_\alpha < \aleph_\alpha$. As $\operatorname{cf}(\operatorname{cf} \aleph_\alpha) = \operatorname{cf} \aleph_\alpha$, the cofinality is always a regular cardinal. Each successor cardinal $\aleph_{\alpha+1}$ is regular, as is the countable cardinal $\aleph_0$.

One consequence of König's Theorem is that for every α and every β, $\operatorname{cf} \aleph_\alpha^{\aleph_\beta} > \aleph_\beta$. It follows easily that when $\aleph_\beta \geq \operatorname{cf} \aleph_\alpha$ then $\aleph_\alpha^{\aleph_\beta} \neq \aleph_\alpha$, and therefore $\aleph_\alpha^{\aleph_\beta} > \aleph_\alpha$. Using this as well as rather elementary rules for cardinal exponentiation due to Hausdorff and Tarski, one obtains a complete set of rules for exponentiation under the assumption of GCH:

$$\aleph_\alpha^{\aleph_\beta} = \begin{cases} \aleph_\alpha & \text{if } \aleph_\beta < \operatorname{cf} \aleph_\alpha \\ \aleph_{\alpha+1} & \text{if } \operatorname{cf} \aleph_\alpha \leq \aleph_\beta < \aleph_\alpha \\ \aleph_{\beta+1} & \text{if } \aleph_\alpha \leq \aleph_\beta \end{cases}$$

Note that even in the presence of GCH, singular cardinals exhibit different behavior: the second clause does not apply if $\aleph_\alpha$ is regular.

§3. Consistency and independence of the generalized continuum hypothesis. The continuum problem remained open until 1939 when a significant progress came from Gödel who showed in [25] that GCH is consistent with the axioms of set theory ZFC (Zermelo-Fraenkel's axioms with the axiom of choice). Gödel produced the model L of *constructible sets* and proved that GCH holds in the model L. In addition to the consistency proof of GCH, Gödel's method introduced the important concept of *inner models*; an interested reader can learn more about this subject in the article [35], in this issue of the Bulletin.

In 1963, Cohen proved the independence of the continuum hypothesis [7], [8]. Cohen constructed a model of ZFC in which CH fails. Moreover, Cohen's method of *forcing* proved to be a powerful tool for obtaining other independence results, and in particular was used to show that cardinal arithmetic of *regular* cardinals can behave arbitrarily, within the limits imposed by König's Theorem. Shortly after Cohen's breakthrough, Solovay showed that $2^{\aleph_0}$ can take any value not excluded by König's theorem, i.e., one can have $2^{\aleph_0} = \aleph_\alpha$ for any α as long as $\operatorname{cf}\aleph_\alpha > \aleph_0$. Then Easton produced a model [13] in which the function $2^{\aleph_\alpha}$, for regular $\aleph_\alpha$, can behave in any prescribed way consistent with König's theorem.

The natural question arose whether the freedom enjoyed by the function $2^{\aleph_\alpha}$ on regular cardinals extends as well to singular cardinals. In particular, can a singular cardinal be the least cardinal at which GCH fails? Or, specifically, is it possible to have $2^{\aleph_n} = \aleph_{n+1}$ for every n while $2^{\aleph_\omega} = \aleph_{\omega+2}$? These questions (first asked by Solovay in the mid-sixties, see [43]) are a part of what has become known as the *Singular Cardinals Problem*.

At first the consensus among set theorists was that an improvement in Cohen's method will lead to a general consistency result along the lines of Easton's theorem. The truth however turned out to be much more interesting.

§4. **The singular cardinals problem.** By Easton's theorem, the only rules for the continuum function $2^{\aleph_\alpha}$ on regular cardinals that are provable in ZFC state that $2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$ if $\alpha \leq \beta$, and that $\operatorname{cf} 2^{\aleph_\alpha} > \aleph_\alpha$. The situation is different for singular cardinals. If $\aleph_\alpha$ is singular then an easy use of cardinal arithmetic shows that $2^{\aleph_\alpha} = (\sum_{\gamma<\alpha} 2^{\aleph_\gamma})^{\operatorname{cf}\aleph_\alpha}$. As a consequence, if $\aleph_\alpha$ is singular and if $2^{\aleph_\gamma}$ has the same value $2^{\aleph_{\gamma_0}}$ for eventually all $\gamma < \alpha$ (i.e., $\gamma_0 \leq \gamma < \alpha$) then $2^{\aleph_\alpha} = 2^{\aleph_{\gamma_0}}$ as well. Thus additional rules are needed if one hopes for an extension of Easton's theorem. This was observed, e.g., in [3] and [27].

For cardinal arithmetic on singular cardinals, it is not the continuum function $2^{\aleph_\alpha}$ alone that determines all exponentiation. It turns out that it is the function $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$ that is crucial for cardinal arithmetic. (Note that if $\aleph_\alpha$ is regular then $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha} = \aleph_\alpha^{\aleph_\alpha} = 2^{\aleph_\alpha}$; for singular cardinals we only have $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha} \leq 2^{\aleph_\alpha}$.)

The knowledge of the function $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$ is sufficient for all cardinal exponentiation. That all cardinal arithmetic, including infinite products $\prod$, can be reduced to the function $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$ was already known to Gödel who in [26] so remarked pointing to Tarski's work [69].

The continuum function is determined as follows: If $\aleph_\alpha$ is regular then $2^{\aleph_\alpha} = \aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$. So let $\aleph_\alpha$ be singular. If $2^{\aleph_\gamma} = 2^{\aleph_{\gamma_0}}$ for eventually all $\gamma < \alpha$, then $2^{\aleph_\alpha} = 2^{\aleph_{\gamma_0}}$. If the continuum function $2^{\aleph_\gamma}$ is not eventually constant below $\aleph_\alpha$ then $2^{\aleph_\alpha} = \kappa^{\operatorname{cf}\kappa}$ where $\kappa = \sum_{\gamma<\alpha} 2^{\aleph_\gamma}$.

The exponentiation $\aleph_\alpha^{\aleph_\beta}$ is then determined from the continuum function and the function $\kappa^{\operatorname{cf}\kappa}$ inductively, the crucial case being when $\operatorname{cf}\aleph_\alpha \le \aleph_\beta < \aleph_\alpha$ and $\sum_{\gamma<\alpha} \aleph_\gamma^{\aleph_\beta} = \aleph_\alpha$. Then we have $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$.

Thus the key to cardinal arithmetic is the function $\kappa^{\operatorname{cf}\kappa}$. A consequence of König's theorem is that $\kappa^{\operatorname{cf}\kappa} \ne \kappa$ and so $\kappa^{\operatorname{cf}\kappa} \ge \kappa^+$. Of course if $2^{\operatorname{cf}\kappa} \ge \kappa$ then $\kappa^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa}$. Hence what is decisive (in addition to the continuum function on regulars) is the behavior of $\kappa^{\operatorname{cf}\kappa}$ for those κ for which $2^{\operatorname{cf}\kappa} < \kappa$. The simplest possibility is when

$$2^{\operatorname{cf}\kappa} < \kappa \quad \text{implies} \quad \kappa^{\operatorname{cf}\kappa} = \kappa^+.$$

This is known as the *Singular Cardinal Hypothesis* (SCH).

Assuming SCH, the continuum function on regular cardinals itself determines cardinal exponentiation: if $\aleph_\alpha$ is singular then $2^{\aleph_\alpha}$ is either κ or κ^+ where $\kappa = \sum_{\gamma<\alpha} 2^{\aleph_\gamma}$, depending on whether $2^{\aleph_\gamma}$ is or is not eventually constant. Similarly for $\aleph_\alpha^{\aleph_\beta}$ where $\operatorname{cf}\aleph_\alpha \le \aleph_\beta < \aleph_\alpha$.

With this analysis of cardinal arithmetic it is now clear that the fundamental question related to the singular cardinals problem is whether SCH can fail. In the simplest case mentioned in Section 3, we can ask whether SCH can fail for $\aleph_\omega$: can we have $2^{\aleph_0} < \aleph_\omega$ and at the same time $\aleph_\omega^{\aleph_0} \ge \aleph_{\omega+2}$? (This is a somewhat finer question than whether $\aleph_\omega$ can be a strong limit cardinal while $2^{\aleph_\omega} \ge \aleph_{\omega+2}$; if $2^{\aleph_n} < \aleph_\omega$ for all $\aleph_n$ then $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$.)

Using basic cardinal arithmetic and König's theorem, it is possible to derive several additional rules for the behavior of the function $\kappa^{\operatorname{cf}\kappa}$ (see [28]). For instance, if κ is strong limit, then $\operatorname{cf}(\kappa^{\operatorname{cf}\kappa}) > \kappa$; or if $\kappa \le \lambda^{\operatorname{cf}\lambda}$ for some $\lambda < \kappa$ with $\operatorname{cf}\lambda \ge \operatorname{cf}\kappa$, then $\kappa^{\operatorname{cf}\kappa} \le \lambda^{\operatorname{cf}\lambda}$. It turns out however that more dramatic restrictions are in store for arithmetic of singular cardinals.

§5. Silver's theorem and Jensen's covering theorem. Until 1974 most set theorists believed that the restriction to regular cardinals in Easton's theorem was due to the weakness of its proof and that analogous results for singular cardinals would be forthcoming. In particular, it was expected that it was possible for a singular cardinal to be the least counterexample to GCH.

This changed dramatically when Silver proved the following theorem [67]:

If κ is a singular cardinal of uncountable cofinality, and if $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.

Following Silver's result, several theorems appeared that further restricted the behavior of the function 2^κ for singular cardinals of uncountable cofinality, most notably the theorem of Galvin and Hajnal [15].

If $\aleph_\lambda$ *is a strong limit cardinal of uncountable cofinality then* $2^{\aleph_\lambda} < \aleph_{(2^{|\lambda|})^+}$.

In another direction, Jensen was able to combine Silver's ideas with his previous work [34] on the fine structure of L, to prove his remarkable Covering Theorem [10]:

If $0^\#$ *does not exist then every uncountable set of ordinals can be covered by a constructible set of the same cardinality.*

(The existence of $0^\#$ is a large cardinal axiom that we shall return to in the next section.) An easy argument using the Covering Theorem shows that unless $0^\#$ exists, $2^{\operatorname{cf}\kappa} < \kappa$ implies $\kappa^{\operatorname{cf}\kappa} = \kappa^+$, i.e., SCH holds. Thus in order to violate SCH we need large cardinals.

A related corollary of the Covering Theorem states that if $0^\#$ does not exist then for every $\lambda \geq \aleph_2$, if λ is a regular cardinal in L then $\operatorname{cf}\lambda = |\lambda|$. In particular, a regular cardinal cannot be changed into a singular cardinal in the absence of large cardinals. (The assumption $\lambda \geq \aleph_2$ is necessary, as Bukovský [4] and Namba [46] produced a forcing extension of L—in which $0^\#$ cannot exist—where $|\omega_2^L| = \aleph_1$ and $\operatorname{cf}\omega_2^L = \omega$.)

The assumption of Silver's Theorem that κ has uncountable cofinality figures prominently in the proof. To let the reader appreciate the significance of uncountable cofinality, I shall give a brief outline of the methods used in Silver's and Galvin-Hajnal's theorems.

For simplicity, let us consider $\aleph_{\omega_1}$. For Silver's Theorem, assume that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$. The main idea of the proof is that there exists a generic extension $V[G]$ of the universe, in which there exists a *normal* ultrafilter U on $P^V(\omega_1)$. Normality means, as usual, that the diagonal is the least nonconstant function; here the fact that ω_1 is uncountable is essential. Working in $V[G]$, we can calculate the size of $(2^{\aleph_{\omega_1}})^V$ as follows. Consider the ultrapower M of V by U, and the elementary embedding $j\colon V \to M$. M is not well founded, but thanks to normality, the M-ordinal κ represented by the function $\alpha \mapsto \aleph_\alpha$ is the supremum of the M-ordinals $j(\aleph_\gamma)$, $\gamma < \omega_1$. It follows that the linearly ordered set $(\kappa^+)^M$ has size at most $((\aleph_{\omega_1})^V)^+$. A further argument gives $|P^V(\aleph_{\omega_1})| \leq |P^M(\kappa)|$, and since $M \models 2^\kappa = \kappa^+$, we have $|(2^{\aleph_{\omega_1}})^V| \leq ((\aleph_{\omega_1})^V)^+$. This is calculated in $V[G]$, but since the forcing extension is "mild," the same inequality holds in V.

Silver's proof has been reworked in [1], replacing the forcing technique by a purely combinatorial argument. A further improvement of the combinatorial method led to the above mentioned theorem of Galvin and Hajnal. In the Galvin-Hajnal theorem, the uncountability of the cofinality is again essential. Roughly speaking, the generic ultrapower of Silver is replaced by a well-founded partial ordering of ordinal-valued functions. If f and g are ordinal functions on ω_1, let $f < g$ denote the relation

$$\{\, \alpha < \omega_1 : f(\alpha) < g(\alpha) \,\} \text{ contains a club.}$$

Due to normality of the club filter, the relation $<$ is a well-founded partial ordering. Hence every ordinal function f on ω_1 can be assigned its rank in $<$, the *Galvin-Hajnal norm* $\|f\|$ of f. The essence of Galvin-Hajnal's proof is that the size of $2^{\aleph_{\omega_1}}$ (if $\aleph_{\omega_1}$ is strong limit) is related to the Galvin-Hajnal norm of the function $\alpha \mapsto 2^{\aleph_\alpha}$. The analysis of this relationship yields the upper bound $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$.

§6. Large cardinals. As the singular cardinals problem is so closely related to large cardinal axioms, we shall now review the basics of the theory of large cardinals.

An uncountable cardinal number κ is *inaccessible* if it is regular and a strong limit cardinal. An immediate consequence of inaccessibility is that V_κ, the collection of all sets of rank less than κ, is a model of ZFC; another immediate consequence is that $\kappa = \aleph_\kappa$ is a fixed point of the aleph sequence. By Gödel's 2nd Incompleteness Theorem it follows that the existence of inaccessible cardinals is unprovable in ZFC. In fact, a slightly more involved argument shows that the relative consistency of inaccessible cardinals is unprovable. Thus the existence of inaccessibles is to ZFC as the existence of an infinite set is to Peano arithmetic. For that reason, large cardinal axioms are sometimes referred to as *strong axioms of infinity*.

Modern large cardinal theory recognizes a substantial number of large cardinal axioms. Interestingly enough, these axioms form a linearly ordered scale, on which the relation of a stronger axiom to the weaker theories is just as described above in the case of inaccessible cardinals and ZFC. This scale of large cardinals serves as a measure of consistency strength of various set theoretic assumptions (including assumptions on singular cardinals).

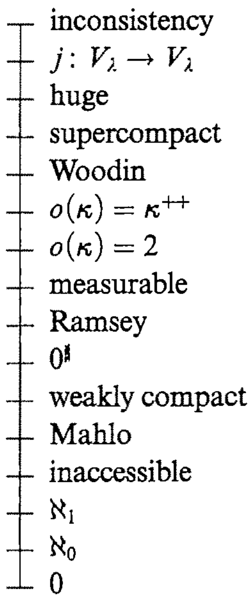
One of the most prominent large cardinals is a measurable cardinal. Measurable cardinals were introduced by Ulam while working on the *measure problem*, the question whether some set E can carry a nontrivial countably additive measure such that every subset of E is measurable. An uncountable cardinal κ is *measurable* if κ carries an atomless κ-additive 2-valued measure; equivalently, if κ carries a nonprincipal κ-complete ultrafilter. (The terms "κ-additive" and "κ-complete" refer to unions and intersections of *fewer than* κ sets.) In [50], Scott applied to measurable cardinals the method of ultraproducts, thus laying the groundwork for the modern large cardinal theory. One of the basic observations is that the existence of measurable cardinals is equivalent to the existence of a nontrivial elementary embedding $j\colon V \to M$ where M is some transitive model. Every measurable cardinal is not only inaccessible, but on the scale of large cardinals is above Mahlo, weakly compact and Ramsey cardinals, to mention just the most important categories of large cardinals (see picture).

Scott's theorem [50] states that measurable cardinals don't exist in L. The subsequent work of Gaifman, Rowbottom [49] and Silver [66] established the strength of measurable cardinals *vis à vis* constructible sets, isolating the principle "$0^\#$ exists." The existence of $0^\#$ (between weakly compact and Ramsey on our scale) states that there aren't many constructible sets, and the truth definition for the model L exists. Equivalently, there exists a nontrivial elementary embedding $j\colon L \to L$. One consequence of $0^\#$ is that *every* uncountable cardinal (in V) is inaccessible in L. In particular, $\aleph_\omega$ is regular in L; recall that by the Covering Theorem, if $0^\#$ does not exist then $\aleph_\omega$ (and every singular cardinal) must be singular in L.

Among large cardinals stronger than measurable, *supercompact* cardinals are most prominent. There exists at present a good classification of large cardinals between measurables and *Woodin cardinals*, using the theory of *inner models* (we refer the reader to Jensen's article [35], in this Bulletin). In particular, measurable cardinals are classified by their *order* $o(\kappa)$, where $o(\kappa) = 1$ means that κ is measurable, $o(\kappa) = 2$ means that κ carries a normal measure in which almost all $\alpha < \kappa$ are measurable, and so on, up to $o(\kappa) = \kappa^{++}$.

Above supercompact and huge cardinals, the scale approaches its end with the existence of an elementary embedding $j\colon V_\lambda \to V_\lambda$, as by a theorem of Kunen, $j\colon V \to V$ is inconsistent.

For the benefit of nonspecialists, as well as a reference for the forthcoming sections we present a scale of the more prominent large cardinals:



§7. **Large cardinals and the singular cardinals problem.** Prior to Jensen's Covering Theorem there had been scattered results indicating that there might be some link between the behavior of singular cardinals and the large cardinal axioms. For instance, Solovay proved in [68] that if κ is a supercompact cardinal and $\lambda > \kappa$ is singular then $\lambda^{\aleph_0} \leq \lambda^+$. This means that the SCH holds above the least supercompact cardinal. On the other hand, Prikry introduced in [47] a method of forcing that changes the cofinality of a measurable cardinal to ω without collapsing it. The measurable cardinal κ

remains a cardinal but has $\operatorname{cf}\kappa = \omega$ in the extension. Subsequently, Silver devised a forcing method that, starting with a supercompact cardinal κ, produced a model in which κ is measurable and $2^\kappa > \kappa^+$. This, combined with the Prikry forcing, yields the consistency of the negation of SCH, relative to a supercompact cardinal: a model of ZFC in which κ is a strong limit cardinal, $\operatorname{cf}\kappa = \omega$, and $2^\kappa > \kappa^+$.

About the same time that Jensen established, by the Covering Theorem, the necessity of large cardinals for the negation of the SCH, Magidor obtained the first in a series of consistency results using large cardinals. In [40], he proved the consistency of $2^{\aleph_\omega} > \aleph_{\omega+1}$ (and $\aleph_\omega$ strong limit) from a supercompact cardinal, and in [41] the consistency of $2^{\aleph_\omega} = \aleph_{\omega+2}$ along with GCH below $\aleph_\omega$, from a 2-huge cardinal. When it soon became clear that large cardinals are indeed necessary, Magidor's method was refined to yield other consistency results. The general idea in all these proofs is as follows: start with a large cardinal κ, blow up 2^κ, change the cofinality of κ by adjoining a new set of cardinals C cofinal in κ, and destroy all cardinals below κ that are not in the set C. The technique employed to change the cofinality evolved from Prikry's method, progressing through [42] to [48]. More recently, a new method for building models for the negation of SCH was developed in [20] and [22].

In Magidor's model [40], $2^{\aleph_\omega}$ could be as large as $\aleph_{\omega+\omega+1}$, which was improved by Shelah [54] as to have $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for any $\alpha < \omega_1$. Woodin [14] constructed a model in which $2^\kappa = \kappa^{++}$ for all infinite cardinals κ, and Cummings [9] found a model in which GCH fails exactly at all limit cardinals.

As for the consistency strength of the failure of SCH, a series of results proved it to be exactly a measurable cardinal of order $o(\kappa) = \kappa^{++}$. Magidor's assumption of supercompactness was first weakened by Woodin, and eventually Gitik obtained a model of $\neg$SCH in [17] using $o(\kappa) = \kappa^{++}$. The other direction, namely that $o(\kappa) = \kappa^{++}$ is a necessary assumption, is also due to Gitik ([18]), and uses the technique of inner models.

The technique of inner models is an outgrowth of Jensen's Covering Theorem. The first step up from $0^\#$ was the core model K and the covering theorem for K ([11], [12]), followed by inner models and covering theorems for measurable cardinals [44] and beyond. Details can be found in Jensen's article [35]. This technique produced a number of results, notably [45], [21], [19] and [23] showing the necessity of large cardinal axioms for various violations of SCH.

A major open problem in cardinal arithmetic is whether $2^{\aleph_\omega}$ can be greater than $\aleph_{\omega_1}$ (while $\aleph_\omega$ is strong limit). The analysis provided by the pcf theory indicates that an entirely new approach would be needed, and the inner model technique shows that its consistency strength is enormous.

§8. Upper bounds. Following Silver's Theorem [67], one of the main directions of research in the theory of singular cardinals has been the search for upper bounds on $2^{\aleph_\alpha}$ for strong limit singular cardinals $\aleph_\alpha$; or more precisely, on the function $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$ for those cardinals that satisfy $2^{\operatorname{cf}\aleph_\alpha} < \aleph_\alpha$. Let us recall that the Galvin-Hajnal Theorem gives a bound

$$2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$$

if $\aleph_{\omega_1}$ is a strong limit cardinal. In full generality, if $\aleph_\delta$ is a strong limit cardinal of uncountable cofinality then $2^{\aleph_\delta} < \aleph_{(|\delta|^{\operatorname{cf}\delta})^+}$.

The theorem leaves open the question whether an upper bound exists for $2^{\aleph_\omega}$ (and other cardinals of cofinality ω). The method does not give any information in this case, as it depends heavily on the use of closed unbounded sets.

The other question left open is the case when $\aleph_\delta$ is a fixed point of the aleph function, i.e., when $\delta = \aleph_\delta$.

Several papers extended the Galvin-Hajnal result under large cardinal assumptions: Magidor [39] proved $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$ from Chang's Conjecture (which has the consistency strength of approximately Ramsey cardinals); Jech and Prikry [31] proved $2^{F(\aleph_1)} < F(\aleph_2)$, where $F(\aleph_\alpha)$ is the $\aleph_\alpha$th fixed point of the $\aleph$ function, from a saturated ideal (approximately supercompact); and Shelah [52] proved the same result from Chang's Conjecture. In [55] and [56] Shelah eliminated the large cardinal assumptions, obtaining somewhat weaker bounds, such as $2^{F(\aleph_1)} < F((2^{2^{\aleph_1}})^+)$. Interestingly, his method broke down for the ωth iteration of the fixed point operation; eventually it turned out that in this case no a priori bounds are provable.

To obtain upper bounds for $2^{\aleph_\omega}$, a different approach is needed. The major breakthrough came when Shelah discovered the significance of cofinalities of ultraproducts $\prod_{n=0}^{\infty} \aleph_n / D$, where D is an ultrafilter on ω. Making use of these, he obtained the analog of the Galvin-Hajnal Theorem: if $\aleph_\omega$ is strong limit then

$$2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$$

(cf. [53], Chapter XIII). The proof of this theorem led Shelah to a systematic study of cofinalities of reduced products of sets of cardinals. In a sequence of papers [52], [57], [58] and [61] (Chapters II, VIII and IX), he developed a beautiful theory that brought a number of unexpected results and yielded deep applications to cardinal arithmetic. The crowning achievement so far is the following result that we shall discuss in the following section: if $2^{\aleph_0} < \aleph_\omega$, then

$$\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}.$$

§9. Shelah's pcf theory. In this last section we shall outline Shelah's theory of possible cofinalities (pcf). The central concept of the theory is the cofinality of an ultraproduct of a set of regular cardinals:

Let A be a set of regular cardinals, and let D be an ultrafilter on A. The ultraproduct $\prod A/D$ consists of equivalence classes of ordinal functions f such that $\operatorname{dom}(f) = A$ and $f(\alpha) \in \alpha$ for all $\alpha \in A$; two functions f and g are equivalent mod D if $\{\, \alpha : f(\alpha) = g(\alpha) \,\} \in D$. The ultraproduct is linearly ordered by the relation $f <_D g$ defined by $\{\, \alpha : f(\alpha) < g(\alpha) \,\} \in D$, and the *cofinality* of $\prod A/D$ is the unique regular cardinal $\kappa = \operatorname{cof} \prod A/D$ such that $(\prod A/D, <_D)$ has a cofinal subset of order type κ.

If $\kappa = \operatorname{cof} \prod A/D$ for some ultrafilter D, we say that κ is a *possible cofinality* (of $\prod A$), and we let

$$\operatorname{pcf} A = \left\{ \operatorname{cof} \prod A/D : D \text{ an ultrafilter on } A \right\}$$

be the set of all possible cofinalities of $\prod A$.

As a typical case, let us consider the set $A = \{\aleph_n\}_{n=0}^{\infty}$. Of course, every $\aleph_n$ is a possible cofinality, by a principal ultrafilter. If $\kappa = \operatorname{cof} \prod A/D$ where D is nonprincipal then clearly $\kappa \geq \aleph_{\omega+1}$, and since $|\prod A| = \aleph_\omega^{\aleph_0}$, we have $\kappa \leq \aleph_\omega^{\aleph_0}$. So unless $\aleph_\omega^{\aleph_0} > \aleph_{\omega+1}$, the set pcf A has only one element outside A, and so a meaningful theory of possible cofinalities requires the negation of the singular cardinals hypothesis. It turns out that the pcf theory is more fundamental than cardinal arithmetic.

In [51] Shelah proved, among others, the following result that was a precursor of the future pcf theory:

THEOREM 9.1. *For every singular cardinal λ of cofinality ω there exists an increasing sequence $\{\lambda_n\}_{n=0}^{\infty}$ of regular cardinals with limit λ such that $\operatorname{cof} \prod_{n=0}^{\infty} \lambda_n/D = \lambda^+$ for every nonprincipal ultrafilter D.*

This result shows, e.g., that $\aleph_{\omega+1}$ is a possible cofinality of $\prod_{n=0}^{\infty} \aleph_n$ regardless of the value of $\aleph_\omega^{\aleph_0}$.

Shortly thereafter, Shelah established the relation between possible cofinalities and cardinal arithmetic, and obtained an upper bound for $2^{\aleph_\omega}$, cf. [53]. We say that A is an *interval* of regular cardinals if whenever $\alpha < \beta < \gamma$ are regular cardinals and $\alpha, \gamma \in A$, then $\beta \in A$. The following theorem reduces the simplest cases of the singular cardinals problem to the problem of possible cofinalities:

THEOREM 9.2.

(a) *If A is an interval of regular cardinals such that $|A| < \min A$ then pcf A is an interval.*

(b) *If, moreover, $(\min A)^{|A|} < \sup A$ then $\kappa = |\prod A|$ is a possible cofinality.*

This immediately gives an upper bound on $2^{\aleph_\omega}$ if $\aleph_\omega$ is strong limit: since $|\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty| \leq 2^{2^{\aleph_0}}$ (the number of ultrafilters on a countable set), and since we assume that $2^{2^{\aleph_0}} < \aleph_\omega$, we have $2^{\aleph_\omega} = |\prod_{n=0}^\infty \aleph_n| < \aleph_{\aleph_\omega}$. It also reduces the problem of evaluating $\aleph_\omega^{\aleph_0}$ (and similar products) to finding the maximum of possible cofinalities of $\prod_{n=0}^\infty \aleph_n$. (There are analogous reductions for cases not covered by Theorem 9.2.) Notice another consequence of Theorem 9.2: as $|\prod A|$ is a possible cofinality, it must be a regular cardinal. Hence if $\aleph_\omega$ is a strong limit, $2^{\aleph_\omega}$ is regular. This does not follow from König's Theorem.

If $2^{\aleph_0} < \aleph_\omega$ then all regular cardinals between $\aleph_{\omega+1}$ and $\aleph_\omega^{\aleph_0}$ are possible cofinalities of $\prod_{n=0}^\infty \aleph_n$. It is interesting to see how they are attained in the models where $2^{\aleph_\omega} > \aleph_{\omega+1}$. A detailed analysis of this was done in [29] for Magidor's model [40]; a similar analysis is possible for other models. Let κ_n, $n < \omega$, be the cardinals in the Prikry sequence; e.g., in the model where $2^{\aleph_\omega} = \aleph_{\omega+2}$ these are $\kappa_n = \aleph_{3n}$. Then $\operatorname{cof}\prod_{n<\omega}\kappa_n^+/D = \aleph_{\omega+1}$ and $\operatorname{cof}\prod_{n<\omega}\kappa_n^{++}/D = \aleph_{\omega+2}$, for every nonprincipal ultrafilter. Hence $\prod_n \aleph_{3n+1}$ has cofinality $\aleph_{\omega+1}$ while $\prod_n \aleph_{3n+2}$ has cofinality $\aleph_{\omega+2}$!

Fundamentals of the pcf theory. The theory developed by Shelah in [51], [57] and [58], and described in detail in the monograph [61] analyzes possible cofinalities of products $\prod A$ where A is a set of regular cardinals with the property that $|A| < \min A$. (For rather trivial reasons, the general pcf theory breaks down in the case when $\sup A$ is a fixed point of the aleph function.) We shall now present the main points of the theory; the reader interested in proofs of the theorems stated here and in the techniques involved might find either [5] or [30] helpful.

Let A be a set of regular cardinals and assume that $|A| < \min A$, and let

$$\operatorname{pcf} A = \left\{ \operatorname{cof}\prod A/D : D \text{ an ultrafilter on } A \right\}.$$

First we mention the trivial facts about pcf:

- $A \subseteq \operatorname{pcf} A$,
- if $A_1 \subseteq A_2$ then $\operatorname{pcf} A_1 \subseteq \operatorname{pcf} A_2$,
- $\operatorname{pcf}(A_1 \cup A_2) = \operatorname{pcf} A_1 \cup \operatorname{pcf} A_2$.

Another not so difficult observation is that when $|\operatorname{pcf} A| < \min A$ then $\operatorname{pcf}\operatorname{pcf} A = \operatorname{pcf} A$.

The key result of the pcf theory is the following:

THEOREM 9.3 (Existence of generators). *There exists a collection* $\{B_\nu : \nu \in \operatorname{pcf} A\}$ *of subsets of* A *such that for every* ν,

(i) $\nu = \max\operatorname{pcf} B_\nu$,
(ii) $\nu \notin \operatorname{pcf}(A - B_\nu)$.

(The sets B_ν are called *generators* of pcf A.)

In other words:

(i) for every ultrafilter D on B_ν, $\operatorname{cof} \prod A/D \leq \nu$; and there exists some D on B_ν such that $\operatorname{cof} \prod A/D = \nu$,

(ii) for every D, if $\operatorname{cof} \prod A/D = \nu$ then $B_\nu \in D$.

Therefore, the cofinality of $\prod A/D$ is determined by which generators belong to D:

$$\operatorname{cof} \prod A/D = \text{the least } \lambda \text{ such that } B_\lambda \in D.$$

Let us note some consequences of Theorem 9.3:

(iii) $|\operatorname{pcf} A| \leq 2^{|A|}$.

This is because $|\operatorname{pcf} A|$ does not exceed the number of generators, which is at most $2^{|A|}$. An immediate consequence is a better bound on $2^{\aleph_\omega}$ if $\aleph_\omega$ is a strong limit: $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.

(iv) pcf A has a maximal element.

If not, then $\{ A - B_\nu : \nu \in \operatorname{pcf} A \}$ has the finite intersection property; let D be an ultrafilter extending it, and let $\lambda = \operatorname{cof} \prod A/D$. Then $B_\lambda \in D$, a contradiction.

The same argument proves the following important property of generators:

THEOREM 9.4 (Compactness). *For every $X \subseteq A$ there exists a finite set of cardinals $\nu_1, \dots, \nu_k \in \operatorname{pcf} X$ such that $X \subseteq B_{\nu_1} \cup \dots \cup B_{\nu_k}$.*

Transitive generators and the localization theorem. Generators of pcf A are not necessarily unique, as (e.g.) changing B_ν by a finite set won't affect properties (i) or (ii). The B_ν's are however unique up to the B_η's for $\eta < \nu$. More precisely: for every $\kappa \leq \max \operatorname{pcf} A$, let

$$\begin{aligned} J_\kappa = {} & \text{the ideal (of subsets of } A\text{) generated by the sets } B_\eta, \\ & \text{for } \eta < \kappa \text{ and } \eta \in \operatorname{pcf} A. \end{aligned}$$

Properties (i) and (ii) imply that for every $X \subseteq A$,

$$X \in J_\kappa \quad \text{if and only if} \quad \operatorname{cof} \prod X/D < \kappa \text{ for every ultrafilter } D \text{ on } X,$$

and that each generator B_ν is unique mod J_ν.

An important feature of the pcf theory is that pcf A has *transitive* generators (incidentally, this theorem is highly nontrivial):

THEOREM 9.5. *There exist generators B_ν, $\nu \in \operatorname{pcf} A$, such that whenever $\eta \in B_\nu$ then $B_\eta \subseteq B_\nu$.*

Transitivity of generators is particularly useful when applied to sets $A = \operatorname{pcf} A$. A fairly straightforward use of transitivity and compactness yields the following important result:

THEOREM 9.6 (Localization Theorem). *Let $X \subseteq \operatorname{pcf} A$ and let $\lambda \in \operatorname{pcf} X$. Then X has a subset X_0 such that $|X_0| \leq |A|$, and such that $\lambda \in \operatorname{pcf} X_0$.*

An upper bound for $|\operatorname{pcf} A|$. The strength of the Localization Theorem is best illustrated by its application providing an upper bound on the size of $\operatorname{pcf} A$, and consequently, on the function $2^{\aleph_\alpha}$ for singular $\aleph_\alpha$. We shall now outline this application of the pcf theory.

First we should mention that one application of the techniques used in the pcf theory is the following analog of Theorem 9.1 for uncountable cofinality:

THEOREM 9.7. *If λ is a singular cardinal of uncountable cofinality then there exists a closed unbounded set $C \subseteq \lambda$ such that $\operatorname{cof} \prod_{\alpha \in C} \alpha^+ / D = \lambda^+$ for every ultrafilter D concentrating on end-segments of C.*

Now let $A = \{\aleph_n\}_{n=0}^{\infty}$. By theorem 9.2, $\operatorname{pcf} A$ is an interval of regular cardinals and, if $2^{\aleph_0} < \aleph_\omega$ then $\max \operatorname{pcf} A = \aleph_\omega^{\aleph_0}$. We claim that $\max \operatorname{pcf} A < \aleph_{\omega_4}$, and therefore $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$.

The structure of $\operatorname{pcf} A$ induces a closure operation on Θ, where $\aleph_{\Theta+1} = \max \operatorname{pcf} A$: if $X \subseteq \Theta$, we let

$$\alpha \in \overline{X} \quad \text{if and only if} \quad \aleph_{\alpha+1} \in \operatorname{pcf}\{\, \aleph_{\xi+1} : \xi \in X \,\}.$$

From Theorem 9.7 it follows that if $\lambda < \Theta$ is an ordinal of uncountable cofinality then there exists a club $C \subseteq \lambda$ such that $\sup \overline{C} = \lambda$. From the Localization Theorem it follows that every set $X \subseteq \Theta$ of order type ω_1 has a countable initial segment X_0 such that $\sup \overline{X_0} \geq \sup X$.

Shelah now proceeds to show that if $\Theta \geq \aleph_4$ then no closure operation on Θ with those two properties exists. Therefore $\Theta < \aleph_4$, and $\max \operatorname{pcf}\{\aleph_n\}_{n=0}^{\infty} < \aleph_{\omega_4}$!

There are some elements of the proof that are specific to $\aleph_4$, making it impossible at present to bring down the upper bound to, say, $\aleph_{\omega_3}$. The method definitely does not work for $\aleph_{\omega_1}$, cf. [33], but it is still an open problem whether $2^{\aleph_\omega}$ can be greater than $\aleph_{\omega_1}$ (with $\aleph_\omega$ strong limit). In fact, it is still unknown whether $|A| < |\operatorname{pcf} A|$ is possible (together with $|A| < \min A$).

Reduced products of ordinals. The main technique used in the analysis of possible cofinalities involves reduced products of ordinal numbers. We shall conclude the article with a brief description of this topic.

Let A be an infinite set, and let I be an ideal on A. For ordinal functions f, g on A we define:

$$f =_I g \quad \text{if} \quad \{a \in A : f(a) \neq g(a)\} \in I,$$
$$f <_I g \quad \text{if} \quad \{a \in A : f(a) \geq g(a)\} \in I.$$

Let λ_a, $a \in A$, be limit ordinal numbers. The *reduced product*

$$\prod_{a \in A} \lambda_a / I$$

consists of equivalence classes of ordinal functions f mod $=_I$, such that $f(a) \in \lambda_a$ for all $a \in A$.

Let κ be a regular cardinal. We say that $P = \prod_{a \in A} \lambda_a / I$ is *κ-directed* if every subset X of P of size less than κ has an upper bound in P, i.e., a function $f \in P$ such that $g <_I f$ for every $g \in X$. We say that P has *true cofinality* κ if there exists an increasing sequence $\langle f_\alpha : \alpha < \kappa \rangle$ cofinal in P, i.e., $f_0 <_I f_1 <_I \cdots$, and for every $g \in P$ there is an α such that $g <_I f_\alpha$. Note that if P has true cofinality κ then P is κ-directed.

We recall that for a given set A of regular cardinals, J_κ ($\kappa \leq \max \operatorname{pcf} A$) denotes the ideal on A generated by the B_ν, $\nu < \kappa$. The crucial property of the generators which enables the analysis of pcf A is this:

THEOREM 9.8. *If A is a set of regular cardinals such that $|A| < \min A$ and if $\kappa \in \operatorname{pcf} A$ then $\prod_{\nu \in B_\kappa} \nu / J_\kappa$ has true cofinality κ.*

In fact, the technology provided by the pcf theory goes beyond analyzing the ideals J_κ. Perhaps the most general result on reduced products is the following. (This is explicitly stated for $\kappa > 2^{|A|}$ in Lemma 2.4 of [30], and can be proved for $\kappa > |A|^+$ as in Theorems 7.3 and 7.9 of [5]):

THEOREM 9.9 (Splitting Theorem). *Let I be an ideal on A, let λ_a, $a \in A$, be limit ordinals of cofinality greater than $|A|^+$ and let κ be a regular cardinal, $\kappa > |A|^+$. If $\prod_{a \in A} \lambda_a / I$ is κ-directed, then either $\prod_{a \in A} \lambda_a / I$ is κ^+-directed, or it has true cofinality κ, or there exist A_1, $A_2 \notin I$, $A_1 \cup A_2 = A$, such that $\prod_{a \in A_1} \lambda_a / I$ is κ^+-directed and $\prod_{a \in A_2} \lambda_a / I$ has true cofinality κ.*

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THE COMPLEXITY OF PROPOSITIONAL PROOFS

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§1. **Introduction.** The classical propositional calculus has an undeserved reputation among logicians as being essentially trivial. I hope to convince the reader that it presents some of the most challenging and intriguing problems in modern logic.

Although the problem of the complexity of propositional proofs is very natural, it has been investigated systematically only since the late 1960s. Interest in the problem arose from two fields connected with computers, automated theorem proving and computational complexity theory. The earliest paper in the subject is a ground-breaking article by Tseitin [62], the published version of a talk given in 1966 at a Leningrad seminar. In the three decades since that talk, substantial progress has been made in determining the relative complexity of proof systems, and in proving strong lower bounds for some restricted proof systems. However, major problems remain to challenge researchers.

The present paper provides a survey of the field, and of some of the techniques that have proved successful in deriving lower bounds on the complexity of proofs. A major area only touched upon here is the proof theory of bounded arithmetic and its relation to the complexity of propositional proofs. The reader is referred to the book by Buss [10] for background in bounded arithmetic. The forthcoming book by Krajíček [40] also gives a good introduction to bounded arithmetic, as well as covering most of the basic results in complexity of propositional proofs.

§2. **Proof systems and simulation.** The literature of mathematical logic contains a very wide variety of proof systems. To compare their efficiency, we need a general definition of a proof system. In this section, we give such a definition, together with another that formalizes the relation holding between two proof systems when one can simulate the other efficiently. The definitions are adapted from Cook and Reckhow [20].

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Let Σ be a finite alphabet; we write Σ^* for the set of all finite strings over Σ. A *language* is defined as a subset of Σ^*, that is, a set of strings over a fixed alphabet Σ. The length of a string x is written as $|x|$.

DEFINITION 2.1. If Σ_1 and Σ_2 are finite alphabets, a function f from Σ_1^* into Σ_2^* is in $\mathcal{L}$ if it can be computed by a deterministic Turing machine in time bounded by a polynomial in the length of the input.

The class $\mathcal{L}$ of polynomial-time computable functions is a way of making precise the vague notion of "feasibly computable function".

DEFINITION 2.2. If $L \subseteq \Sigma^*$, a *proof system for* L is a function $f : \Sigma_1^* \to L$ for some alphabet Σ_1, where $f \in \mathcal{L}$ and f is onto. A proof system f is *polynomially bounded* if there is a polynomial $p(n)$ such that for all $y \in L$, there is an $x \in \Sigma_1^*$ such that $y = f(x)$ and $|x| \leq p(|y|)$.

The intention of this definition is that $f(x) = y$ is to hold if x is a proof of y. The crucial property of a proof system as defined above is that, given an alleged proof, there is a feasible method for checking whether or not it really is a proof, and if so, of what it is a proof. A standard axiomatic proof system for the tautologies, for example, can be brought under the definition by associating the following function f with the proof system $\mathcal{F}$: if a string of symbols σ is a legitimate proof in $\mathcal{F}$ of a formula A, then let $f(\sigma) = A$; if it is not a proof in $\mathcal{F}$ then let $f(\sigma) = T$, where T is some standard tautology, say $P \vee \neg P$.

Let us recall here some of the basic definitions in computational complexity theory (for details the reader is referred to [32, 36, 46]). A set of strings is in the class $\mathcal{P}$ ($\mathcal{NP}$) if it is recognized by a deterministic (non-deterministic) Turing machine in time polynomial in the length of the input. A set of strings is in the class *co-*$\mathcal{NP}$ if it is the complement of a language in $\mathcal{NP}$. In more logical terms, a set S of strings is in $\mathcal{P}$ if its characteristic function is in $\mathcal{L}$, while it is in $\mathcal{NP}$ if the condition $y \in S$ can be expressed in the form $(\exists x)(|x| \leq p(|y|) \wedge R(x, y))$, where p is a polynomial, and R is a polynomial-time computable relation. Thus $\mathcal{P}$ is the polynomial-time analogue of the recursive sets, while $\mathcal{NP}$ corresponds to the recursively enumerable sets. Thus the basic question $\mathcal{P} =? \mathcal{NP}$ is the polynomial-time analogue of the halting problem.

The importance of our main question for theoretical computer science lies in the following result of Cook and Reckhow [20].

THEOREM 2.1. *$\mathcal{NP}$ = co-$\mathcal{NP}$ if and only if there is a polynomially-bounded proof system for the classical tautologies.*

PROOF. If $\mathcal{NP} = co\text{-}\mathcal{NP}$ then since the set TAUT of classical tautologies is in $co\text{-}\mathcal{NP}$, TAUT would be in $\mathcal{NP}$, that is to say, there would be a nondeterministic Turing machine M accepting TAUT. Let f be the function such that $f(x) = y$ if and only if x encodes a computation of M that accepts y; then f is a polynomially-bounded proof system for TAUT.

Conversely, let us assume that there is a polynomially-bounded proof system for TAUT. Let L be a language in $\mathcal{NP}$. By the basic $\mathcal{NP}$-completeness result of Cook [16], L is reducible to the complement of TAUT in the sense that there is a function $f \in \mathcal{L}$ so that for any string x, $x \in L$ if and only if $f(x) \notin TAUT$. Hence a nondeterministic polynomial-time procedure for accepting the complement of L is: on input x, compute $f(x)$ and accept x if $f(x)$ has a proof in the proof system. Hence, $\mathcal{NP}$ is closed under complementation, so $\mathcal{NP} = co\text{-}\mathcal{NP}$. ⊣

This equivalence result underlines the very far-reaching nature of the widely believed conjecture $\mathcal{NP} \neq co\text{-}\mathcal{NP}$. The conjecture implies that even ZFC, together with any true axioms of infinity that are thought desirable (provided that they have a sufficiently simple syntactic form) is not a polynomially-bounded proof system for the classical tautologies (where we take a proof of $TAUT(\ulcorner A \urcorner)$ as a proof of the tautology A).

We can say nothing of interest about the complexity of such powerful proof systems as the above (in effect, the strongest we can imagine). We can, however, order proof systems in terms of complexity, and prove some non-trivial separation results for systems low down in the hierarchy.

DEFINITION 2.3. If $f_1 : \Sigma_1^* \to L$ and $f_2 : \Sigma_2^* \to L$ are proof systems for L, then f_2 *p-simulates* f_1 provided that there is a polynomial-time computable function $g : \Sigma_1^* \to \Sigma_2^*$ such that $f_2(g(x)) = f_1(x)$ for all x.

Thus g is a feasible translation function that translates proofs in f_1 into proofs in f_2. We have assumed in the above definition that the language of both proof systems is the same. Reckhow's thesis [52, §5.1.2] contains a more general definition of p-simulation that eliminates this restriction. It is easy to see that the p-simulation relation is reflexive and transitive, and also that the following theorem can be proved from the definitions.

THEOREM 2.2. *If a proof system f_2 for L p-simulates a polynomially bounded proof system f_1, then f_2 is also polynomially bounded.*

The intersection of the p-simulation relation and its converse is an equivalence relation; thus we can segregate classes of proof systems into equivalence classes within which the systems are "equally efficient up to a polynomial."

§3. A map of proof systems.

Since the complexity class $\mathcal{P}$ is closed under complementation, it follows that if $\mathcal{P} = \mathcal{NP}$ then $\mathcal{NP} = co\text{-}\mathcal{NP}$. This suggests that we might attack the problem $\mathcal{P} =?\mathcal{NP}$ by trying to prove that $\mathcal{NP} \neq co\text{-}\mathcal{NP}$; by Theorem 2.1, this is the same as trying to show that there is no polynomially-bounded proof system for the classical tautologies. This line of research was first suggested in papers by Cook and Reckhow [19, 20]. At the moment, the goal of settling the question $\mathcal{NP} \neq co\text{-}\mathcal{NP}$ seems rather distant. However, progress has been made in classifying the relative complexity of well known proof systems, and in proving lower bounds for restricted systems. An attractive feature of the research programme is that we can hope to approach the goal step by step, developing ideas and techniques for simpler systems first.

The diagram in Figure 1 is a map showing the relative efficiency of various systems. The boxes in the diagram indicate equivalence classes of proof systems under the p-equivalence relation. Systems below the dotted line have been shown to be not polynomially bounded, while no such lower bounds are known for those that lie above the line. Hence, the dotted line represents the current frontier of research on the main problem. Although systems below the line are no longer candidates for the role of a polynomially bounded proof system, there are still some interesting open problems concerning the relative complexity of such systems. Questions of this sort, although not directly related to such problems as $\mathcal{NP} =?co\text{-}\mathcal{NP}$, have some relevance to the more practical problem of constructing efficient automatic theorem provers. Although the more powerful systems above the dotted line are the current focus of interest in the complex of questions surrounding the $\mathcal{NP} =?co\text{-}\mathcal{NP}$ problem, the systems below allow simple and easily mechanized search strategies, and so are still of considerable interest in automated theorem proving.

An arrow from one box to the other in the diagram indicates that any proof system in the first box can p-simulate any system in the second box. In the case of cut-free Gentzen systems, this simulation must be understood as referring to a particular language on which both systems are based. An arrow with a slash through it indicates that no p-simulation is possible between any two systems in the classes in question. If a simulation is possible in the reverse direction, then we can say that systems in one class are strictly more powerful than systems in the other (up to a polynomial). The diagram shows that all such questions of relative strength have been settled for systems below the dotted line, with the exception of the case of the relative complexity of resolution and cut-free Gentzen systems where connectives other than the biconditional and negation are involved.

The diagram shows only a selection from the wide variety of proof systems that have been considered in the literature of logic, automatic theorem proving and combinatorics. A more detailed diagram, showing a wider selection

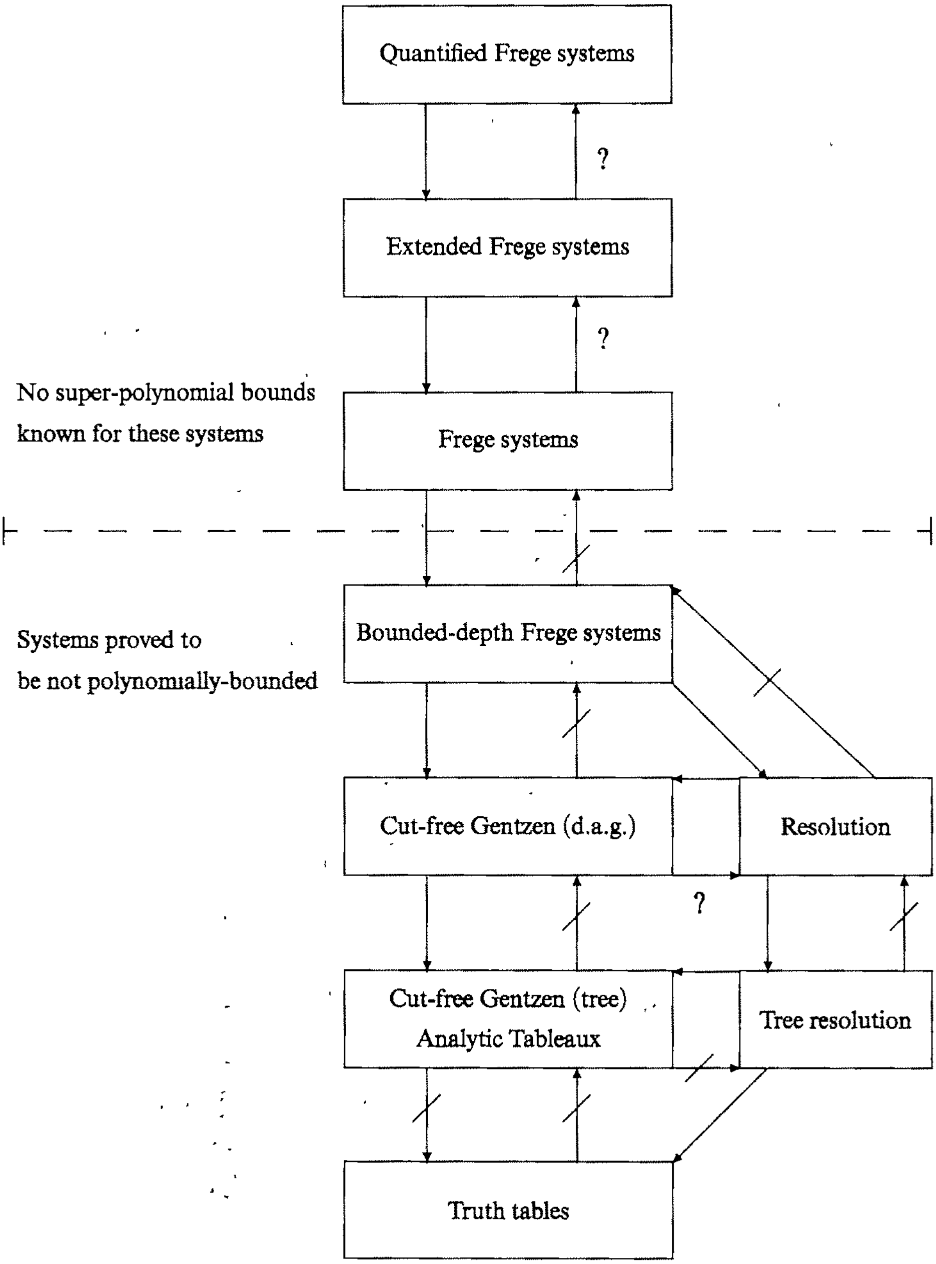

FIGURE 1. Proof system map.

of proof systems, though not reflecting work after 1976, is to be found in Reckhow [52].

Before proceeding to consider particular proof systems, let us fix our notation. We assume an infinite supply of propositional variables and their negations; a variable or its negation is a *literal*. We say that a variable P and its negation $\sim P$ are *complements* of each other; we write the complement of a literal l as $\bar{l}$. A finite set of literals is a *clause*; it is to be interpreted as the disjunction of the literals contained in it. A set of clauses is to be interpreted as their conjunction. A clause *mentions* a literal l if either l or $\bar{l}$ is in the clause. The *length* of a clause is the number of literals in it. We shall sometimes write a clause by juxtaposing the literals in it.

An *assignment* is an assignment of truth-values to a set of propositional variables; some variables may remain unset under an assignment. If Σ is a set of clauses, and ϕ an assignment, then we write $\Sigma \restriction \phi$ for the set of clauses that results from Σ by replacing variables by their values under ϕ and making obvious simplifications. That is to say, if a clause in Σ contains a literal made true by ϕ, then it is removed from the set, while if a literal in a clause is falsified by ϕ then it is removed from the clause. The notation $[l := 1]$ denotes the assignment that sets the literal l to 1 and is otherwise undefined, similarly for $[l := 0]$.

It is useful to fix terminology relating to graphs and trees here. A *graph* consists of a finite set of vertices, a finite set of edges and an incidence relation so that every edge is incident with exactly two distinct vertices (the endpoints of the edge). That is to say, the graphs considered here can contain multiple edges, but not loops; a graph is *simple* if it has at most one edge between any two vertices. Trees should be visualized as genealogical trees, with the root at the top; the nodes immediately below a given node in a tree are its children. The *depth of a tree* T, written $Depth(T)$, is the maximum length of a branch in T.

Derivations in a proof system can be represented either as trees, or as sequences of steps (where a step could be a formula or a sequent). It is normal in the proof-theoretic literature to represent derivations as trees. It is clear, though, that this representation is inefficient, since a step must be repeated every time it is used. If S is a proof system, we denote the corresponding proof system in which derivations are represented as trees by S_{Tree}, reserving the notation S for the system in which derivations are represented as sequences.

§4. **Analytic tableaux.** The method of analytic tableaux, or truth trees, is employed in many introductory texts; it is given a particularly elegant formulation in Smullyan's monograph [59]. Here we shall only consider the simple form of the method where all formulas are clauses. If Σ is a contradictory set of clauses, then a *tableau for* Σ is a tree in which the interior nodes are associated with clauses from Σ; if a node is associated with a given clause, then the children of that node are labeled with the literals in the clause. Note that the node associated with a clause is not labeled with that clause itself, so that the root of the tree remains unlabeled. A tableau for Σ is a *refutation of* Σ if every branch in the tableau is closed (i.e. contains a literal and its negation). We define the *size* of a tableau refutation as the number of interior nodes in the tableau (this measure of complexity, omitting the leaves of the tree, is convenient for inductive proofs). If Σ is a set of clauses, then $t(\Sigma)$ is defined to be the minimum size of a tableau refutation of Σ. Because of the simple structure of tableau refutations, it is possible to prove exact lower bounds on their complexity. The basic tools are the following lemmas.

LEMMA 4.1. *In a tableau refutation of minimal size, no branch contains repeated literals.*

PROOF. If a tableau refutation contains a branch with repeated literals, then it can be pruned as follows. Let T be a subtree of the tableau whose root is associated with a clause containing a literal l, and this literal l labels a node in the tableau on the path from the root of the tableau to T. Replace T with the immediate subtree of T whose root is labeled with l, but replacing the label on this subtree with the label on the root of T. The resulting tableau is still closed, and is smaller than the original. ⊣

LEMMA 4.2. *If* Σ *is an unsatisfiable set of clauses, then* $t(\Sigma)$ *satisfies the recursive equation*

$$t(\Sigma) = \min\{t(\Sigma\restriction[l_1 := 1]) + \cdots + t(\Sigma\restriction[l_n := 1]) + 1 : l_1 \vee \cdots \vee l_n \in \Sigma\}.$$

PROOF. For $C = l_1 \vee \cdots \vee l_n \in \Sigma$, let T be a tableau refutation of Σ that is minimal among refutations that have C associated with their root. Let $T_1, \dots, T_n$ be the immediate subtrees of T having $l_1, \dots, l_n$ as labels on their roots. By Lemma 4.1, the literal l_i does not occur in T_i below the root of T_i; the complement of l_i may occur as the label of at least one leaf in T_i. Thus if we remove from T_i the leaves labeled with $\overline{l_i}$, the result is a refutation of $\Sigma\restriction[l_i := 1]$. Hence the size of T_i is $t(\Sigma\restriction[l_i := 1])$, so that the size of T is

$$t(\Sigma\restriction[l_1 := 1]) + \cdots + t(\Sigma\restriction[l_n := 1]) + 1.$$

Choosing C to minimize this function, we obtain the equation of the lemma. ⊣

A truth table for a formula with n variables, represented as a vector of 0's and 1's, has length 2^n, so that the truth table method is inefficient for large values of n. Of course, we are only considering asymptotic complexity measures here. In practice, the truth table method may be quite efficient for formulas containing a small number of variables, given a reasonably sophisticated implementation. It is easy, however, to find contradictory sets of clauses containing n variables that can be refuted quickly by elementary proof methods, for example the sets A_n containing all the variables $P_1, \dots, P_n$ together with the formula $\sim P_1 \vee \cdots \vee \sim P_n$. The set A_n has a tableau refutation of size $n + 1$.

Somewhat surprisingly, there are cases where truth tables are more efficient than analytic tableaux. This fact was first observed by Marcello D'Agostino, who proved the next result [22].

THEOREM 4.1. *The analytic tableau proof system cannot p-simulate the method of truth tables.*

PROOF. Let Π_n be the set of all clauses of length n in n variables. For any literals l_1 and l_2 in Π_n, the sets of clauses $\Pi_n \restriction [l_1 := 1]$ and $\Pi_n \restriction [l_2 := 1]$ are logically isomorphic (that is to say, one can be obtained from the other by permuting variables and replacing literals by their complements). Hence, $t(\Pi_n \restriction [l_1 := 1]) = t(\Pi_n \restriction [l_2 := 1])$. It follows by Lemma 4.2 that $t(\Pi_n)$ can be computed by the recursion: $t(\Pi_1) = 2$, $t(\Pi_{n+1}) = (n+1)t(\Pi_n) + 1$. This leads to the explicit formula

$$t(\Pi_n) = n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right),$$

asymptotic to $e.n!$. By Stirling's approximation, $(2^n)^c = o(n!)$ for any fixed c, completing the proof. ⊣

Although analytic tableaux work well on simple examples, there are cases where any tableau refutation necessarily contains a great deal of repetition. This is shown by a set of examples due to Cook [17]. Cook's construction associates a set of clauses with a labeled binary tree as follows. Let T be a binary tree in which the interior nodes are labeled with distinct variables. We associate a set of clauses $\Sigma(T)$ with T, in such a way that each branch b in $\Sigma(T)$ has a clause $C_b \in \Sigma(T)$ associated with it. The variables in C_b are those labeling the nodes in b; if P is such a variable, then P is included in C_b if b branches to the left below the node labeled with P, otherwise C_b contains $\sim P$. Figure 2 shows a simple example.

Cook's clauses are the sets of clauses $\Sigma_n = \Sigma(T_n)$ associated with the complete binary tree T_n of depth n. To include the case where $n = 0$, we take T_0 to consist of a single node, counted as an interior node; the set of clauses $\Sigma(T_0)$ is $\{\Lambda\}$, where Λ is the empty clause.

If one of the variables in $\Sigma(T)$ is set to 0 or 1, then the resulting simplified set of clauses is also of the form $\Sigma(T')$ for some binary tree T'. Let l be a literal in $\Sigma(T)$, and P the variable in l. Define $T \restriction [l := 1]$ to be the tree

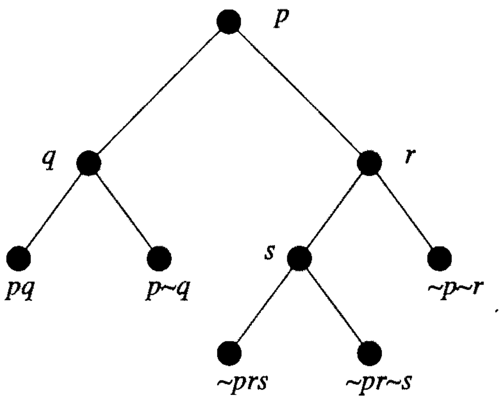

FIGURE 2. $\Sigma(T) = \{pq, p{\sim}q, {\sim}prs, {\sim}pr{\sim}s, {\sim}p{\sim}r\}$

resulting from T by replacing the subtree whose root is labeled with P by either its immediate left or right subtree, depending on whether l is negated or not. Then it is easy to see that $\Sigma(T)\restriction[l := 1] = \Sigma(T\restriction[l := 1])$.

The next lemma allows us to compute $t(T) = t(\Sigma(T))$ directly from the structure of T.

LEMMA 4.3. *The function $t(T)$ satisfies the following recursion equations:*

1. *If T has only one node, then $t(T) = 1$;*
2. *If T has immediate subtrees U and V, then*

$$t(T) = t(U).t(V) + \min\{t(U), t(V)\}.$$

PROOF. If T has only one node, then $\Sigma(T) = \{\Lambda\}$, so $t(T) = 1$ (recall that by our convention, the unique node in a one-node tree counts as an interior node).

Assume the recursion equations hold for trees of size less than that of T, and let T have immediate subtrees U and V. Let $C = l_1 \vee \cdots \vee l_k$ be a clause in $\Sigma(T)$ that is associated with a branch ending in a leaf in U (the argument for branches in V is symmetrical). Define U_j for $2 \leq j \leq k$ to be the labeled tree $U\restriction[l_j := 1]$. Let $t_C(T)$ be the size of a minimal tableau in which C is associated with the root. Then by Lemma 4.2,

$$\begin{aligned} t_C(T) &= t(T\restriction[l_1 := 1]) + \cdots + t(T\restriction[l_k := 1]) + 1 \\ &= t(V) + \textstyle\sum_{j=2}^{k}[t(V).t(U_j) + \min\{t(V), t(U_j)\}] + 1 \\ &\qquad \text{(by the induction hypothesis)} \\ &= t(V)[1 + \textstyle\sum_{j=2}^{k} t(U_j)] + \sum_{j=2}^{k} \min\{t(V), t(U_j)\} + 1. \qquad (1) \end{aligned}$$

By Lemma 4.2 again,

$$1 + \sum_{j=2}^{k} t(U_j) \geq t(U), \tag{2}$$

so by (1),

$$\begin{aligned} t_C(T) &\geq t(V)[1 + \textstyle\sum_{j=2}^{k} t(U_j)] + \min\{t(V), [1 + \textstyle\sum_{j=2}^{k} t(U_j)]\} \\ &\geq t(V).t(U) + \min\{t(V), t(U)\}. \end{aligned} \tag{3}$$

For the opposite inequality, assume that $t(U) \leq t(V)$ and that P is the variable labeling the root of T. Let l_1 be P or $\sim P$ according to whether U is the left or right subtree of T, let $l_2 \vee \cdots \vee l_k$ be the clause associated with the root of a minimal tableau refutation of $t(U)$, and C be the clause $l_1 \vee \cdots \vee l_k$. Then for every j, $t(U_j) \leq t(V)$, so that by (1),

$$\begin{aligned} t_C(T) &= t(V)[1 + \textstyle\sum_{j=2}^{k} t(U_j)] + \textstyle\sum_{j=2}^{k} t(U_j) + 1 \\ &= t(V).t(U) + t(U), \end{aligned} \tag{4}$$

completing the proof. ⊣

THEOREM 4.2.

1. *The clauses* Σ_n *satisfy the recursion equations*

$$t(\Sigma_0) = 1, \quad t(\Sigma_{n+1}) = t(\Sigma_n).[t(\Sigma_n) + 1];$$

2. *The asymptotic behaviour of the function* $t(\Sigma_n)$ *is given by* $t(\Sigma_n) \sim 2^{c2^n}$, *where* $0.67618 < c < 0.67819$.

PROOF. The left and right subtrees of the complete binary tree T_n are isomorphic, so the first claim follows immediately from Lemma 4.3.

Let $z_n = t(\Sigma_n)$; we wish to estimate the growth of z_n. Taking logarithms to the base 2, we have by the first part of the lemma,

$$\log z_{n+1} = 2 \log z_n + \log(1 + 1/z_n), \tag{5}$$

hence

$$\log z_{n+1} = 2^{n-1} + 2^{n-2} \log(1 + 1/z_1) + \cdots + \log(1 + 1/z_n), \tag{6}$$

so that

$$\frac{\log z_{n+1}}{2^{n+1}} = \frac{1}{2} + \frac{\log(1 + 1/z_1)}{4} + \cdots + \frac{\log(1 + 1/z_n)}{2^{n+1}}. \tag{7}$$

he right hand side of (7) converges rapidly; we find $c = 0.67618634966\ldots$ from $n = 5$. ⊣

The preceding theorem is due to Cook; a version of it appeared without proof in [19]. Cook's original unpublished proof [17] contains a gap; the proof above is joint work of Cook and the present author. Murray and Rosenthal [45] prove a lower bound of $2^{2^{n-1}}$ for the size of analytic tableau refutations of Σ_n.

Theorem 4.2 has some significance for automated theorem proving based on simple tableau methods. The set Σ_6 contains only 64 clauses of length 6, but the minimal tableau refutation for Σ_6 has 10,650,056,950,806 interior nodes. This shows that any practical implementation of the tableau method must incorporate routines to eliminate repetition in tableau construction.

§5. **Resolution.** The resolution rule is a simple form of the familiar cut rule. If Al and $B\bar{l}$ are clauses, then the clause AB may be inferred by the resolution rule, *resolving on* the literal l. A *resolution refutation* of a set of clauses Σ is a derivation of the empty clause from Σ, using the resolution rule. Refutations can be represented as trees or as sequences of clauses; the worst case complexity differs considerably depending on the representation. We shall distinguish between the two by describing the first system as "tree resolution," the second simply as "resolution."

Although resolution operates only on clauses, it can be converted into a general purpose theorem prover for tautologies by employing an efficient method of conversion to conjunctive normal form, first used by Tseitin [62]. Let A be a formula containing various binary connectives such as $\rightarrow$ and $\equiv$; associate a literal with each subformula of A so that the literal associated with a subformula $\sim B$ is the complement of the literal associated with B. If the subformula is a propositional variable, then the associated literal is simply the variable itself. We write l_B for the literal associated with the subformula B. If B is a subformula having the form $C \circ D$, where $\circ$ is a binary connective, then $Cl(B)$ is the set of clauses making up the conjunctive normal form of $l_B \equiv (C \circ D)$. For example, if B has the form $(C \equiv D)$, then $Cl(B)$ is the set of clauses

$$\{\, \overline{l_B}\,\overline{l_C}\,l_D,\ \overline{l_B}\,l_C\,\overline{l_D},\ l_B\,\overline{l_C}\,\overline{l_D},\ l_B\,l_C\,l_D \,\}.$$

The set of clauses $Def(A)$ is defined as the union of all $Cl(B)$, where B is a compound subformula of A.

If A is a tautology, then the set $Def(A) \cup \{\overline{l_A}\}$ is contradictory. Thus we define a proof of A in the resolution system to be a proof of Λ from $Def(A) \cup \{\overline{l_A}\}$. Such a proof of A we shall refer to as a proof by *resolution with limited extension* for the set of connectives (other than $\sim$) occurring in A. In particular, we shall discuss below the system of resolution with limited extension for the biconditional; we refer to this system as *Res*($\equiv$). Note that the size of the set of clauses $Def(A) \cup \{\overline{l_A}\}$ is linear in the size of A, whereas the same is not true for the conjunctive normal form of $\sim A$ itself (the conjunctive normal form of $P_1 \equiv P_2 \equiv \cdots \equiv P_n$ has size $2^{O(n)}$).

The size of a tree resolution proof is defined as the number of leaves in the tree; if Σ is a contradictory set of clauses, then $tr(\Sigma)$ is defined as the minimal size of a tree resolution refutation of Σ. We shall refer to the clauses at the leaves of a tree resolution derivation as the "input clauses" of the derivation.

THEOREM 5.1.

1 *Tree resolution p-simulates the method of analytic tableaux.*

2 *The method of analytic tableaux cannot p-simulate tree resolution.*

PROOF. (1) We shall show for any inconsistent set of clauses Σ that $tr(\Sigma) \leq t(\Sigma)$. The proof is by induction on the number of variables in Σ. If $\Sigma = \{\Lambda\}$, then $t(\Sigma) = tr(\Sigma) = 1$. Let Σ contain at least one variable, and let $l_1 \vee \cdots \vee l_k$ be the clause associated with the root of a minimal tableau refutation of Σ. Let $T_1, \ldots, T_k$ be the sub-tableaux whose roots are labeled with $l_1, \ldots, l_k$. By Lemma 4.1, none of the clauses associated with the interior nodes of T_i contain the literal l_i, so that the only occurrence of the literal l_i as a label in T_i apart from the label on the root is on leaves labeled $\overline{l_i}$. Hence, by removing these leaves, and the label on the root, the tree T_i becomes a tableau refutation of $\Sigma[l_i := 1]$. By induction hypothesis, $\Sigma[l_i := 1]$ has a tree resolution refutation U_i whose size is less than equal to that of T_i. By adding $\overline{l_i}$ to the appropriate clauses labeling the nodes of U_i, we obtain a tree resolution proof of $\overline{l_i}$ from Σ. Starting from $l_1 \vee \cdots \vee l_k$ and resolving successively on $l_1, \ldots, l_k$, we can combine $U_1, \ldots, U_k$ to obtain a tree resolution refutation of Σ whose size is bounded by the sum of the sizes of $U_1, \ldots, U_k$, completing the induction step.

(2) The examples Σ_n of the preceding section have very small tree resolution refutations. In fact, we can label the interior nodes of T_n with clauses so that it is a tree refutation of Σ_n. ⊣

A sequence of clauses $C_1, \ldots, C_k$ in a resolution derivation is an *irregularity* if each C_i, $i < k$, is a premiss for C_{i+1}, and there is a literal l that appears in C_1 and C_k, but does not appear in any clause C_j, where $1 < j < k$. That is to say, the literal l is removed by resolution from C_1, and is then later re-introduced in a clause depending on C_2. A derivation is *regular* if it contains no irregularity.

LEMMA 5.1. *A tree resolution refutation of minimal size is regular.*

PROOF. Let $C_1, \ldots, C_k$ be an irregularity in a tree refutation; we shall show how this may be removed while decreasing the size of the refutation. This is accomplished by discarding the first resolution on l, so that for every i, $i < k$, C_i is replaced by a clause D_i, where D_i is a subclause of $C_i l$. If at any point in the new refutation, the literal resolved upon in the original inference of C_{i+1} from C_i is missing from D_i, then we simply set $D_{i+1} = D_i$.

The resulting refutation is smaller than the original (we have discarded at least one subtree). Since no new irregularities are introduced in the process of removal, the Lemma follows. ⊣

The corresponding lemma for resolution fails. Andreas Goerdt [33] shows that there is an infinite sequence of contradictory sets of clauses having polynomial-size resolution refutations for which the size of any regular resolution refutation grows faster than any fixed polynomial. Goerdt's examples are modified versions of the pigeonhole clauses described in Section 7 below.

LEMMA 5.2. *If T is a regular resolution proof of a literal l from a set of clauses Σ, then the result of deleting all occurrences of l from T is a regular resolution refutation of* $\Sigma \restriction [l := 0]$.

PROOF. Since T is regular, it can contain no application of resolution where the literal l is resolved on. Hence, $\bar{l}$ cannot occur in any input clause in T, so that the tree resulting from the deletion of l is a refutation of $\Sigma \restriction [l := 0]$. ⊣

Regular tree resolution is closely related to the method of *semantic trees* introduced by Robinson [54] and Kowalski and Hayes [37]. A semantic tree is a binary tree in which the nodes have assignments associated with them. The assignment associated with the root is empty. If ϕ is an assignment associated with an interior node in the tree then the assignments associated with the children of the node are the assignments ϕ_1 and ϕ_2 extending ϕ with $\phi_1(P) = 0$ and $\phi_2(P) = 1$, where P is a variable not in the domain of ϕ. A semantic tree T is a *refutation* of a set of clauses Σ if the variables assigned values in T all belong to Σ and each of the assignments at the leaves of T falsify a clause in Σ.

We can rewrite a regular tree resolution refutation of a set of clauses as a semantic tree by the following technique. First, associate the empty assignment with the root. Second, if $A \lor B$ is a clause in the tree derived by resolution from $A \lor P$ and $B \lor {\sim}P$, and ϕ is associated with the conclusion of the inference, then we associate with the premisses the extensions of ϕ obtained by setting P to 0 and 1 respectively. Conversely, a semantic tree refutation of minimal size can be converted into a resolution refutation by associating with a leaf a clause falsified at that leaf, and then performing resolutions by resolving on the literals labeling the edges.

Regular refutations of a special kind are produced by the *Davis-Putnam procedure*. Given a set of Σ of input clauses, this procedure involves choosing a variable and then forming all possible non-tautologous resolvents from Σ that result from eliminating the chosen variable. This procedure is repeated until the empty clause is produced or no more resolvents can be formed (in which case the input set must be satisfiable). Clearly the refutation produced depends uniquely on the order of elimination adopted. The name of the procedure derives from a well known paper by Davis and Putnam on automated theorem proving [24].

The phrase "Davis-Putnam procedure" is unfortunately ambiguous, since in the literature of automated theorem proving, it refers to a decision procedure for satisfiability involving the recursive construction of a semantic tree. The confusion stems from the fact that during the implementation of the algorithm described in [24], Davis, Logemann and Loveland [23] replaced the original method by this second one, mainly for reasons of space efficiency. In the present article, the phrase "Davis-Putnam procedure" refers to the restricted version of the resolution proof procedure where the refutations are produced by the first method described above.

In the remainder of this section, the lower bounds proved for various forms of resolution are given for the graph-based examples introduced by Tseitin [62]. This paper of Tseitin is a landmark as the first to give non-trivial lower bounds for propositional proofs; although it pre-dates the first papers on $\mathcal{NP}$-completeness, the distinction between polynomial and exponential growth of proofs is already clear in it.

If G is a graph, then a *labeling* G' of G is an assignment of literals to the edges of G, so that distinct edges are assigned literals that are distinct and not complements of each other, together with an assignment $\text{Charge}(x) \in \{0, 1\}$ to each of the vertices x in G. If G' is a labeled graph, and x a vertex in G', and $l_1, \ldots, l_k$ the literals labeling the edges attached to x, then $\text{Clauses}(x)$ is the set of clauses equivalent to the conjunctive normal form of the modulo 2 equation $l_1 \oplus \cdots \oplus l_k = \text{Charge}(x)$. That is to say, a clause C in $\text{Clauses}(x)$ contains the literals $l_1, \ldots, l_k$, and the parity of the number of complemented literals in $\{l_1, \ldots, l_k\}$ in C is opposite to that of $\text{Charge}(x)$. The set of clauses $\text{Clauses}(G')$ is the union of all the sets $\text{Clauses}(x)$, for x a vertex in G. Let us write $\text{Charge}(G')$ for the sum modulo 2 of the charges on the vertices of G'; a labeling G' of G is *even* or *odd* depending on whether $\text{Charge}(G')$ is 0 or 1.

LEMMA 5.3. *If G is a connected graph, then* $\text{Clauses}(G')$ *is contradictory if and only if the labeling G' is odd.*

PROOF. Assume that the labeling G' is odd. If we sum the left-hand sides of all the mod 2 equations associated with the vertices of G, the result is 0, because each literal is attached to exactly two vertices, and so appears twice in the sum. On the other hand, the right-hand sides sum to 1, by assumption, so the set of equations, and so the set of clauses, are contradictory.

Conversely, suppose $\text{Charge}(G') = 0$. Let x and y be vertices in G connected by an edge e. The set of clauses $\text{Clauses}(G')$ is unchanged if we perform the following operation: replace the literal labeling e by its complement, and replace Charge(x) and Charge(y) by their complements. Let us refer to this operation as *transferring a charge* between x and y. If x and y are two vertices in G with Charge(x) = Charge(y) = 1, then there is a chain of vertices $x = v_1, \dots, v_j = y$ forming a path from x to y. If we successively transfer a charge from v_1 to v_2 to ... v_j, the result is a set of clauses associated with a labeling in which two fewer vertices have an odd charge. Repeating this process, we obtain a labeling in which all vertices have the charge 0. A satisfying assignment is obtained by setting all the literals on the edges to 0. ⊣

For the remainder of this section, we assume that G' is a connected graph with an odd labeling; we identify an edge with the literal labeling it. The proof of the preceding lemma shows that any two sets of clauses associated with an odd labeling of a connected graph G are logically isomorphic, so we shall sometimes write Clauses(G) to represent any such set.

Let G' be a labeled graph, and l an edge in G'. Define $G' \restriction [l := 0]$ to be the labeled graph resulting from G' by deleting l, and $G' \restriction [l := 1]$ the labeled graph resulting from G' by deleting l and complementing the charges on the vertices incident with l.

Lemma 5.4. *For any graph G' with an odd labeling,* Clauses $(G' \restriction [l := 0]) =$ Clauses $(G') \restriction [l := 0]$, *and* Clauses $(G' \restriction [l := 1]) =$ Clauses $(G') \restriction [l := 1]$.

Proof. By definition. ⊣

Regular resolution refutations of sets of clauses based on graphs can be visualized in terms of joining together connected subgraphs, as we show in the next two lemmas.

Lemma 5.5. *Let G' be a labeled graph. If T is a resolution proof of a clause C from* Clauses(G'), *then there is a connected component H' of G' so that T is a resolution proof of C from* Clauses(H').

Proof. By induction on the size of T. ⊣

Let G be an unlabeled connected graph. A *deletion tree for G* is a binary tree labeled with connected subgraphs of G so that the root is labeled with G, and if an interior node is labeled with a subgraph G_1, then the children of that node are labeled with graphs resulting from the deletion of an edge e in G_1. That is, if the deletion of e results in the disconnection of G_1, then the two children are labeled with the two resulting connected subgraphs G_2 and G_3; otherwise, the children are labeled with two copies of G_1 with e deleted.

Lemma 5.6. *Let G be an unlabeled connected graph, and G' an odd labeling of G.*

1. *A deletion tree for G can be labeled with clauses so that it becomes a tree resolution refutation of* Clauses(G').
2. *A tree resolution refutation of* Clauses(G') *can be labeled with subgraphs of G so that it becomes a deletion tree for G.*

Proof. (1) We prove this by induction on the number of edges in G. If G has no edges, then a deletion tree for G consists of a single node; label this node with Λ. Let T be a deletion tree for a graph G with immediate subtrees T_1 and T_2 whose roots are labeled with graphs G_1 and G_2 obtained by deleting an edge labeled with the literal l in G'. Let G_1' and G_2' be the odd labeled components of $G'\restriction[l := 0]$ and $G'\restriction[l := 1]$ respectively. When the deletion of l disconnects G these components must be distinct, and so G_1 and G_2 correspond to the graphs immediately below G in the deletion tree. By induction hypothesis, T_1 and T_2 can be labeled with clauses so that they are regular resolution refutations R_1, R_2 of Clauses(G_1') and Clauses(G_2'). Since by Lemma 5.4, Clauses $(G_1') \subseteq$ Clauses $(G')\restriction[l := 0]$, we can add l to the appropriate clauses in R_1 so that it is a regular resolution proof of l from Clauses(G'). Similarly, we can add $\bar{l}$ to R_2 to produce a regular resolution proof of $\bar{l}$ from Clauses(G'). Hence, by labeling the root of T with Λ, we have produced a regular resolution refutation of Clauses(G').

(2) We prove this by induction on the depth of the refutation of Clauses(G'). If the refutation has depth 0, then it consists of Λ alone, so G consists of a single node. Label the root with this node. Let R be a regular tree resolution refutation of Clauses(G') in which the immediate subtrees R_1 and R_2 are proofs of l and $\bar{l}$. By Lemma 5.2 and Lemma 5.4, if we delete l and $\bar{l}$ from R_1 and R_2 respectively, we obtain resolution refutations R_1' and R_2' of Clauses$(G'\restriction[l := 0])$ and Clauses$(G'\restriction[l := 1])$. By Lemmas 5.3 and 5.5, R_1' and R_2' are refutations of Clauses(G_1') and Clauses(G_2'), where G_1' and G_2' are connected components of $G'\restriction[l := 0]$ and $G'\restriction[l := 1]$ with an odd labeling. By induction hypothesis, the nodes of R_1' and R_2' can be labeled with subgraphs of G_1 and G_2 so that they become deletion trees for G_1 and G_2. If the deletion of l disconnects G, then G_1 and G_2 are distinct components of the resulting disconnected graph. Hence, if we label the root of R with G, the result is a deletion tree for G. ⊣

The above lemma shows that we can compute the refutation complexity function tr(Clauses(G)) directly from the graph G. Thus, we have reduced the problem of proving lower bounds for tree resolution to that of finding graphs that require large deletion trees. The next result is due in its essentials to Tseitin [62].

THEOREM 5.2. *Tree resolution cannot p-simulate the Davis-Putnam procedure.*

PROOF. For $n > 0$, let the graph G_n consist of $N = 2^n$ vertices $v_1, \dots, v_N$ with adjacent vertices v_i and v_{i+1} joined by n edges. The set of clauses Clauses(G_n) contains $2^{3n}(1/2 - O(2^{-n}))$ clauses of size at most $2n$.

Let T be a minimal deletion tree for G_n. Define a branch in T as follows: whenever the children of a node in T are labeled with distinct graphs resulting from the disconnection of the parent graph, then the branch contains the larger of the two sibling graphs. Since it requires the deletion of n edges to disconnect such components, it follows that there are at least $n(n-1)$ interior nodes q in T along the chosen path where the deletion of the chosen edge does not disconnect the graph at the node. At such a node q, the complexity of the subtree rooted at q must be twice that of either of its subtrees. Thus the size of T is at least $2^{n(n-1)}$, showing that $tr(\text{Clauses}(G_n)) = 2^{n(n-1)}$, a function that is not bounded by a polynomial in the size of Clauses(G_n).

On the other hand, there are Davis-Putnam refutations of Clauses(G_n) with sizes linear in the size of the input clauses. The order of elimination is first to eliminate all edges between v_1 and v_2, then between v_2 and v_3 and so on. Since the size of the clauses produced by this procedure is at most $2n$, the result is a refutation whose size is linear in the size of Clauses(G_n). ⊣

By considering a different sequence of graphs, we can find a family of clauses for which the smallest resolution refutations are exponentially big. The basic idea of the lower bound proof given below, from Urquhart [63], is due to Armin Haken [35], who introduced an ingenious "bottle-neck" counting argument to prove the corresponding result for the pigeonhole clauses.

The analysis of refutations of Clauses(G') is facilitated by considering those assignments that make exactly one clause in Clauses(G') false. These are easy to describe. If ϕ is an assignment to the edges of G, and x a vertex in G, then Charge(ϕ, x) is defined to be the sum modulo 2 of the values assigned by ϕ to the edges attached to x. An assignment to the edges of G is *x-critical* if Charge$(\phi, y) =$ Charge(y) for all vertices y in G except for x. An x-critical assignment falsifies exactly one clause in Clauses(x), while all other clauses in Clauses(G') are satisfied; it is easy to see that this property characterizes x-critical assignments for G.

If G is a connected graph, we say that an assignment ϕ of truth-values to some of the edges of G is *non-separating* if the graph that results by deleting the edges assigned a value by ϕ is connected.

LEMMA 5.7. *If ϕ is a non-separating assignment to some of the edges of a connected graph G with an odd labeling, and x is any vertex of G, then ϕ may be extended to an x-critical assignment for G.*

PROOF. Fix a spanning tree for G that does not contain any edge assigned a value by ϕ. Assign values arbitarily to any edge not in the spanning tree that has not yet been assigned a value. Fix x as the root of the spanning tree; proceeding from the leaves of the tree inward towards x, assign values to the edges attached to vertices other than x so that Clauses(y) is satisfied. The resulting assignment must be x-critical since Clauses(G) is contradictory. ⊣

The graphs used in the lower bound for resolution are the expander graphs used by Galil [31] to prove an exponential lower bound for regular resolution, with a small modification to simplify the proof. The expander graph H_m is a simple bipartite graph in which each vertex has degree at most 5 and each side contains m^2 vertices (for brevity we write $n = m^2$). The particular family of expander graphs used here was first defined by Margulis [44]. The exact definition of the graphs is not needed; for the lower bound all that is needed is the expanding property proved by Margulis and stated in the next lemma.

LEMMA 5.8. *There is a constant $d > 0$ such that if V_1 is contained in one side of H_m, $|V_1| \leq n/2$, and V_2 consists of all the vertices in the other side of H_m that are connected to vertices of V_1 by an edge, then $|V_2| \geq (1 + d)|V_1|$.*

PROOF. See Gabber and Galil [30], who also provide a numerical lower bound for the expansion factor d. ⊣

The graph G_m is obtained from H_m by the following modification. We add $n - 1$ edges to each side of the graph so that each side forms a connected chain. We call the new edges *side edges* and the edges in the original graph *middle edges*. The resulting graph still satisfies Lemma 5.8, and each vertex in it has degree at most 7. Let Ω_m be Clauses(G_m); Ω_m contains at most $128n$ clauses of length at most 7, so the entire set of clauses has size $O(n)$.

We now specify for each m a set of *restrictions* $\mathcal{R}_m$, a family of assignments to some of the edges in G_m. Let d be the constant in Lemma 5.8, and let f be $d/16$. A restriction in $\mathcal{R}_m$ is specified by choosing a set of $\lceil fn \rceil$ middle edges, and then assigning truth-values arbitrarily to the chosen edges. For $P \in \mathcal{R}_m$, we write $E(P)$ for the set of chosen edges in P. Every restriction P in $\mathcal{R}_m$ is non-separating because at least one middle edge must be unset by P.

If C is a clause, and $P \in \mathcal{R}_m$, then we define $Crit(C, P)$ as the set of vertices $x \in G_m$ such that there is an x-critical assignment ϕ extending P so that $\phi(C) = 0$. In a resolution refutation of Ω_m, a clause C is *P-complex,* where $P \in \mathcal{R}_m$, if C is the first clause in the refutation for which $|Crit(C, P)| \geq n/4$; a clause is *complex* if it is P-complex for some P. For every $P \in \mathcal{R}_m$, a P-complex clause must exist in a refutation of Ω_n, because by Lemma 5.7, $|Crit(\Lambda, P)| = 2n$. The complex clauses in a refutation form a "bottle-neck" in that a given clause can only be P-complex for an exponentially small fraction of all $P \in \mathcal{R}_m$.

LEMMA 5.9. *If C is a P-complex clause, then at least $\lfloor fn \rfloor$ middle edges are mentioned in C.*

PROOF. If C is an input clause in Clauses(x), then $Crit(C, P) = \{x\}$, so we can assume that C is inferred from earlier clauses D and E by the resolution rule. Since the resolution rule is sound, $Crit(C, P) \subseteq Crit(D, P) \cup Crit(E, P)$. Because both $|Crit(D, P)|$ and $|Crit(E, P)|$ are less than $n/4$, $|Crit(C, P)| < n/2$. Let $Crit(C, P) = W_1 \cup W_2$, where W_1 and W_2 are contained in opposite sides of G_m, and $|W_1| \geq |W_2|$; let Y_2 be the set of vertices not in W_2 that are connected to W_1 by a middle edge. Since $|Crit(C, P)| \geq n/4$, $|W_1| \geq n/8$, so by Lemma 5.8, $|Y_2| \geq dn/8$.

Let e be an edge connecting a vertex x in W_1 with a vertex y in Y_2 that is not one of the chosen edges in the restriction P. By the definition of a restriction, there are at least $\lfloor fn \rfloor$ such edges. Since $x \in Crit(C, P)$, there is an x-critical assignment ϕ extending P so that $\phi(C) = 0$. If e is not mentioned in C, then the assignment ϕ' obtained from ϕ by reversing the truth-value of e also falsifies C. The assignment ϕ' is y-critical, and since e is not a chosen edge in P, ϕ' also extends P. This contradicts the assumption that y is not in W_2. ⊣

THEOREM 5.3. *There is a constant $c > 1$ such that for sufficiently large m any resolution refutation of Ω_m contains c^n distinct clauses.*

PROOF. Let us fix a resolution refutation of Ω_m, with respect to which the notion of complex clause is defined. Every $P \in \mathcal{R}_m$ is associated with exactly one P-complex clause in the refutation, but a given complex clause may have a number of restrictions associated with it. We show that the number of restrictions associated with a given complex clause is exponentially small, so that there must be exponentially many complex clauses in the refutation.

Let C be a complex clause. By Lemma 5.9, there is a set $E(C)$ of middle edges mentioned in C, where $|E(C)| = \lfloor fn \rfloor$. The fraction of restrictions P with $|E(P) \cap E(C)| = i$ with respect to which C is P-complex is at most 2^{-i}, since the edges in P are set independently. Hence, the ratio between the number of restrictions associated with C and the total number of restrictions is bounded by

$$\sum_{i=0}^{s} \binom{M}{i} \binom{N-M}{s-i} \binom{N}{s}^{-1} 2^{-i}, \tag{8}$$

where $M = |E(C)| = \lfloor fn \rfloor$, $s = |E(P)| = \lceil fn \rceil$, and N is the number of middle edges in H_m.

We can find a bound for (8) by adapting Chvátal's elegant bound [14] for the tail of the hypergeometric distribution. First, we establish an inequality:

$$\begin{aligned}
&\sum_{i=0}^{s-j}\binom{M}{i}\binom{N-M}{s-i}\binom{s-i}{j}\binom{N}{s}^{-1}\\
&\quad= \tbinom{N-M}{j}\textstyle\sum_{i=0}^{s-j}\tbinom{M}{i}\tbinom{N-M-j}{s-i-j}\tbinom{N}{s}^{-1}\\
&\quad= \tbinom{N-M}{j}\tbinom{N-j}{s-j}\tbinom{N}{s}^{-1}\\
&\quad= \tbinom{N-M}{j}\tbinom{s}{j}\tbinom{N}{j}^{-1}\\
&\quad\leq \tbinom{s}{j}\left(\tfrac{N-M}{N}\right)^{j}.
\end{aligned}\tag{9}$$

The bound on the ratio (8) follows from this inequality by the computation:

$$\begin{aligned}
&\sum_{i=0}^{s}\binom{M}{i}\binom{N-M}{s-i}\binom{N}{s}^{-1}2^{-i}\\
&\quad= 2^{-s}\textstyle\sum_{i=0}^{s}\tbinom{M}{i}\tbinom{N-M}{s-i}\tbinom{N}{s}^{-1}\sum_{j=0}^{s-i}\tbinom{s-i}{j}\\
&\quad= 2^{-s}\textstyle\sum_{j=0}^{s}\sum_{i=0}^{s-j}\tbinom{M}{i}\tbinom{N-M}{s-i}\tbinom{s-i}{j}\tbinom{N}{s}^{-1}\\
&\quad\leq 2^{-s}\textstyle\sum_{j=0}^{s}\tbinom{s}{j}\left(\tfrac{N-M}{N}\right)^{j}\\
&\quad= 2^{-s}\left(2-\tfrac{M}{N}\right)^{s}\\
&\quad= \left(1-\tfrac{M}{2N}\right)^{s}.
\end{aligned}\tag{10}$$

Since $N \leq 5n$, $M/2N \geq f/11$, for sufficiently large m. It follows that there must be at least c^n complex clauses in the refutation, where $c = (1-f/11)^{-f}$. ⊣

The basic properties of the graph-based clauses that are exploited in the lower bound argument are an "expansion" property (reflected in Lemma 5.9), and an "independence" property (reflected in the fact that in a restriction the chosen middle edges can be set independently). Chvátal and Szemerédi, in a far reaching generalization of the preceding theorem, proved a lower bound for sets of randomly chosen clauses, by showing that sets of random clauses satisfy appropriately generalized forms of these properties. Define the *random family of m clauses of length k over n variables* to consist of a random sample of size m chosen with replacement from the set of all clauses of length k with variables chosen from a set of n variables. Chvátal and Szemerédi [15] prove the following result.

THEOREM 5.4. *For every choice of positive integers n, c, k such that $k \geq 3$ and $c2^{-k} \geq 0.7$, there is a positive number ε such that, with probability tending* *to one as n tends to infinity, the random family of cn clauses of size k over n variables is unsatisfiable and its resolution complexity is at least* $(1+\varepsilon)^n$.

§6. **Cut-free Gentzen systems.** Cut-free Gentzen systems have proved popular in work on automated deduction, since they allow simple search strategies in constructing derivations. In the present section, we consider a sequent calculus G based on the biconditional as the only connective. A sequent has the form $\Gamma \vdash \Delta$, where Γ and Δ are sequences of formulas. The axioms of G are sequents of the form $A \vdash A$. The rules of inference of G are as follows:

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (\mathit{Permutation})$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ (\mathit{Contraction})$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Theta \vdash \Delta, \Xi}\ (\mathit{Thinning})$$

$$\frac{\Gamma, A, B \vdash \Delta \qquad \Gamma \vdash \Delta, A, B}{\Gamma, (A \equiv B) \vdash \Delta}\ (\equiv \vdash)$$

$$\frac{\Gamma, A \vdash \Delta, B \qquad \Gamma, B \vdash \Delta, A}{\Gamma \vdash \Delta, (A \equiv B)}\ (\vdash \equiv)$$

An alternative formulation of G is possible in which the axioms are sequents of the form $\Gamma, A \vdash A, \Delta$, and the thinning rule is omitted. We denote this alternative formulation by G'. It is the version adopted by Smullyan [59, pp. 105-106], and is usually employed in automatic theorem provers; the system of Wang [67] is of this type. The Leningrad group headed by Shanin in their work on computer search for natural logical proofs [55] used a formulation of the second type for the proof search, but then transformed the resulting derivations into simplified derivations in a system of the first type by a pruning procedure.

It is natural to use a system of the second type in a computer search, because if the usual 'bottom-up' search procedure is employed, the thinning rule can (when employed in reverse) result in potentially useful information being discarded. However, as we show below, the two formulations are quite distinct from the point of view of worst case complexity. There are certain sequents for which short proofs can be found only by employing the thinning rule.

Derivations in G have the *subformula property*, that is, any formula occurring in the derivation must occur as a subformula in the conclusion of the derivation. In fact, an analysis of derivations in G shows that *occurrences* of formulas in the derivation can be identified with *occurrences* of formulas in the conclusion. This can be seen by tracing occurrences of formulas step by step up the derivation from the conclusion. Thus in an application of ($\vdash\equiv$), for example, the displayed occurrences of A in the premisses are to be identified with the displayed occurrence of A in the conclusion of the inference. Similarly, in an application of Contraction, both occurrences of the displayed formula A in the premiss are to be identified with the occurrence of A in the conclusion. This identification of occurrences will be used subsequently to prove lower bounds for the proof systems.

The Cut rule

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash \Delta, A}{\Gamma \vdash \Delta}\ (Cut)$$

is not necessary for completeness, but in some cases results in much shorter derivations. The formula A in the Cut rule is said to be the *cut formula*. The subformula property fails for derivations in the system $G + Cut$ that results by adding the Cut rule to G. However, the property is preserved if we restrict the Cut rule appropriately. We shall say that a derivation of a sequent $\Gamma \vdash \Delta$ is a derivation in G *with the analytic Cut rule* if the derivation belongs to $G + Cut$, and all cut formulas are subformulas of formulas in the conclusion $\Gamma \vdash \Delta$.

We shall now prove some results from [65] that settle the relative complexity of resolution and cut-free Gentzen systems, at least for the case of tautologies involving the biconditional. As in the case of tree resolution, we define the complexity of a derivation in G_{Tree} to be the number of leaves in the derivation (that is, the number of occurrences of axioms). The simulation in the following theorem is due to Tseitin [62].

THEOREM 6.1. *The system* $Res(\equiv)_{Tree}$ *p-simulates* G_{Tree}.

PROOF. If D is a derivation in G_{Tree} of a sequent $\vdash A$, and $\Theta \vdash \Xi$ is a sequent in D, then any formula B that occurs as an antecedent or consequent formula in $\Theta \vdash \Xi$ is a subformula of A, and hence is assigned a literal l_B as part of $Def(A)$. We construct a clause corresponding to the sequent $\Theta \vdash \Xi$ by forming the disjunction of all the propositional variables $\overline{l_B}$ if B is an antecedent formula, together with the l_B if B is a consequent formula.

The resulting tree of clauses is not a resolution proof, but is easily converted into a resolution derivation of Λ from $Def(A) \cup \{\overline{l_A}\}$ by inserting some added resolvents. In every case except when a sequent is an axiom, we show that we can derive a subclause of the clause corresponding to the sequent. The rules ($\equiv\vdash$) and ($\vdash\equiv$) are simulated by forming two resolvents by resolving the clauses corresponding to the premisses against clauses in $Def(A)$, and then resolving these in turn to derive a subclause of the clause corresponding to the conclusion of the inference. The result is a resolution refutation having complexity $O(n)$, where n is the complexity of D. ⊣

We now define the sequence of biconditional tautologies that form the basis of the lower bounds in this section. For any $n > 0$, let U_n be the formula

$$P_n \equiv P_{n-1} \equiv \ldots \equiv P_1 \equiv P_n \equiv P_{n-1} \equiv \ldots \equiv P_1$$

where we are omitting parentheses according to the convention of association to the right; for example, $A \equiv B \equiv C$ abbreviates $(A \equiv (B \equiv C))$. All the variables in U_n occur exactly twice, so that U_n is a tautology. To distinguish between two occurrences of the same variable P_k; we shall write the first occurrence as P_k^1, the second occurrence as P_k^2. The subformula of U_n beginning with the subformula occurrence P_k^i will be denoted by U_k^i. Thus U_k^2 contains k occurrences of variables, while U_k^1 contains $n+k$ occurrences; in particular, $U_n^1 = U_n$.

If $\Gamma \vdash \Delta$ is a sequent, we use the term *O-assignment* to refer to an assignment of truth-values $\{0, 1\}$ to the occurrences of the variables in $\Gamma \vdash \Delta$. An O-assignment is extended to all the occurrences of subformulas in the sequent by the usual truth table method. The entire sequent takes the value 0 under an O-assignment if all occurrences of formulas in Γ take the value 1, and all the occurrences of formulas in Δ the value 0. It is essential to the notion of O-assignment that distinct truth values can be assigned to different occurrences of the same variable. In particular, by choosing an appropriate O-assignment, it is possible to falsify a tautological sequent. If D is a cut-free derivation of a sequent $\Gamma \vdash \Delta$, then any O-assignment for $\Gamma \vdash \Delta$ can be extended to all the sequents in D; this is possible because of the identification noted earlier between occurrences of formulas in the conclusion and occurrences of formulas in D. A given occurrence of a subformula in the conclusion can correspond to multiple occurrences in a sequent earlier in the derivation; the form of the rules, however, guarantees that all of these occurrences have the same value as the occurrence in the conclusion. The notion of O-assignment was used earlier in a somewhat different form in [64] to prove an exponential lower bound for cut-free Gentzen systems. An exponential lower bound for the tree version of a cut-free Gentzen system was proved earlier by Statman [61].

The formula U_n has 2^{2n} O-assignments associated with it. We are interested only in certain of these. We shall call an O-assignment to U_n *critical* if there is exactly one variable in U_n whose occurrences in U_n are assigned different values. If this variable is P_k, then we say that the O-assignment in question is *k-critical*. All critical O-assignments falsify the formula U_n; a critical O-assignment is uniquely determined by k, and the values the O-assignment gives to the occurrences P_i^1, for $1 \leq i \leq n$, so that there are $n \cdot 2^n$ distinct critical O-assignments for the sequent U_n.

THEOREM 6.2. *The minimal complexity of a derivation of U_n in the system G_{Tree} is $n \cdot 2^n$.*

PROOF. Any critical O-assignment for $\vdash U_n$ falsifies the conclusion of the derivation, $\vdash U_n$, and if it falsifies the conclusion of an inference, then it also falsifies one of the premisses. Hence, if ϕ is any critical O-assignment for $\vdash U_n$, we can trace a branch in the derivation from the conclusion to an axiom, so that all the sequents in the branch are falsified by ϕ. If ϕ is a k-critical O-assignment, then the only subformula of U_n whose occurrences have distinct values assigned to them under ϕ is P_k. Thus the axiom at the tip of the branch must have the form $P_k \vdash P_k$, where the antecedent and consequent occurrences of P_k correspond to distinct occurrences of P_k in U_n. It follows from this that if ϕ and ψ are respectively k-critical and j-critical for $j \neq k$, then the branches for ϕ and ψ are distinct. Furthermore, since the axiom at the tip of the branch for ϕ must contain P_k^2, it follows that the branch must contain occurrences of all P_i^1, for $1 \leq i \leq n$, as whole formulas on some sequent in the branch. Since distinct k-critical O-assignments give distinct sequences of values to the occurrences P_i^1, all these branches must also be distinct. There are $n \cdot 2^n$ O-assignments for $\vdash U_n$, so that the complexity of the derivation is at least $n \cdot 2^n$. It is easily verified that there is a derivation of this complexity. ⊣

We now show that there are relatively short proofs of U_n in the resolution system. To describe the proofs, it is useful to give a graphical representation of these tautologies. The sets of clauses derived from the sequence of formulas U_n will be represented in the form Clauses(G_n), for a sequence of graphs G_n.

The graph G_n associated with the formula U_n is a planar graph that we describe by giving the co-ordinates of its nodes. G_n has as its nodes the set of points $\{(i, 1) : 0 \leq i \leq n\} \cup \{(i, 0) : 1 \leq i \leq n-1\}$. The following nodes are joined in G_n: $(i, 1)$ to $(i+1, 1)$, $(i, 0)$ to both $(i, 1)$ and $(i+1, 0)$, $(n, 1)$ to both $(n-1, 0)$ and $(1, 0)$. The graph may be described as a ladder with a few extra attachments. The labels attached to G_n are as follows. The vertical lines, and the line joining $(n, 1)$ to $(n-1, 0)$ are labeled with the variables P_n to P_1 from left to right. The horizontal lines joining the points with y co-ordinate 1 are labeled with the variables Q_n^1 to Q_1^1 from left to right; the horizontal lines joining points with y co-ordinate 0 are labeled with the variables Q_{n-1}^2 to Q_2^2 from left to right. The line joining $(n, 1)$ to $(1, 0)$ is labeled with the variable Q_n^2. The node $(0, 1)$ is labeled with 0; all other nodes are labeled 1. The accompanying figure shows the labeled graph corresponding to U_5. A node is shown filled in only if it is labeled with 0.

The set of clauses $Def(U_n) \cup \{\sim Q_n^1\}$, where the variable Q_k^i is correlated with the subformula U_k^i, is identical with Clauses(G_n). The graphs G_n are similar to examples used by Galil ([31, Fig. 3.2.4]) to show that the Davis-Putnam procedure is very sensitive to the order of elimination adopted in forming resolvents; with one order of elimination, ladder-like graphs result

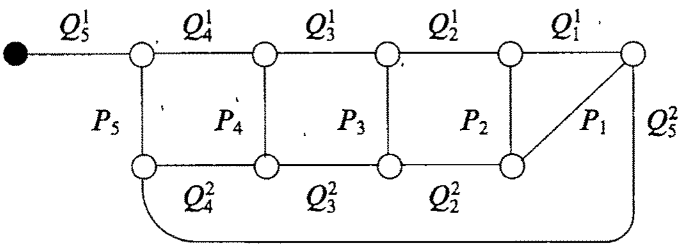

FIGURE 3. The labeled graph G_5.

in exponential-size refutations, while a different order gives rise to linear-size refutations.

THEOREM 6.3. *The tautologies U_n have proofs in $Res(\equiv)_{Tree}$ of complexity $O(n^2)$.*

PROOF. Because the set of clauses $Def(U_n) \cup \{\sim Q_n^1\}$ can be described in terms of the graph G_n, Lemma 5.6 shows that it is sufficient to find a deletion process for G_n in which the underlying tree has $O(n^2)$ leaves. Such a process can be constructed as follows: first, remove four edges so that the result is a $2 \times (n-1)$ grid graph. Now delete a top edge and the corresponding bottom edge in such a way as to break the graph into two components that are as nearly equal in size as possible; repeat this process till subgraphs are reached that consist of two nodes linked by a vertical line, then delete the vertical line. A branch in the resulting tree has length at most $2\log n + c$, for some constant c, so that the tree has $O(n^2)$ leaves. ⊣

COROLLARY 6.1. *The system G_{Tree} cannot p-simulate $Res(\equiv)_{Tree}$.*

In contrast to the foregoing results, if derivations are presented in linear form, then resolution and cut-free Gentzen systems are equally powerful systems (up to a polynomial) when pure biconditional tautologies are considered.

THEOREM 6.4. *Each of the following systems can p-simulate any of the others: G, G + analytic Cut, $Res(\equiv)$.*

PROOF. It is trivially true that G + *analytic Cut* can p-simulate G. In addition, the simulation of G_{Tree} by $Res(\equiv)_{Tree}$ in Theorem 6.1 can be extended readily to a simulation of G + *analytic Cut* by $Res(\equiv)$.

The simulation of G + *analytic Cut* by $Res(\equiv)$ can be reversed as follows. Given a refutation of $Def(A) \cup \{\overline{l_A}\}$ in $Res(\equiv)$, we can convert it into a shorter derivation of l_A in $Res(\equiv)$ by omitting any resolution involving the literal $\overline{l_A}$. We can then simulate this derivation in G + *analytic Cut* by using the reverse of the translation employed earlier; the analytic cut rule can be employed to simulate resolution inferences.

It remains only to show that G can p-simulate G + *analytic Cut*. We shall show how to replace an analytic cut inference by a sequence of inferences using the inference rules of G so that the number of inferences of G used is a linear function of the length of the conclusion of the derivation. Thus let D be a derivation of a sequent $\Gamma \vdash \Delta$ in G + *analytic Cut*. Let

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash \Delta, A}{\Gamma \vdash \Delta}\ (Cut)$$

be an inference by the analytic Cut rule in D. The formula A is a subformula of a formula B occurring as an antecedent or consequent formula in the conclusion of the derivation. Thus there is a sequence of formulas $A = B_0 \cdots B_k = B$ so that B_i is an immediate subformula of B_{i+1}. By using the rules of G, we can derive $\Gamma, B_i \vdash \Delta$ and $\Gamma \vdash B_i, \Delta$ for any i. Thus, let us suppose that B_{i+1} has the form $(B_i \equiv C)$, and that we have already derived the sequents $\Gamma, B_i \vdash \Delta$ and $\Gamma \vdash B_i, \Delta$. By the weakening rule, we can derive $\Gamma, B_i, C \vdash \Delta$ and $\Gamma \vdash B_i, C, \Delta$ and so $\Gamma, B_{i+1} \vdash \Delta$ by $(\equiv \vdash)$. Symmetrically, we can derive $\Gamma \vdash B_{i+1}, \Delta$. This involves 6 extra steps in the proof, so it is possible to derive $\Gamma, B \vdash \Delta$ and $\Gamma \vdash B, \Delta$ in $6k$ extra steps. By repeating this manoeuvre, for any sequent $\Theta \vdash \Xi$ in the derivation D, we can derive a corresponding sequent in G of the form $\Theta, \Gamma' \vdash \Xi, \Delta'$, where Γ' and Δ' are subsets of Γ and Δ. The derivation of $\Gamma \vdash \Delta$ in G that results has complexity $O(k.m)$, where k is the complexity of D, and m is the number of symbols in $\Gamma \vdash \Delta$. ⊣

This somewhat unexpected simulation result depends on the special features of the inference rules for $\equiv$ in G. It extends easily to include negation, but does not appear to extend to conjunction and disjunction. Whether the simulation result holds when the cut-free Gentzen system includes these connectives is open.

We now sketch a result mentioned earlier, that the addition of the Thinning rule results in an exponential shortening of derivations in some cases.

THEOREM 6.5. *A derivation of U_n in the system G' must contain at least $n.2^n$ distinct sequents.*

PROOF. The proof of this result is essentially the same as the proof of Theorem 6.2. As in the earlier proof, we can trace an O-assignment back up a derivation of U_n to an axiom. Axioms corresponding to distinct O-assignments must be distinct, by the argument given earlier to show that the branches of the tree must be distinct. ⊣

Techniques similar to those used in the lower bound for resolution can be used to prove exponential lower bounds for cut-free Gentzen systems. The following result is proved in Urquhart [64].

THEOREM 6.6. *There is a sequence F_n of biconditional tautologies, where each formula has length $O(n^2)$, but the shortest proof of F_n in G contains at least $2^{n/16}$ distinct sequents.*

This result can be improved to a lower bound exponential in the size of a family of biconditional tautologies based on expander graphs by adapting the proof of Theorem 5.3.

We conclude this section with the observation that a cut-free Gentzen system for a given set of connectives and the corresponding analytic tableau system are p-equivalent. This can be seen most easily by using the form of Gentzen system where the thinning rule is omitted. Then (as Dowd [26] first observed) there is a straightforward and efficient translation procedure between the two systems; the details are to be found in Smullyan's book [59, Ch. XI]. The proof of equivalence is completed by showing that (in contrast to the case where proofs are represented as sequences) the system without thinning can simulate the system with the thinning rule in an efficient way.

§7. **Frege systems.** Proof systems p-equivalent to axiomatic systems with schematic axioms and rules form a natural and important class. This family of systems are called "Frege systems" in the literature of proof complexity. Strictly speaking, this is a misnomer, since Frege's original system of propositional logic [28] included a tacitly applied rule of substitution; according to Church [13, p. 158], von Neumann [66] was the first to use axiom schemes to avoid the use of a substitution rule. However, the term "Frege system" seems well entrenched in the complexity literature, so it is employed here.

We assume in this and the following sections a language for propositional logic based on a functionally complete set of connectives, for example the language based on binary disjunction $\vee$ and negation $\sim$. We shall include in addition the propositional constants 0 and 1 standing for "false" and "true" respectively. As we shall see below, the exact choice of language is in many cases not crucial. If A is a formula and $p_1, \ldots, p_m$ a sequence of variables then we write $A[B_1/p_1, \ldots, B_m/p_m]$ for the formula resulting from A by substituting $B_1, \ldots, B_m$ for $p_1, \ldots, p_m$.

A *Frege rule* is defined to be a sequence of formulas written in the form $A_1, \ldots, A_k \vdash A_0$. In the case that the sequence $A_1, \ldots, A_k$ is empty, the rule is referred to as an *axiom scheme*. The rule is *sound* if $A_1, \ldots, A_k \models A_0$, that is, if every truth-value assignment satisfying $A_1, \ldots, A_k$ also satisfies A_0. If $A_1, \ldots, A_k \vdash A_0$ is a Frege rule, then C_0 is *inferred from* $C_1, \ldots, C_k$ by this rule if there is a sequence of formulas $B_1, \ldots, B_m$ and variables $p_1, \ldots, p_m$ so that for all i, $0 \leq i \leq k$, $C_i = A_i[B_1/p_1, \ldots, B_m/p_m]$.

If $\mathcal{F}$ is a set of Frege rules and A a formula, then a *proof of A in $\mathcal{F}$ from $A_1, \ldots, A_m$* is a finite sequence of formulas such that every formula in the sequence is one of $A_1, \ldots, A_m$ or inferred from earlier formulas in the sequence by a rule in $\mathcal{F}$, and the last formula is A. The formulas in the sequence are the *lines* in the proof.

If $\mathcal{F}$ is a set of Frege rules, then it is *implicationally complete* if whenever $A_1, \ldots, A_m \models A_0$ then there is a proof of A_0 in $\mathcal{F}$ from $A_1, \ldots, A_m$. A *Frege system* is defined to be a finite set of sound Frege rules that is implicationally complete.

EXAMPLE 7.1. Shoenfield's system [56, p. 21], in which the primitive connectives are $\vee$ and $\sim$:

Excluded middle: $\vdash \sim p \vee p$;
Expansion rule: $p \vdash q \vee p$;
Contraction rule: $p \vee p \vdash p$;
Associative rule: $p \vee (q \vee r) \vdash (p \vee q) \vee r$;
Cut rule: $p \vee q, \sim p \vee r \vdash q \vee r$.

We define the *size* of a Frege proof as the number of occurrences of symbols in it. Another measure is that of the number of lines in a proof; we shall refer to this measure as the *length* of a proof. The length and size measures of a proof may not be polynomially related, since it is possible for a Frege proof to contain lines that are exponentially large, as a function of the proof's length. The *complexity* of a Frege rule is the number of distinct formulas occurring in the rule; for example, the Cut rule in Shoenfield's system has complexity 7.

Many types of systems familiar in the logical literature are p-equivalent to Frege systems. Among these are systems obtained by adding the cut rule to cut-free Gentzen systems, and systems of natural deduction containing the deduction theorem as a primitive rule. The statement of this equivalence forms one of the main results in Reckhow's thesis [52].

THEOREM 7.1. *Any two systems from the following classes are p-equivalent: Frege systems, natural deduction systems, Gentzen systems with cut.*

The proof of this theorem is straightforward when the two systems are based on the same connectives, or when there is a direct translation possible (for example, such a translation is possible between the connective sets $\{\rightarrow, \sim\}$ and $\{\vee, \sim\}$). However, an efficient direct translation is not possible, for example, in the case of the connective sets $\{\rightarrow, \sim, \equiv\}$ and $\{\wedge, \sim\}$. In this case, it is necessary to employ a technique of indirect translation based on the well known construction of Spira [60], [68, pp. 218-221], [27, pp.

68-73]. The reader is referred to Reckhow [52] or the book by Krajíček [40] for details of the proof.

For general Frege systems only very weak lower bounds on the size of proofs are known. This failure in proving lower bounds mirrors the corresponding lack of success in proving strong lower bounds on the size of formulas or circuits computing explicitly defined Boolean functions (the books by Wegener and Dunne [27, 68] provide good surveys of work in this area). Strong lower bounds have been proved only in the case where significant restrictions are placed on the structure of the proofs considered. These restrictions are a counterpart in proof theory of restricted classes of circuits for which strong lower bounds are known.

We conclude this section with a sketch of the lower bounds just mentioned. We give a full outline of the proof, but omit the rather intricate details of the central combinatorial lemmas on which the proof rests. For the purpose of these lower bounds, we employ the language based on disjunction and negation, together with the constants 0 and 1; a conjunction $A \wedge B$ is treated as an abbreviation for the formula $\sim(\sim A \vee \sim B)$.

A formula can be represented by its formation tree in which the leaves are labeled with propositional variables or constants, and an interior node is labeled with $\vee$ if it is the parent of two nodes, and with $\sim$ if it is the parent of only one. A branch in the tree representing a formula, when traversed from the root to the leaf at the end of the branch is labeled with a block of operators of one kind (say $\sim$), followed by a block of the other kind (say $\vee$), ... , ending with a variable or constant. The *logical depth* of a branch is defined to be the number of blocks of operators labeling the branch. The *depth* of a formula is the maximum logical depth of the branches in its formation tree.

EXAMPLE 7.2. The formula $(\sim p \vee \sim\sim 1) \vee \sim(\sim q \vee r)$ has depth 4.

The depth of a proof in a Frege system is the maximum depth of a line in the proof. The lower bound sketched below is for proofs of bounded depth, in which all formulas have depth bounded by a fixed constant.

The lower bound is based on the propositional pigeon-hole principle, mentioned above as the basis for Haken's exponential lower bound [35] for resolution. Let D, R be finite non-empty sets where $D \cap R = \emptyset$, and let $S = D \cup R$. A *matching* between D and R is a set of mutually disjoint unordered pairs $\{i, j\}$, where $i \in D$, $j \in R$ (that is to say, a matching in the complete bipartite graph $D \times R$). A matching *covers* a vertex i if $\{i, j\}$ belongs to the matching for some vertex j; a matching covers a set X if it covers all the vertices in X. If π is a matching then we denote by $V(\pi)$ the set of vertices covered by π. A matching between D and R is *perfect* if it covers all of the vertices in $D \cup R$.

The pigeon-hole principle states that if $|D| = n + 1$, $|R| = n$ then there is no perfect matching between D and R. To formalize this as a tautology in propositional logic we introduce propositional variablesP_{ij} for $i \in D$, $j \in R$. The language built from these variables and the constants 0,1 using the connectives $\vee, \sim$ we shall refer to as $L(D, R)$; we also refer to the language as L_n in contexts where D, R are understood as the basic sets. The tautology $PHP(D, R)$ is the disjunction

$$\bigvee_{\substack{i \neq j \in D \\ k \in R}} (P_{ik} \wedge P_{jk}) \vee \bigvee_{\substack{i \neq j \in R \\ k \in D}} (P_{ki} \wedge P_{kj}) \vee \bigvee_{i \in D} \bigwedge_{k \in R} {\sim} P_{ik} \vee \bigvee_{k \in R} \bigwedge_{i \in D} {\sim} P_{ik}.$$

We shall also refer to this as PHP_n when the underlying sets are understood. The negation of PHP_n is equivalent to the conjunction of a set of clauses; Haken [35] showed that this set of clauses requires resolution refutations of size c^n, for $c > 1$.

The most important step so far in our understanding of the complexity of propositional proofs was taken by Ajtai in a remarkable paper [1] in which he proved the following result.

THEOREM 7.2. *For a given Frege system $\mathcal{F}$, natural numbers c, d, and sufficiently large n, any depth c proof of PHP_n in $\mathcal{F}$ must have size greater than n^d.*

This theorem serves to separate bounded-depth Frege systems from Frege systems in the map of proof systems, since Buss [11] shows that the pigeon-hole tautologies have polynomial-size proofs in a Frege system (this result improves on the earlier result of Cook and Reckhow [20] showing the same result for extended Frege systems).

Ajtai's proof is a highly ingenious blend of non-standard models for number theory and combinatorics. Subsequent work by a number of authors simplified Ajtai's proof, first eliminating the use of non-standard models [9], second improving the lower bound from super-polynomial to exponential [39, 7, 49, 38]. Krajíček [39] proved the first truly exponential lower bounds for bounded depth proofs, using modified versions of the pigeon-hole formulas for each depth d. In the same paper, he also showed that depth d Frege systems cannot p-simulate depth $d + 1$ Frege systems. Shortly afterwards, Pitassi, Beame and Impagliazzo [7, 49] and (independently) Krajíček, Pudlák and Woods [7, 38] improved Ajtai's lower bound for the pigeon-hole principle from super-polynomial to exponential; their proof is sketched here.

Let D, R be fixed, where $|D| = n + 1$, $|R| = n$. The set of matchings between D and R we shall denote by M_n. A matching π determines a *restriction* ρ_π of the variables of L_n by the following definition. For a variable P_{ij}, if i or j is covered by π then $\rho_\pi(P_{ij}) = 1$ if $\{i, j\} \in \pi$, $\rho_\pi(P_{ij}) = 0$ if $\{i, j\} \notin \pi$; otherwise $\rho_\pi(P_{ij})$ is undefined. Since a matching uniquely determines and is determined by the corresponding restriction, we shall identify a matching with the restriction it determines, and refer to it according to context as a matching or a restriction. If ρ_1 and ρ_2 are two matchings in M_n, and $\rho_1 \cup \rho_2$ is also a matching, then we say that they are *compatible*. If ρ_1 and ρ_2 are compatible matchings, then their union will be written as $\rho_1\rho_2$. If ρ is a matching, then $D\restriction\rho = D \setminus V(\rho)$, $R\restriction\rho = R \setminus V(\rho)$ and $S\restriction\rho = S \setminus V(\rho)$. If M is a set of matchings, and ρ a matching, then $M\restriction\rho$ is defined to be $\{\rho' \setminus \rho : \rho' \in M,\ \rho' \text{ compatible with } \rho\}$.

If A is a formula of L_n, and $\rho \in M_n$, then we denote by $A\restriction\rho$ the formula resulting from A by substituting for the variables in A the constants representing their value under ρ. That is to say, if P_{ij} is set to 1 or 0 by ρ, then we substitute 1 or 0 for P_{ij}, otherwise the variable is unchanged. If Γ is a set of formulas and $\rho \in M_n$ then $\Gamma\restriction\rho$ is $\{A\restriction\rho : A \in \Gamma\}$. The formula $A\restriction\rho$ can be simplified by eliminating the constants by the rules $\neg 0 \equiv 1$, $\neg 1 \equiv 0$, $(0 \vee A) \equiv A$, $(A \vee 0) \equiv A$, $(1 \vee A) \equiv 1$, $(A \vee 1) \equiv 1$. If a formula A can be simplified to a formula B using these rules, then we write $A \equiv B$.

The language L_n contains only binary disjunction. However, in the proofs that follow it is convenient to introduce an auxiliary language that uses unbounded conjunctions and disjunctions. We shall distinguish the order of the terms in such conjunctions and disjunctions.

Let A be an unbounded conjunction each of whose conjuncts is a variable of L_n or a constant. We shall say that A is a *matching term* if the set of pairs $\{i, j\}$ for P_{ij} a variable in A forms a matching. The *size* of a matching term is the cardinality $|\pi|$ of the matching π corresponding to it; the set of vertices $V(A)$ associated with a matching term A is the set of vertices mentioned in the variables in A, that is, the set $V(\pi)$. If π is a matching, then we shall write $\wedge\pi$ for the matching term that describes it, the conjunction containing the set of variables P_{ij} for $\{i, j\} \in \pi$ as conjuncts.

An unbounded disjunction of matching terms we shall call a *matching disjunction*; it is a *matching disjunction over S* if all the vertices mentioned in it are in S. If all of the matching terms in a matching disjunction have size bounded by r, then it is an *r-disjunction.*

Let A be a disjunction in the language L_n, and A_i, $i \in I$, those subformulas of A that are not disjunctions, but every subformula of A properly containing them is a disjunction. Then the *merged form of A* is the unbounded disjunction $\bigvee_{i\in I} A_i$.

The proof of the lower bound is significantly complicated in this case by the fact that we are dealing with a system in which all the steps are tautologies. In contrast, the lower bound for resolution (for example) exploits the fact that a refutation can be considered as making progress towards a contradiction. It is plain that to have a similar notion of "progress" in this case, we have to employ a non-standard "truth" definition. The solution to this problem is to assign each step in a derivation its own space of assignments with respect to which it is a "tautology"; if we choose the spaces in the right way, the rules of inference are sound with respect to these "local tautologies."

The spaces of local assignments are provided by matching trees, that is, decision trees in which the branches represent matchings. We assume that the space of matchings is the set M_n of matchings between D and R, where $|D| = n + 1$, $|R| = n$, $S = D \cup R$.

DEFINITION 7.1. A *matching tree* over S is a tree T satisfying the conditions:

1. The nodes of T other than the leaves are labeled with vertices in S;
2. If a node in T is labeled with a vertex $i \in S$, then the edges leading out of the node are labeled with distinct pairs of the form $\{i, j\}$ where $j \in R$ if $i \in D$, $j \in D$ if $i \in R$;
3. No node or edge label is repeated on a branch of T;
4. If p is a node of T then the edge labels on the path from the root of T to p determine a matching $\pi(p)$ between D and R.

We shall use the notation $Br(T)$ for the set of matchings determined by the branches of T, that is, $\{\pi(l) : l \text{ a leaf in } T\}$. If M is a set of matchings, then T is said to be *complete for M* if for any node p in T labeled with a vertex $i \in S$, the set of matchings $\{\pi(q) : q \text{ a child of } p\}$ consists of all matchings in M of the form $\pi(p) \cup \{\{i, j\}\}$. If the space of matchings is M_n, we shall use the abbreviation "complete" instead of "complete for M_n."

LEMMA 7.1. *Let T be a complete matching tree over the space* $S = D \cup R$, $|D| = n + 1$, $|R| = n$, *and* ρ *a matching in* M_n *such that* $|\rho| + Depth(T) \leq n$. *Then there is a* $\pi \in Br(T)$ *such that* $\pi \cup \rho \in M_n$.

PROOF. We show that by successively choosing nodes in T starting at the root we can find a branch in T so that the required π labels the chosen path. Let us suppose that the nodes have been chosen as far as a node p that is not a leaf. By assumption, $\rho \cup \pi(p) \in M_n$; since $|\rho| + Depth(T) \leq n$, $|\rho \cup \pi(p)| < n$. Let i be the vertex in S labeling node p; there exists at least one matching extending $\rho \cup \pi(p)$ that covers i. Since T is complete, at least one edge below p is labeled with a pair that extends $\rho \cup \pi(p)$ to a matching in M_n. Then we can extend the path by choosing the node at the end of this edge. ⊣

If the leaves of a matching tree T are each labeled with 0 or 1, then it is a *matching decision tree*. We define for $i = 0, 1$,

$$Br_i(T) = \{\pi(l) : l \text{ is a leaf of } T \text{ labeled } i\}.$$

If T is a matching decision tree, then T^c is the matching decision tree that results by changing the leaf labels of T from 0 to 1 and 1 to 0, while $Disj(T)$ is

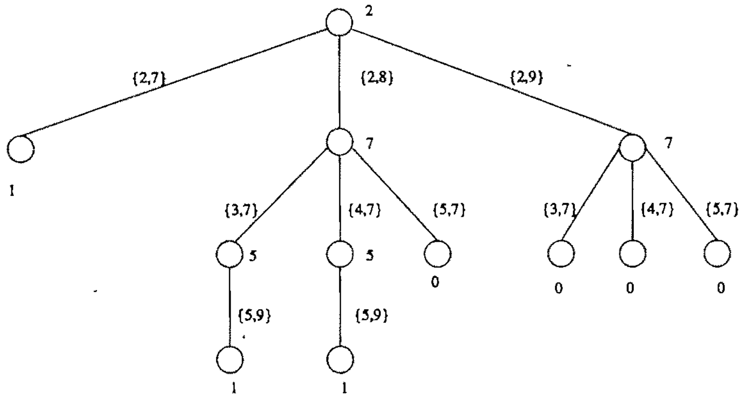

FIGURE 4. A matching decision tree.

the unbounded disjunction $\bigvee\{\bigwedge\pi : \pi \in Br_1(T)\}$. Figure 4 shows a matching decision tree where $D = \{1,2,3,4,5\}$ and $R = \{6,7,8,9\}$.

LEMMA 7.2. *If T is a matching decision tree, and ρ extends a matching $\pi(l) \in Br(T)$, then $Disj(T)\restriction\rho \equiv 0$ or 1 according to whether l is labeled 0 or 1.*

PROOF. If l is labeled 1, then since ρ extends $\pi(l)$, the term $\bigwedge\pi(l)$ is set to 1 by ρ, so that $Disj(T)\restriction\rho \equiv 1$. If l is labeled 0, then we need to establish that for any leaf l' labeled 1, $\bigwedge\pi(l')\restriction\rho \equiv 0$. Let p be the node at which the branches ending in l and l' diverge. If i is the vertex in S labeling p, then $\pi(l)$ and $\pi(l')$ must disagree on the vertex matched with i. Thus $\bigwedge\pi(l')\restriction\rho \equiv 0$, showing that $Disj(T)\restriction\rho \equiv 0$. ⊣

If F is a matching disjunction, and T a matching decision tree, then we say that T *represents* F if for every $\pi(l) \in Br(T)$, $F\restriction\pi(l) \equiv 1$ if l is labeled 1, and $F\restriction\pi(l) \equiv 0$ if l is labeled 0.

We now introduce the basic concept of a k-evaluation; a k-evaluation can be considered as a kind of non-standard truth-definition for a set of formulas. The notion of k-evaluation is due to Krajíček, Pudlák and Woods [38]. The definition of k-evaluation used here differs from that of [38]; in that paper a more general definition is used in which formulas are assigned sets of restrictions rather than complete decision trees.

DEFINITION 7.2. Let Γ be a set of formulas of L_n, closed under subformulas, where $S = D \cup R$, $|D| = n+1$, $|R| = n$. Let $k > 0$. A k-evaluation T is an assignment of complete matching decision trees $T(A)$ to formulas $A \in \Gamma$ so that:

1. $T(A)$ has depth $\leq k$;

2. $T(1)$ is the tree with a single node labeled 1, and $T(0)$ is a tree with a single node labeled 0;
3. $T(P_{ij})$ is the full matching tree for $\{i, j\}$ over S, with a leaf l labeled 1 if $\pi(l)$ contains $\{i, j\}$, otherwise 0;
4. $T(\neg A) = T(A)^c$;
5. If A is a disjunction, and $\bigvee_{i \in I} A_i$ is the merged form of A then $T(A)$ represents $\bigvee_{i \in I} Disj(T(A_i))$.

If T is a k-evaluation for a set of formulas Γ, then the set of matchings $Br(T(A))$ can be considered as a space of truth-value assignments for A; thus if $T(A)$ has all its leaves labeled 1, we can think of A as a kind of "tautology" relative to this space. However, in contrast to the classical notion of tautology, this notion is not preserved under classically sound inferences (this fact is the key to the lower bound argument).

Example 7.3. Let $D = \{1, 2, 3\}$ and $R = \{4, 5\}$, and let $\Gamma = \{P_{14} \vee P_{15}, \neg P_{15} \vee \neg P_{25}, P_{14} \vee \neg P_{25}\}$. Then there is a 2-evaluation for Γ so that the first two formulas in Γ have 1 on all their leaves, but the third formula does not, although it is a logical consequence of the first two.

The following lemma shows that examples like this do not exist if the depth of a k-evaluation is small enough relative to the size of the inference rules of the proof system.

Lemma 7.3. *Let $\mathcal{F}$ be a Frege system in which the complexity of the rules is bounded by f, and $\mathcal{P}$ a proof in $\mathcal{F}$ in the language $L(D, R)$, where $S = D \cup R$, $|R| = n$. If T is a k-evaluation for all the formulas in $\mathcal{P}$ and $k \leq n/f$, then for any line A in $\mathcal{P}$,*

$$\forall \pi (\pi \in Br(T(A)) \Rightarrow Disj(T(A)) \restriction \pi \equiv 1),$$

that is, $T(A)$ has all of its leaves labeled 1.

Proof. The lemma is proved by induction on the number of lines in the proof $\mathcal{P}$. Let

$$\frac{A_1(B_1/p_1, \dots, B_m/p_m), \dots, A_k(B_1/p_1, \dots, B_m/p_m)}{A_0(B_1/p_1, \dots, B_m/p_m)}$$

be an instance of a rule of $\mathcal{F}$, and assume that the lemma holds for all of the premisses of the inference. Let Γ be the set of formulas $A(B_1/p_1, \dots, B_m/p_m)$, where $A(p_1, \dots, p_m)$ is a subformula of some A_i. By assumption, $|\Gamma| \leq f$; let $M = \{\pi_1 \cup \dots \cup \pi_j \in M_n : \pi_i \in Br(T(C_i))\}$, where $\Gamma = \{C_1, \dots, C_j\}$. By Lemma 7.1, if $\pi_i \in Br(T(C_i))$, then there is a $\pi \in M_n$ so that $\pi_i \subseteq \pi$. Let us abbreviate $Disj(T(A))$ as $D(A)$. Then for $\pi \in M$ and $A, B \in \Gamma$,

1. $D(A) \restriction \pi \equiv 0$ or $D(A) \restriction \pi \equiv 1$;
2. $D(0) \restriction \pi \equiv 0$ and $D(1) \restriction \pi \equiv 1$;

3. If $\neg A \in \Gamma$ then $D(\neg A)\restriction \pi \equiv 1 \Leftrightarrow D(A)\restriction \pi \equiv 0$;
4. If $(A \vee B) \in \Gamma$ then $D(A \vee B)\restriction \pi \equiv 1 \Leftrightarrow D(A)\restriction \pi \equiv 1$ or $D(B)\restriction \pi \equiv 1$.

These equivalences follow from the definition of a k-evaluation and from Lemmas 7.1 and 7.2.

For any $\pi \in M$, define an assignment V_π of truth-values to the formulas in Γ by setting $V_\pi(C_i) = 1$ if $D(C_i)\restriction \pi \equiv 1$, $V_\pi(C_i) \equiv 0$ if $D(C_i)\restriction \pi \equiv 0$. The list of equivalences above shows that V_π respects the rules of classical logic. By Lemma 7.2, the premisses of the inference are all assigned the value 1 by V_π; since the rule of inference is sound, the conclusion of the inference is also assigned 1 by V_π. Now let $\sigma \in Br(T(A_0(B_1/p_1, \dots, B_m/p_m)))$. There is a $\pi \in M$ extending σ, so $V_\pi(A_0(B_1/p_1, \dots, B_m/p_m)) = 1$, equivalently, $D(A_0(B_1/p_1, \dots, B_m/p_m))\restriction \sigma \equiv 1$, concluding the proof of the lemma. ⊣

The next lemma shows that, relative to a k-evaluation, $k < n - 1$, the pigeon-hole tautology PHP_n is a "contradiction."

LEMMA 7.4. *Let* $D \cup R = S$, $|D| = n + 1$, $|R| = n$, $PHP_n = PHP(D, R)$. *If* T *is a* k*-evaluation for a set of formulas containing* PHP_n, $k < n - 1$, *then all the leaves of* $T(PHP_n)$ *are labeled 0.*

PROOF. Left as an exercise for the reader; the proof uses Lemma 7.2. ⊣

We now state without proof the central lemma showing that if a set of bounded depth formulas of L_n is subjected to a random restriction then, provided the set is not too large, the set of restricted formulas has associated decision trees of small depth. From this result the lower bound on the size of propositional proofs follows by earlier lemmas.

LEMMA 7.5. *Let* d *be an integer,* $0 < \varepsilon < 1/5$, $0 < \delta < \varepsilon^d$ *and* Γ *a set of formulas of* L_n *of depth* $\leq d$, *closed under subformulas. If* $|\Gamma| < 2^{n^\delta}$, $q = \lceil n^{\varepsilon^d} \rceil$ *and* n *is sufficiently large, then there exists* $\rho \in M_n^q$ *so that there is a* $2n^\delta$*-evaluation of* $\Gamma\restriction\rho$.

This lemma is proved by induction on the depth d. The induction step is handled by a "switching lemma" that says (roughly speaking) that if a matching disjunction is simplified by a random restriction, then with high probability the resulting simplified disjunction can be represented by a small depth decision tree. The name "switching lemma" derives from the corresponding combinatorial lemmas in circuit theory [29, 34], showing that with high probability, the application of a random restriction makes it possible to switch efficiently between conjunctive and disjunctive normal form ("efficiently" in the sense that a large blow-up in formula size does not occur). These lemmas allow the proof of strong lower bounds on the size of bounded depth circuits computing functions such as parity. The reader is referred to papers of Razborov [51] and Beame [5] for elegant proofs of various switching lemmas.

THEOREM 7.3. *Let $\mathcal{F}$ be a Frege system and $d > 4$. Then for sufficiently large n every depth d proof in $\mathcal{F}$ of PHP_n must have size at least 2^{n^δ}, for $\delta < (1/5)^d$.*

PROOF. Let the rules of $\mathcal{F}$ have complexity bounded by f, $0 < \delta < (1/5)^d$, and let $A_1, \ldots, A_t$ be a proof in $\mathcal{F}$ of depth d and size $\leq 2^{n^\delta}$.

Choose ε so that $\varepsilon < 1/5$, $\delta < \varepsilon^d$. By Lemma 7.5, there exists $\rho \in M_n^q$, $q = \lceil n^{\varepsilon^d} \rceil$, and a $2n^\delta$-evaluation T of $\Gamma \upharpoonright \rho$, where Γ is the set of subformulas in the proof $A_1, \ldots, A_t$. Then $A_1 \upharpoonright \rho, \ldots, A_t \upharpoonright \rho$ is a proof in $\mathcal{F}$ in the language $L(D \upharpoonright \rho, R \upharpoonright \rho)$.

Since $\delta < \varepsilon^d$ and n is sufficiently large, $2n^\delta \leq n^{\varepsilon^d}/f$, so by Lemma 7.3, for every step A_k in the proof, $T(A_k \upharpoonright \rho)$ has all its leaves labeled 1. On the other hand, $PHP_n \upharpoonright \rho \equiv PHP(D \upharpoonright \rho, R \upharpoonright \rho)$, so by Lemma 7.4, if PHP_n were the last line A_t of the proof, all the leaves of $T(PHP_n \upharpoonright \rho)$ would be labeled 0. It follows that $A_1, \ldots A_t$ cannot be a proof of PHP_n. Hence, any proof in $\mathcal{F}$ of PHP_n must have size at least 2^{n^δ}. ⊣

In a subsequent paper [2], Ajtai extended his lower bound to a system obtained from a bounded depth Frege system by adding certain axiom schemes. The pigeonhole principle PHP_n states that there is no perfect matching in the bipartite graph $D \times R$, where $|D| = n + 1$, and $|R| = n$. Let PAR_n be the tautology defined in a similar way stating that there is no perfect matching in the complete graph K_{2n+1}. Ajtai proves that even when we add to a Frege system all of the pigeonhole formulas PHP_n as new axiom schemes, the tautologies PAR_n still require bounded depth proofs that grow faster than any polynomial in n, when the proofs are restricted to a fixed depth (the pigeonhole formulas can be derived very easily by proofs of bounded depth when the formulas PAR_n are taken as axiom schemes). Beame and Pitassi [8] recently extended Ajtai's result by showing an exponential lower bound on the size of bounded depth Frege proofs of PAR_n in Frege systems with the added pigeonhole schemes (Søren Riis [53] gave an independent proof of this result).

Ajtai provided a further extension of these results in recent work [3, 4] on the independence of modular counting principles. The modulo q counting principle states that no finite set whose cardinality is not divisible by q can be partitioned into q-element classes. For a fixed cardinal number N, this principle can be stated as a propositional tautology $Count_q^N$; in this notation, the principle PAR_n can be expressed as $Count_2^{2n+1}$. Ajtai proved that whenever p, q are distinct primes, the propositional formulas $Count_q^{qn+1}$ do not have polynomial size, bounded depth Frege proofs from instances of $Count_p^m$, where $m \not\equiv 0 \pmod{p}$. Beame, Impagliazzo, Krajíček, Pitassi and Pudlák [6] extended this result to composite p and q.

The preceding results are significant not just from the point of view of propositional complexity theory, but also as providing independence results in systems of bounded arithmetic. The system $I\Delta_0$ of first order bounded arithmetic introduced by Parikh [47] has been extensively studied; in it, the induction scheme is restricted to formulas containing only bounded quantifiers. Let $I\Delta_0(f)$ be the system obtained from $I\Delta_0$ by adding a new function symbol f that is allowed to appear in the induction scheme. Let $PHP(f)$ be the formula in the expanded language expressing the fact that f is not a bijection between $\{0,\dots,n\}$ and $\{1,\dots,n\}$ for any n. Then Ajtai's lower bound for bounded depth proofs shows that $PHP(f)$ is unprovable in $I\Delta_0(f)$; similar independence results can be proved for the modular counting principles. This follows from the fact that for statements S of an appropriate syntactic form, there is a corresponding sequence of tautologies expressing restricted versions of S so that if S is provable in bounded arithmetic, the sequence of tautologies has polynomial-size proofs in a bounded-depth Frege system. This connection between bounded arithmetic and propositional logic was first observed by Paris and Wilkie [48]. They showed that if $PHP(f)$ were provable in $I\Delta_0(f)$ then there would be polynomial size Frege proofs of the pigeon-hole tautologies; Buss [11] strengthened the conclusion of this theorem to apply to bounded-depth Frege systems. The reader is referred to Buss's paper [11] and the book by Krajíček [40] for details of this connection.

§8. **The extension and substitution rules.** A natural way to extend a Frege system is to allow the possibility of abbreviating formulas by definitions. This idea was first proposed by Tseitin [62] in the context of resolution, but is perhaps more natural in the context of axiomatic systems.

Let $\mathcal{F}$ be a Frege system; for convenience, let us assume that the language of $\mathcal{F}$ contains a symbol $\equiv$ for the biconditional. If $\Gamma \cup \{A\}$ is a set of formulas of $\mathcal{F}$, then a sequence of formulas ending in A is a *proof of A from Γ in $\mathcal{F}$ with extension* if each formula in the sequence either belongs to Γ, or is inferred from earlier formulas in the sequence by one of the rules of $\mathcal{F}$ or has the form $P \equiv A$, where P is a variable not in appearing in $\Gamma \cup \{A\}$, nor in any earlier formula in the sequence. In the case of a step of the last type, the variable is said to be introduced by the *extension rule*. We shall refer to the system with the addition of the extension rule as an *extended Frege system*. The extension rule appears to be very powerful; since abbreviations can be iterated, very long formulas can be abbreviated to short ones by using the extension rule.

The substitution rule is another natural rule that appears in the earliest systems for propositional logic, such as those of Frege [28] and Whitehead and Russell [69]. (Since the substitution rule is unsound, in proofs from assumptions, we must disallow substitution for variables appearing in assumptions.) It is not hard to prove that a Frege system with the addition of the substitution rule can p-simulate the same system with the extension rule added. Surprisingly, the converse simulation also holds, a result first proved by Dowd [26].

THEOREM 8.1. *Any two systems from the following classes are p-equivalent: extended resolution, extended Frege systems, Frege systems with substitution.*

PROOF. See Krajíček and Pudlák [41]. ⊣

Extended Frege systems are significant in themselves as a natural class of proof systems, but also because of a connection with another form of bounded arithmetic. This was the first connection to be observed between propositional logic and bounded arithmetic; it appeared in a fundamental paper of Cook [18]. The system PV is a free variable system of arithmetic that bears the same relation to the polynomial-time computable functions as Skolem's recursive arithmetic bears to the primitive recursive functions. Whereas Skolem's system has a function symbol for each primitive recursive function, PV has one for each polynomial-time computable function; for details, see [21].

Cook introduced PV as a way of formalizing the intuitive notion of 'feasibly constructive proof': feasibly constructive proof is to polynomial-time algorithm as constructive proof is to algorithm. The next theorem (from Cook [18]) emphasizes the power of extended Frege systems; it shows that if a combinatorial principle has a feasibly constructive proof, then the corresponding family of tautologies has polynomial-size proofs in an extended Frege system.

THEOREM 8.2. *If $t = u$ is an equation of PV then there is a polynomially growing family of propositional formulas $|t = u|_n$ so that:*

1. *The formula $|t = u|_n$ is a tautology iff $t = u$ is true when restricted to numerals of length n or less;*
2. *If $\vdash_{PV} t = u$ then there is a polynomial $p(n)$ so that $|t = u|_n$ has an extended Frege proof of length at most $p(n)$ for all n.*

PROOF. A detailed proof is contained in Dowd's thesis [25]. ⊣

The next theorem shows that an extended Frege system can p-simulate any proof system whose soundness is provable in PV. In Section 2, a proof system was defined as a polynomial-time computable function; thus any proof system is represented by a primitive function symbol of PV.

In particular, let EF be the function symbol of PV representing a fixed extended Frege system. Let $TRUE(x, y)$ be the arithmetical function with range $\{0, 1\}$ such that $TRUE(m, n) = 1$ if and only if m is the encoding of a propositional formula A and n the encoding of an assignment under which A is true. If P is a function symbol of PV representing a proof system, then the soundness of P can be expressed in the form $\forall x \forall y[TRUE(P(x), y) = 1]$.

THEOREM 8.3. *If* $PV \vdash TRUE(P(x), y) = 1$, *then there is a function symbol* G *of* PV *so that* $PV \vdash EF(G(x)) = P(x)$.

This is the main result of Cook [18]. A detailed proof is in Dowd [25]. Combined with Dowd's observation that the soundness of a Frege system with substitution is provable in an extended Frege system, it leads immediately to Theorem 8.1.

The foregoing results all emphasize the power of extended Frege systems, and tend to show that proving lower bounds for such systems is a formidable challenge. There are also some theoretical reasons to think that such results will be hard to obtain. Cook and Urquhart [21] show that in a precisely defined sense there can be no feasibly constructive proof of a super-polynomial lower bound for an extended Frege system. Buss [12] and Krajíček and Pudlák [42] have proved further results along the same lines.

We conclude this survey of proof systems by mentioning the quantified propositional calculus, the form of second order logic obtained by adding rules for propositional quantifiers to a Frege system [13, §28]. By restricting our attention to theorems not containing propositional quantifiers, we can consider such systems as proof systems for tautologies in the usual sense. Such systems can simulate Frege systems with substitution, but otherwise little is known about their complexity. There are significant connections between complexity questions about such systems and problems in bounded arithmetic. The reader is referred to the work of Dowd [25] and Krajíček and Pudlák [43].

It is not known whether in the p-simulation ordering there is a greatest element, that is, whether or not there is a propositional proof system that p-simulates all propositional proof systems. Krajíček and Pudlák [41] discuss the relation of this question to other well known open problems in computational complexity theory.

§9. **Open problems and acknowledgments.** The major questions in the area of the complexity of propositional proofs remain unsolved. Of these, perhaps the most important and challenging is that of proving super-polynomial lower bounds on the length of proofs in Frege and extended Frege systems. Even substantial improvements on the currently known weak lower bounds would be of considerable interest.

The best known lower bound on the size of Frege proofs is quadratic. It rests on the observation that in a Frege proof, each application of a schematic rule can involve only a finite number of "active" subformulas. Hence, in a Frege proof of a tautology that is not a substitution instance of a smaller tautology, all the subformulas must occur as active subformulas somewhere in the proof. (For the details of this result, see Krajíček [40, Ch. 4].)

PROBLEM 9.1. Prove a lower bound on the size of Frege proofs that is better than quadratic.

The next problem is probably not too difficult, but might require a new idea.

PROBLEM 9.2. Can a cut-free Gentzen system based on the connectives $\{\vee, \sim\}$ p-simulate resolution as a system for refuting contradictory sets of clauses?

A natural extension of the results on bounded depth Frege proofs would be to prove lower bounds for proofs of bounded depth where we allow not only unbounded disjunctions and conjunctions, but also unbounded connectives computing the parity function. That is to say, we alter the definition of depth above to allow unbounded logical gates $(x_1 \oplus \cdots \oplus x_n)$ computing addition modulo 2 to count as formulas of depth 1. Strong lower bounds have been proved by Razborov and Smolensky using the corresponding model of bounded depth circuits [50, 58].

PROBLEM 9.3. Can we prove superpolynomial lower bounds on the complexity of bounded depth Frege proofs, using the modified definition of depth?

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RECENT ADVANCES IN ORDINAL ANALYSIS: Π^1_2 – CA AND RELATED SYSTEMS

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§1. Introduction. The purpose of this paper is, in general, to report the state of the art of ordinal analysis and, in particular, the recent success in obtaining an ordinal analysis for the system of Π^1_2-analysis, which is the subsystem of formal second order arithmetic, $\mathbf{Z}_2$, with comprehension confined to Π^1_2-formulae.[1] The same techniques can be used to provide ordinal analyses for theories that are reducible to iterated Π^1_2-comprehension, e.g., Δ^1_3-comprehension. The details will be laid out in [28].

Ordinal-theoretic proof theory came into existence in 1936, springing forth from Gentzen's head in the course of his consistency proof of arithmetic. Gentzen fostered hopes that with sufficiently large constructive ordinals one could establish the consistency of analysis, i.e., $\mathbf{Z}_2$. Considerable progress has been made in proof theory since Gentzen's tragic death on August 4th, 1945, but an ordinal analysis of $\mathbf{Z}_2$ is still something to be sought. However, for reasons that cannot be explained here, Π^1_2-comprehension appears to be the main stumbling block[2] on the road to understanding full comprehension, giving hope for an ordinal analysis of $\mathbf{Z}_2$ in the foreseeable future.

Roughly speaking, *ordinally informative* proof theory attaches ordinals in a recursive representation system to proofs in a given formal system; transformations on proofs to certain canonical forms are then partially mirrored by operations on the associated ordinals. Among other things, ordinal analysis of a formal system serves to characterize its provably recursive ordinals, functions and functionals and can yield both conservation and combinatorial independence results. Since there is no wide familiarity among logicians with ordinally informative proof theory, we begin in §2 with an explanation of its current rationale and goals, which take the place of the original

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[1] I would like to thank Tim Carlson for innumerable and invaluable conversations which have helped me find my way through Π^1_2. Carlson is completing work on an approach to Π^1_2 comprehension which was developed independently of that presented here

[2] For more background information see [43, p 259], [11, p. 362], [27, p. 374]

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Hilbert Program, and follow that in §3 with an explanation of its basic technical tools, namely ordinal representation systems and ramified set theory equipped with suitable reflection rules. The connection of the system of Π_2^1-comprehension (Π_2^1 – **CA** hereafter) with set theory comes through the fact that **KP** + Σ_1-separation is a conservative extension of Π_2^1 – **CA** + **BI**, where **BI** is the so-called principle of *Bar Induction*. It is indicated in §4 how this fragment of set theory can be sliced into a hierarchy of reflection principles. Then §5 describes in outline an ordinal representation system which serves for the ordinal analyses of **KP** + Σ_1-separation. Most important in this regard are certain *projection functions* (often called *collapsing functions*). As their existence is rather difficult to prove, in §6 we will indicate a model for them based on very large cardinals, in which they can be construed as inverses of certain partial elementary embeddings.

To set the stage for the following, a very brief history of ordinal-theoretic proof theory since Gentzen reads as follows: In the 1950's proof theory flourished in the hands of Schütte: in [35] he introduced an infinitary system for first order number theory with the so-called ω-rule, which had already been proposed by Hilbert [16]. Ordinals were assigned as lengths to derivations and via cut-elimination he re-obtained Gentzen's ordinal analysis for number theory in a particularly transparent way. Further, Schütte extended his approach to systems of ramified analysis and brought this technique to perfection in his monograph "Beweistheorie" [36]. Independently, in 1964 Feferman [6] and Schütte [37, 38] determined the ordinal bound Γ_0 for theories of autonomous ramified progressions.

A major breakthrough was made by Takeuti in 1967, who for the first time obtained an ordinal analysis of an impredicative theory. In [42] he gave an ordinal analysis of Π_1^1 comprehension, extended in 1973 to Δ_2^1 comprehension in [44] jointly with Yasugi. For this Takeuti returned to Gentzen's method of assigning ordinals (ordinal diagrams, to be precise) to purported derivations of the empty sequent (inconsistency).

The next wave of results, which concerned theories of iterated inductive definitions, were obtained by Buchholz, Pohlers, and Sieg in the late 1970's (see [4]). Takeuti's methods of reducing derivations of the empty sequent ("the inconsistency") were extremely difficult to follow, and therefore a more perspicuous treatment was to be hoped for. Since the use of the infinitary ω-rule had greatly facilitated the ordinal analysis of number theory, new infinitary rules were sought. In 1977 (see [2]) Buchholz introduced such rules, dubbed Ω-rules to stress the analogy. They led to a proof-theoretic treatment of a wide variety of systems, as exemplified in the monograph [5] by Buchholz and Schütte. Yet simpler infinitary rules were put forward a few years later by Pohlers, leading to the *method of local predicativity*, which proved to be a very versatile tool (see [25, 24, 26]). With the work of Jäger and Pohlers (see [17, 18, 20]) the forum of ordinal analysis then switched from the realm of second-order arithmetic to set theory, shaping what is now called *admissible proof theory*, after the models of *Kripke-Platek set theory*, **KP**. Their work culminated in the analysis of the system with Δ^1_2-comprehension plus **BI** [20]. In essence, admissible proof theory is a gathering of cut-elimination techniques for infinitary calculi of ramified set theory with Σ- and/or Π_2-reflection rules[3] that lend itself to ordinal analyses of theories of the form **KP**+ *"there are x many admissibles"* or **KP**+ *"there are many admissibles."* By way of illustration, the subsystem of analysis with Δ^1_2-comprehension and Bar Induction can be couched in such terms, for it is naturally interpretable in the set theory **KPi** $:=$ **KP** $+ \forall y \exists z (y \in z \wedge$ *z is admissible*$)$ (cf. [20]).

After an intermediate step [29], which dealt with a set theory **KPM** which formalizes a recursively Mahlo universe, a major step beyond admissible proof theory was taken in [32]. That paper featured ordinal analyses of extensions of **KP** by Π_n-reflection. A generalization of the methods of [32] underlies the treatment of $\Pi^1_2 -$ **CA** sketched below.

§2. Goals of proof theory. If proof theory is to be applied to stronger and stronger systems, what are the aims of this research? In the author's view, a fruitful way of looking at what has been accomplished and might be accomplished by ordinal analyses is given by the following list:

(I) Hilbert's Program relativized: reduction (interpretation) of impredicative theories to (in) constructive frameworks,
(II) Finding currently untapped resources of set theory: new combinatorial principles
(III) Development of a proof theory of set theory

We shall illustrate the above themes by a few examples.

Ad (I): A modified form of Hilbert's program has, of course, already been pursued by the Hilbert school (Bernays, Gentzen, …). In more elaborate terms, Feferman [9, 10] has put forward a relativized version of Hilbert's program, in which one develops a global picture of *What rests on what?* in mathematics, i.e., what higher-order concepts and principles can be reduced to prima facie more basic terms.

Once one is investigating theories which have at least the strength of $\Delta^1_2 -$ **CA** $+$ **BI**, it is obvious that "securing" ordinary mathematics is not the main concern. Even without knowledge of that program carried out under the rubric of "reverse mathematics," it is easily seen that most of ordinary mathematics can be formalized in Δ^1_2 – **CA** + **BI** without effort (note that in the context of second order arithmetic **BI** serves the same purpose as $\in$-induction in set theory). So we must be interested in these strong impredicative principles for their own sake.

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[3] Recall that the salient feature of admissible sets is that they are models of Δ_0-collection and that Δ_0-collection is equivalent to Σ-reflection on the basis of the other axioms of **KP** (see [1]). Furthermore, admissible sets of the form $\mathbf{L}_\alpha$ also satisfy Π_2-reflection.

The reductions we have in mind, underlies a broadened view of "constructivity." There is no once-and-for-all delineated system of constructivism, rather it is an open-ended framework such as Martin-Löf's type theory, where new types along with new forms of inferences might be added as we widen our horizon by reaching higher levels of reflection (a constructivist's large cardinal program). However, space limitations do not permit us to dwell on a constructivist's manifesto of a liberal persuasion.[4]

Constructive theories of functions and sets that relate to Bishop's constructive mathematics as theories like ZFC relate to Cantorian set theory have been proposed by Myhill, Martin–Löf, Feferman and Aczel. Among those are Feferman's constructive theory of operations and classes, $\mathbf{T_0}$ ([7, 8]), and Martin-Löf's intuitionistic type theory of [23] (the latter does not have Russell's notorious *reducibility axiom*). By employing an ordinal analysis for **KPi** it has been shown that **KPi** and consequently Δ^1_2 – **CA**+**BI** can be reduced to both these theories.

THEOREM 2.1 (Feferman [7], Jäger [19], Jäger and Pohlers [20]). Δ^1_2 – **CA** +**BI**, **KPi**, *and* $\mathbf{T_0}$ *are proof-theoretically equivalent. In particular, these theories prove the same theorems in the negative arithmetic fragment.*

THEOREM 2.2 (Griffor and Rathjen [15]; Setzer [39]). *The soundness of the negative arithmetic fragment of* Δ^1_2 – **CA**+**BI** *and* **KPi** *is provable in Martin-Löf's 1984 type theory.*

Ad (II): It has been shown that certain large cardinal axioms settle questions about definable sets of reals, in that they imply the axiom of projective determinacy (see [22]). The existence of infinitely many Woodin cardinals implies that every uncountable projective set of reals is Lebesgue measurable and has a perfect subset, and thus possesses the same cardinality as the continuum (see [45]). However, it is as yet to be shown that the strength of large cardinal axioms has interesting repercussions in the context of arithmetic. A hope in connection with ordinal analyses is that they lead to new combinatorial principles which encapsulate considerable proof-theoretic strength. Examples are still scarce. One case where ordinal notations led to a new combinatorial result was Friedman's extension of Kruskal's Theorem, EKT, which asserts that finite trees are well-quasi-ordered under gap embeddability (see [40]). The gap condition imposed on the embeddings is directly related to an ordinal notation system that was used for the analysis of Π^1_1-comprehension. The principle EKT played a crucial role in the proof of the graph minor theorem of Robertson and Seymour (see [12]).

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[4] It is remarkable that Gödel seems to espouse a similar view in an unpublished essay from 1961 (see [14, pp. 382–387]), where he makes interesting remarks on the growth of reflection.

THEOREM 2.3 (Robertson, Seymour). *For any infinite sequence $G_0, G_1, G_2, \ldots$ of finite graphs there exist $i < j$ so that G_i is isomorphic to a minor of G_j.*

Ad (III): There is a proof theory for extensions of **KP**. Up till now there does not exist a proof theory of ZFC with its essential use of the power set axiom. As a matter of fact, the flow of ideas has been in the other direction as ordinal-theoretic proof theory has been constructivizing notions of large cardinals. However, if one takes a bold optimistic view that proof theory can be developed up to very large cardinals, then proof theory might shed new light on these lofty regions. For instance, one could hope for new insights as to Kunen's proof of the impossibility of a non-trivial elementary embedding of the universe into itself. All known proofs of $\neg\exists j\,(\,j : V \prec V \wedge j \neq \mathrm{id}\,)$ have an ad hoc flavor.

§3. **Admissible proof theory.** First order number theory is distinguished by the fact that its realm of discourse is clearly delineated, and moreover, that each element n of its realm possesses a canonical name, the nth numeral $\bar{n}$. By contrast, the universe of a set theory is at most implicitly described by its axioms.[5]

When searching for an analogue of the ω-rule, (for all n: $\vdash \phi(\bar{n}) \;\Rightarrow\; \vdash \forall x\phi(x)$), for systems of set theory, one gets confronted with the problem of "naming" sets. However, if a set of ordinals or an ordinal representation system $\mathcal{OR}$ is given at the outset, one can build its associated formal constructible hierarchy, the $RS_{\mathcal{OR}}$-terms, as follows: For each $\alpha \in \mathcal{OR}$, $\mathbb{L}_\alpha$ is an $RS_{\mathcal{OR}}$-term of level α, and, if $s_1, \cdots, s_n$ are $RS_{\mathcal{OR}}$-terms with levels $< \alpha$ and if $F(a, b_1, \cdots, b_n)$ is a formula of the ordinary set-theoretic language, then the formal expression $[x \in \mathbb{L}_\alpha : F(x, s_1, \cdots, s_n)^{\mathbb{L}_\alpha}]$ is an $RS_{\mathcal{OR}}$-term of level α. As for the intended interpretation, $\mathbb{L}_\alpha$ is supposed to "name" L_α and $[x \in \mathbb{L}_\alpha : F(x, s_1, \cdots, s_n)^{\mathbb{L}_\alpha}]$ to "name" the set $\{x \in \mathrm{L}_\alpha : F(x, a_1, \cdots, a_n)^{\mathrm{L}_\alpha}\}$, where the superscript L_α makes all unrestricted quantifiers in F become restricted to L_α and $a_1, \cdots, a_n$ are the sets denoted by $s_1, \cdots, s_n$, respectively. Aside from some notational conventions, we are now prepared to introduce the calculus of ramified set theory, $RS_{\mathcal{OR}}$. The conventions are that for an $RS_{\mathcal{OR}}$-term t, $|\,t\,|$ denotes its level, $t \in \mathcal{T}_\alpha := \{t : |\,t\,| < \alpha\}$, and for terms s, t with $|\,s\,| < |\,t\,|$ we set $s \overset{\circ}{\in} t := B(s)$ if t is $[x \in \mathbb{L}_\beta : B(x)]$ and $s \overset{\circ}{\in} t := s = s$

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[5] This makes ordinal analysis of set theories so intriguing. Ordinals appear both in the guise of ordinal representation systems and as objects of the theory under investigation, and the relation between these appearances is rather subtle.

if t is of the form $\mathbb{L}_\beta$. Observe that $s \in t$ and $s \overset{\circ}{\in} t$ have the same truth value under the standard interpretation in the constructible hierarchy.

DEFINITION 3.1. The *rules* of $RS_{\mathcal{OR}}$ are:

$$
\begin{array}{ll}
(\wedge) & \dfrac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'} \\[2ex]
(\vee) & \dfrac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1 \\[2ex]
(\forall) & \dfrac{\cdots \Gamma, s \overset{\circ}{\in} t \rightarrow F(s) \cdots (s \in \mathcal{T}_{|t|})}{\Gamma, (\forall x \in t) F(x)} \\[2ex]
(\exists) & \dfrac{\Gamma, s \overset{\circ}{\in} t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } s \in \mathcal{T}_{|t|} \\[2ex]
(\notin) & \dfrac{\cdots \Gamma, s \overset{\circ}{\in} t \rightarrow r \neq s \cdots\cdots (s \in \mathcal{T}_{|t|})}{\Gamma, r \notin t} \\[2ex]
(\in) & \dfrac{\Gamma, s \overset{\circ}{\in} t \ \wedge \ r = s}{\Gamma, r \in t} \quad \text{if } s \in \mathcal{T}_{|t|} \\[2ex]
(\text{Cut}) & \dfrac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}
\end{array}
$$

The rules of $RS_{\mathcal{OR}}$ are pure infinitary logic and therefore this calculus enjoys cut-elimination. But unlike the case of first order number theory, in general, we won't be able to embed **KP** into $RS_{\mathcal{OR}}$. We will have to add non-logical rules, Σ-reflection rules (cf. [1, I.4.1]), to $RS_{\mathcal{OR}}$ in order to accommodate **KP**. However, the addition of such rules, providing $\mathcal{OR}$ is recursive, will render the infinitary calculus inconsistent, unless we impose uniformity restrictions on infinitary derivations. To explain this kind of uniformity, rather than construing $\mathcal{OR}$ as an initial segment of the ordinals it is important to conceive of $\mathcal{OR}$ as a subset of the ordinals which has gaps.

To make the preceding more understandable, we shall expound to some extent on a concrete example, the ordinal analysis of **KPi**. To begin with, we shall explain how a sufficiently strong ordinal representation system comes about. Recall that the least standard model of **KPi** is the structure $\mathbf{L_I}$, where **I** denotes the first recursively inaccessible ordinal.

Let $\alpha \mapsto \Omega_\alpha$ be the function that enumerates the admissible ordinals and their limits, and let $\Re$ be the set of admissible ordinals κ so that $\omega < \kappa \leq \mathbf{I}$. Variables κ, π will range over $\Re$. An ordinal representation system for the analysis of **KPi** can be derived from the following sets of ordinals, defined by recursion on α:

$$
(1) \qquad C(\alpha, \beta) = \left\{ \begin{array}{l} \textit{closure of } \beta \cup \{0, \mathbf{I}\} \textit{ under}: \\ \quad +, \quad (\xi \mapsto \omega^\xi), \\ \quad (\xi \mapsto \Omega_\xi)_{\xi < \mathbf{I}}, \quad (\xi\pi \longmapsto \psi_\pi(\xi))_{\xi < \alpha} \end{array} \right.
$$

$$
(2) \qquad \psi_\pi(\alpha) = \min\{\rho : \ C(\alpha, \rho) \cap \pi = \rho \ \wedge \ \pi \in C(\alpha, \rho)\}.
$$

It is non-trivial to show that for any $\pi \in \Re$ satisfying $\pi \in C(\alpha, \pi)$, one has $\psi_\pi(\alpha) < \pi$ (see [30, 31]). Note that if $\rho = \psi_\pi(\alpha) < \pi$, then $[\rho, \pi) \cap C(\alpha, \rho) = \emptyset$, thus the order-type of the ordinals below π which belong to the Skolem hull $C(\alpha, \rho)$ is ρ . In more pictorial terms, ρ is the αth *collapse* of π.

Let $\varepsilon_{\mathbf{I}+1}$ be the least ordinal $\alpha > \mathbf{I}$ such that $\omega^\alpha = \alpha$. $C(\varepsilon_{\mathbf{I}+1}, 0)$ gives rise to an ordinal representation system, i.e., there is a primitive recursive ordinal representation system $\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$, so that

$$\langle C(\varepsilon_{\mathbf{I}+1}, 0), <, \Re, \psi, \ldots \rangle \;\cong\; \langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle. \tag{3}$$

"..." is supposed to indicate that more structure carries over to the ordinal representation system. The ordinal analysis of **KPi** (cf. [20]) establishes a close connection between **KPi** and $\mathcal{OR}$. Thus, in view of of the isomorphism $\cong$ one is led to ask how **KPi** relates to $C(\varepsilon_{\mathbf{I}+1}, 0)$. We will say that an ordinal $\alpha < \mathbf{I}$ has a *good* Σ_1*-definition in* **KPi** if there is a Σ_1-formula $\phi(u)$ such that $\mathbf{L}_\mathbf{I} \models \phi[\alpha]$ and $\mathbf{KPi} \vdash \exists! x \phi(x)$ (see [1, II.5.13]). It turns out that all ordinals $\alpha \in C(\varepsilon_{\mathbf{I}+1}, 0)$ have a canonical good Σ_1-definition in **KPi**, arising from the definition of the Skolem hulls $C(\alpha, \beta)$ (see [30, 31, 34]), but the reverse inclusion doesn't hold. For the time being, the general relation between **KPi** and $C(\varepsilon_{\mathbf{I}+1}, 0)$ remains to be unveiled.

Returning to the main track, to allow for an interpretation of **KPi** in the infinitary calculus, we add the following rules:

$$\begin{array}{lll}
(\neg\mathrm{Ad}) & \dfrac{\cdots \Gamma, \mathbb{L}_\pi \neq t \cdots (\pi \in \Re;\ \pi \leq |t|)}{\Gamma, \neg\mathrm{Ad}(t)} & \\
(\mathrm{Ad}) & \dfrac{\Gamma, \mathbb{L}_\pi = t}{\Gamma, Ad(t)} & \text{if } \pi \in \Re \text{ and } \pi \leq |t| \\
(\mathrm{Ref}_{\Sigma(\mathbf{L}_\pi)}) & \dfrac{\Gamma, A^{\mathbb{L}_\pi}}{\Gamma, (\exists z \in \mathbb{L}_\pi) A^z} & \text{if } A \in \Sigma(\mathbb{L}_\pi).
\end{array}$$

The rules $(\neg\mathrm{Ad})$ and (Ad) just introduce a new predicate whose meaning is the class $\{\mathbf{L}_\pi : \pi \in \Re\}$. New strength is only gained by the rules $(\mathrm{Ref}_{\Sigma(\mathbf{L}_\pi)})$.

An infinitary derivation $\mathfrak{D}$ is said to be a $C(\alpha, \beta)$–derivation if it belongs to the calculus $RS_{\mathcal{OR}}$, where $\mathcal{OR} = C(\alpha, \beta)$. Variations of the parameters α and β will allow us to convey a notion of uniformity. A $C(\alpha, \beta)$–derivation $\mathfrak{D}$ is called α-β-uniform if for any pair (α', β') such that $C(\alpha, \beta) \subseteq C(\alpha', \beta')$, $\mathfrak{D}$ can be *blown up* (or dilated) to a $C(\alpha', \beta')$–derivation $\mathfrak{D}'$. What is meant by blowing-up is best explained by the reverse process, i.e., *mutilation*. If $\mathfrak{D}$ is obtained from $\mathfrak{D}'$ by deleting in all inferences $(\forall)$, $(\notin)$, and $(\neg\mathrm{Ad})$ of $\mathfrak{D}'$ all those premises (together with the subproofs of which they are conclusions) which involve terms that do not belong to $\mathcal{OR} = C(\alpha, \beta)$, then $\mathfrak{D}$ is said to arise by mutilation from $\mathfrak{D}'$. The finite derivations of **KPi** can be unwound to yield $C(0, 0)$–uniform infinitary derivations. The whole process of (partial) cut-elimination, starting with the embedded **KPi** derivations, can then be carried out with $C(\alpha, 0)$–uniform derivations, where α stays in the ordinal representation system $C(\varepsilon_{\mathrm{I}+1}, 0)$, though, as the cut-elimination process progresses, α propagates to ever bigger ordinals.

Hopefully, the preceding has conveyed some understanding of why the ordinal representation systems should be viewed together with their gaps.

Of course, the vocabulary we have just used is familiar from Girard's notion of β-proof and the categorical framework of dilators (see [13]). The problem of forging notions of uniformity for infinitary derivations is ubiquitous in this area of proof theory. β-proofs, however, are too weak a tool, if one aims beyond admissible proof theory. The novel feature needed is that as the cut-elimination procedure progresses we not only require the insertion of more ordinals but also of completely new reflection rules. This means that the cut-elimination process amounts to a very complicated transformation process which cannot be subsumed under the heading of β-proofs.

Fortunately, Buchholz has provided us with a very elegant and flexible setting for describing uniformity in infinitary proofs, called *operator controlled derivations* (see [3]).

Definition 3.2. Let $P(On) = \{X : X \text{ is a set of ordinals}\}$. A class function $\mathcal{H} : P(On) \to P(On)$ will be called an *operator* if $\mathcal{H}$ is a *closure operator*, i.e., monotone, inclusive and idempotent, and satisfies the following conditions for all $X \in P(On)$: $0 \in \mathcal{H}(X)$, and, if α has Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then $\alpha \in \mathcal{H}(X) \iff \alpha_1, \ldots, \alpha_n \in \mathcal{H}(X)$. The latter ensures that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^\sigma$, and decomposition of its members into additive and multiplicative components. For $Z \in P(On)$, the operator $\mathcal{H}[Z]$ is defined by $\mathcal{H}[Z](X) := \mathcal{H}(Z \cup X)$.

If $\mathfrak{X}$ consists of "syntactic material," i.e., terms, formulae, and possibly elements from $\{0, 1\}$, then let $\mathcal{H}[\mathfrak{X}](X) := \mathcal{H}(k(\mathfrak{X}) \cup X)$, where $k(\mathfrak{X})$ is the set of ordinals needed to build this "material." Finally, if s is a term, then define $\mathcal{H}[s]$ by $\mathcal{H}[\{s\}]$.

To facilitate the definition of $\mathcal{H}$–controlled derivations, we assign to each $RS_{\mathcal{OR}}$–formula A, either a (possibly infinite) disjunction $\bigvee(A_\iota)_{\iota \in I}$ or a conjunction $\bigwedge(A_\iota)_{\iota \in I}$ of $RS_{\mathcal{OR}}$–formulae. This assignment will be indicated by $A \cong \bigvee(A_\iota)_{\iota \in I}$ and $A \cong \bigwedge(A_\iota)_{\iota \in I}$, respectively. Define: $r \in t \cong \bigvee(s \mathring{\in} t \wedge r = s)_{s \in \mathcal{T}_{|t|}}$; $\mathrm{Ad}(t) \cong \bigvee(\mathbb{L}_\kappa = t)_{\mathbb{L}_\kappa \in I}$, where $I := \{\mathbb{L}_\eta : \eta \in \mathfrak{R}; \eta \leq |t|\}$; $(\exists x \in t) F(x) \cong \bigvee(s \mathring{\in} t \wedge F(s))_{s \in \mathcal{T}_{|t|}}$; $A_0 \vee A_1 \cong \bigvee(A_\iota)_{\iota \in \{0,1\}}$; $\neg A \cong \bigwedge(\neg A_\iota)_{\iota \in I}$, if $A \cong \bigvee(A_\iota)_{\iota \in I}$. Using this representation of formulae, we can define the *subformulae* of a formula as follows. When $A \cong \bigwedge(A_\iota)_{\iota \in I}$ or $A \cong \bigvee(A_\iota)_{\iota \in I}$, then B is a subformula of A if $B \equiv A$ or, for some $\iota \in I$, B is a subformula of A_ι.

Since one also wants to keep track of the complexity of cuts appearing in derivations, each formula F gets assigned an ordinal rank $rk(F)$ which is the sup of the level of terms in F plus a finite number.

Using the formula representation, in spite of the many rules of $RS_{\mathcal{OR}}$, the notion of $\mathcal{H}$–controlled derivability can be defined concisely. We shall use $I \restriction \alpha$ to denote the set $\{\imath \in I : |\imath| < \alpha\}$.

Definition 3.3. Let $\mathcal{H}$ be an operator and let Γ be a finite set of $RS_{\mathcal{OR}}$–formulae. $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma$ is defined by recursion on α. It is always demanded that $\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}(\emptyset)$. The inductive clauses are:

$$(\bigvee) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Lambda, A_{\imath_0}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Lambda, \bigvee(A_\imath)_{\imath \in I}} \qquad \begin{array}{r} \alpha_0 < \alpha \\ \imath_0 \in I \restriction \alpha \end{array}$$

$$(\bigwedge) \qquad \frac{\mathcal{H}[\imath] \vdash^{\alpha_\imath}_{\rho} \Lambda, A_\imath \text{ for all } \imath \in I}{\mathcal{H} \vdash^{\alpha}_{\rho} \Lambda, \bigwedge(A_\imath)_{\imath \in I}} \qquad |\imath| \leq \alpha_\imath < \alpha$$

$$(\mathrm{Cut}) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Lambda, B \qquad \mathcal{H} \vdash^{\alpha_0}_{\rho} \Lambda, \neg B}{\mathcal{H} \vdash^{\alpha}_{\rho} \Lambda} \qquad \begin{array}{r} \alpha_0 < \alpha \\ rk(B) < \rho \end{array}$$

$$(\mathrm{Ref}_{\Sigma(\mathbf{L}_\pi)}) \qquad \frac{\mathcal{H} \vdash^{\alpha_0}_{\rho} \Lambda, A^{\mathbf{L}_\pi}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Lambda, (\exists z \in \mathbb{L}_\pi) A^z} \qquad \begin{array}{r} \alpha_0 < \alpha \\ \pi \in \mathfrak{R} \\ A^{\mathbf{L}_\pi} \in \Sigma(\mathbb{L}_\pi) \end{array}$$

The specification of the operators needed for an ordinal analysis will of course hinge upon the particular theory and ordinal representation system.

§4. Beyond admissible proof theory. The following gives a heuristic plan for approaching the ordinal analysis of a set theory T:

- Study the standard/canonical models of T in set-theoretic terms.
- Carry out an analysis of the impredicative principles of T.
- Determine an ordering of degrees of impredicativity.
- Distill an ordinal representation system from the previous ordering.
- Develop an infinitary proof system, based on a formal constructible hierarchy, in such a way that degrees of impredicativity are mirrored by reflection rules.
- Show (partial) cut-elimination for the proof system.

As for the degrees of impredicativity, all systems from **KP** through **KPi** have degree 1, **KPM** has degree 2, whereas the system of Π_3 reflection in [32] already has transfinitely many degrees of impredicativity.

In the following, we will gradually slice Π^1_2-comprehension into degrees of reflection. The strength of Π^1_2-comprehension is much greater than that of Δ^1_2-comprehension. In particular, there is no way to describe this comprehension simply in terms of admissibility; on the set–theoretic side, Π^1_2-comprehension corresponds to Σ_1-separation, i.e., the schema of axioms

$$\exists z(z = \{x \in a : \phi(x)\})$$

for all Σ_1 formulas ϕ. Assuming *Infinity* to be among the axioms of **KP**, the precise relationship is as follows:

THEOREM 4.1. **KP** + Σ_1-*separation and* (Π^1_2 – **CA**) + **BI** *prove the same sentences of second order arithmetic.*

The ordinals κ such that $\mathbf{L}_\kappa \models$ **KP** + Σ_1-*separation* are familiar from ordinal recursion theory. They are exactly the nonprojectible ordinals $> \omega$.

DEFINITION 4.2. An admissible ordinal κ is said to be *nonprojectible* if there is no total κ–recursive function mapping κ one–one into some $\beta < \kappa$, where a function $g : \mathbf{L}_\kappa \to \mathbf{L}_\kappa$ is called *κ–recursive* if it is Σ-definable in $\mathbf{L}_\kappa$.

The key to the 'largeness' properties of nonprojectible ordinals is that for any nonprojectible ordinal κ, $\mathbf{L}_\kappa$ is a limit of Σ_1-elementary substructures, i.e., for every $\beta < \kappa$ there exists a $\beta < \rho < \kappa$ such that $\mathbf{L}_\rho$ is a Σ_1–elementary substructure of $\mathbf{L}_\kappa$, written $\mathbf{L}_\rho \prec_1 \mathbf{L}_\kappa$.

Such ordinals satisfying $\mathbf{L}_\rho \prec_1 \mathbf{L}_\kappa$ have strong reflecting properties. For instance, if $\mathbf{L}_\rho \models \phi$ for some set–theoretic sentence ϕ (possibly containing parameters from $\mathbf{L}_\rho$), then there exists a $\gamma < \rho$ such that $\mathbf{L}_\gamma \models \phi$. This is because $\mathbf{L}_\rho \models \phi$ implies $\mathbf{L}_\kappa \models \exists\gamma\phi^{\mathbf{L}_\gamma}$, hence $\mathbf{L}_\rho \models \exists\gamma\phi^{\mathbf{L}_\gamma}$ using $\mathbf{L}_\rho \prec_1 \mathbf{L}_\kappa$.

The last result makes it clear that an ordinal analysis of Π^1_2-comprehension would necessarily involve a proof–theoretic treatment of reflections beyond those surfacing in admissible proof theory.

The notion of stability will be instrumental.

DEFINITION 4.3. α is *δ-stable* if $\mathbf{L}_\alpha \prec_1 \mathbf{L}_{\alpha+\delta}$.

For our purposes we need refinements of this notion, the simplest being provided by:

DEFINITION 4.4. $\alpha > 0$ is said to be *Π_n-reflecting* if $\mathbf{L}_\alpha \models \Pi_n$-reflection. By Π_n-reflection (Σ_n-reflection) we mean the scheme $\phi \to \exists z[Tran(z) \wedge z \neq \emptyset \wedge \phi^z]$, where ϕ is Π_n (Σ_n), and $Tran(z)$ expresses that z is a transitive set.

Π_n-reflection for all n suffices to express one step in the $\prec_1$ relation.

LEMMA 4.5 (cf. [33, 1.18]). $\mathbf{L}_\kappa \prec_1 \mathbf{L}_{\kappa+1}$ *iff* κ *is* Π_n-*reflecting for all* n.

With regard to refining the notion of being δ-stable, however, the constructible hierarchy causes some minor annoyance since we want the Σ_n satisfaction relation on $\mathbf{L}_\beta$ to be uniformly Σ_n-definable in $\mathbf{L}_\beta$ and likewise we want the predicate "$x = \mathbf{L}_\alpha$" to be absolute at all stages. The way out of this dilemma is to use the *Jensen hierarchy* (cf. [21]). Roughly speaking, $\mathbf{J}_\beta$ possesses all the properties of the limit levels of the L-hierarchy. However, in the infinitary proof system we can neglect the difference between the **L** and the **J** hierarchies.

Definition 4.6. κ is said to be δ-Π_n-reflecting if whenever $\phi(u, \vec{x})$ is a set–theoretic Π_n-formula, which may contain parameters from $\mathbf{J}_\kappa$, and $a_1, \dots, a_r \in \mathbf{J}_{\kappa+\delta}$ and $\mathbf{J}_{\kappa+\delta} \models \phi[\kappa, a_1, \dots, a_n]$, then there exists $\kappa_0, \delta_0 < \kappa$ and $b_1, \dots, b_r \in \mathbf{J}_{\kappa_0+\delta_0}$ such that $\mathbf{J}_{\kappa_0+\delta_0} \models \phi[\kappa_0, a_1, \dots, a_n]$ with the parameters of $\phi(u, \vec{x})$ belonging to $\mathbf{J}_{\kappa_0}$.

Putting the previous definition to work, one gets:

Corollary 4.7. *If κ is $\delta + 1$-Π_0-reflecting, then, for all n, κ is δ-Π_n-reflecting.*

At this point let us return to proof theory to explain the need for even further refinements of the preceding notions. Recall that the first nonprojectible ordinal ρ is a limit of smaller ordinals ρ_n such that $\mathbf{L}_{\rho_n} \prec_1 \mathbf{L}_\rho$. In the ordinal representation system for $\Pi^1_2 - \mathbf{CA}$, there will be symbols $\mathfrak{E}_n$ and $\mathfrak{E}_\omega$ for ρ_n and ρ, respectively. The associated infinitary proof system $RS_{\mathcal{OR}}$ will have rules

$$(\mathrm{Ref}_{\Pi_k(\mathbf{L}_{\mathfrak{E}_n+\delta})}) \quad \frac{\Gamma, \phi(\mathbf{s})^{\mathbf{L}_{\mathfrak{E}_n+\delta}}}{\Gamma, (\exists z \in \mathbb{L}_{\mathfrak{E}_n})(\exists \mathbf{x} \in \mathbb{L}_{\mathfrak{E}_n})[Tran(z) \wedge \phi(\mathbf{x})^{z}]}$$

where $\phi(\mathbf{x})$ is Π_n, possibly containing $RS_{\mathcal{OR}}$-terms of levels $< \mathfrak{E}_n$, and $\delta < \mathfrak{E}_\omega$. These rules suffice to bring about the embedding of $\mathbf{KP} + \Sigma_1$-separation in the infinitary proof system, but reflection rules galore will be needed to carry out cut-elimination. For example, there will be "many" ordinals $\pi, \delta \in \mathcal{OR}$ that play the role of δ-Π_{n+1}-reflecting ordinals by virtue of corresponding reflection rules in $RS_{\mathcal{OR}}$. Assume now that there is an infinitary proof $\mathfrak{D}$ of an arithmetic statement and that further $\mathfrak{D}$ contains δ-Π_{n+1}-reflection inferences at π. To eliminate cuts in $\mathfrak{D}$ one has to eliminate those reflection inferences, too. Let $\mathfrak{D}_0$ be a subproof of $\mathfrak{D}$ whose conclusion is the premise of a δ-Π_{n+1}-reflection inference at π. Since $\mathfrak{D}_0$ is a uniform proof, we know more than the sheer truth of its conclusion. Hopefully, this uniformity will enable one to "project" the proof tree below π. This, however, is mostly wishful thinking. To aggravate matters, assume further that $\mathfrak{D}_0$ already contains several δ-Π_{n+1}-reflection inferences at π. Well, then it is, in general, impossible to project $\mathfrak{D}_0$ below π, for otherwise we could repeat this procedure to produce an infinite decending chain of ordinals. The remedy is to replace those inferences by a hierarchy of inferences which are between δ-Π_{n+1}-reflection and δ-Π_n-reflection. This leads to the following:

Definition 4.8. Let $A \subseteq \kappa \times \kappa$. κ is said to be δ-Π_n-reflecting in A if in Definition 4.6 the requirement $\langle \kappa_0, \delta_0 \rangle \in A$ is added.

Lemma 4.9. *Still assuming that π is δ-Π_{n+1}-reflecting, suppose $\vartheta(u)$ is $\Pi_{n+1}(\mathbf{J}_\pi)$ and for every $\alpha < \pi$, $\mathbf{J}_{\pi+\delta} \models \vartheta(\alpha)$. Recursively define*

$$A^\alpha_\vartheta := \{\langle \pi_0, \delta_0 \rangle \in \pi \times \pi : \mathbf{J}_{\pi_0+\delta_0} \models \vartheta(\alpha)\ ; \quad \forall \beta < \alpha\, [\pi_0 \textit{ is } \delta_0\text{-}\Pi_n\textit{-reflecting in } A^\beta_\vartheta \cap \pi_0 \times \pi_0]\}.$$

Then, for all $\alpha < \pi$, π is δ-Π_{n+1}-reflecting in A^α_ϑ.

The proof is by induction on α. As a matter of fact, it will be important to extend the hierarchy beyond π for certain ordinals α. This hierarchy has a counterpart in the ordinal representation system. There the formula $\vartheta(\alpha)$ will refer to the αth Skolem hull of ordinals.

§5. **Skolem hulls and projection functions.** It makes no sense to present an ordinal representation system without giving a semantic interpretation. For ordinal representation systems in impredicative proof theory it is essential to understand the collapsing functions which they encapsulate. In this section, the intention is to give a glimpse, in terms of Skolem hulls, of the ordinal representation system needed for $\Pi^1_2 - \mathbf{CA}$.

Let $(\mathfrak{E}_\alpha)_{\alpha<\theta}$ be a chain of Σ_1-elementary ordinals, i.e., $\mathbf{L}_\alpha \prec_1 \mathbf{L}_\beta$ holds for all $\alpha < \beta < \theta$, and let θ be a primitive recursive ordinal. θ will be kept fixed in the sequel.

Definition 5.1. By recursion on α we shall define sets of ordinals $C(\alpha, \gamma)$ (the α^{th} *Skolem Hull generated from* γ), a set $\mathfrak{R}^\alpha$ of *reflection configurations* and a set $\mathfrak{F}^\alpha$ consisting of triples $(\beta, \mathbb{A}, f)$ such that $\beta \leq \alpha$, $\mathbb{A} \in \mathfrak{R}^\alpha$ and f is a partial function; such triples will be called *collapsing functions*.

Let $\mathfrak{R} = \bigcup_\alpha \mathfrak{R}^\alpha$ and $\mathfrak{F} \doteq \bigcup_\alpha \mathfrak{F}^\alpha$. The definition of $\mathfrak{F}$ will actually guarantee that for any α and $\mathbb{A} \in \mathfrak{R}$ there is at most one f satisfying $(\alpha, \mathbb{A}, f) \in \mathfrak{F}$. This justifies the following definition:

$$\Psi^\alpha_{\mathbb{A}} = \begin{cases} f & \text{if } (\alpha, \mathbb{A}, f) \in \mathfrak{F} \text{ for some } f \\ \emptyset & \text{otherwise.} \end{cases}$$

Let's stipulate some conventions for ease of presentation. Instead of $(\alpha, \mathbb{A}, f) \in \mathfrak{F}$ we shall sloppily write $\Psi^\alpha_{\mathbb{A}} \in \mathfrak{F}$. A reflection configuration $\mathbb{A}$ is related to a certain reflection context which determines its *interval*. Formally, $\mathbb{A}$ is a formal expression built up from ordinals, symbols Π_n and parentheses. By $\mathbb{A} \in C(\alpha, \gamma)$ we mean that any ordinal occurring in $\mathbb{A}$ belongs to $C(\alpha, \gamma)$.

Let $\mathfrak{E} = \{\mathfrak{E}_\sigma : \sigma = 0 \text{ or } \sigma = \mathfrak{E}_{\xi+1} \text{ for some } \xi < \theta\}$. $C_n(\alpha, \gamma)$ is defined by recursion on n as follows:

$$\begin{array}{llcl}
(4) & C_0(\alpha,\gamma) & = & 1+\gamma \,\cup\, \{\mathfrak{E}_\xi : \xi < \theta\} \\
(5) & C_{n+1}(\alpha,\gamma) & = & C_n(\alpha,\gamma) \,\cup\, \left\{\beta : \beta =_{NF} \omega^\xi + \eta;\ \xi, \eta \in C_n(\alpha,\gamma)\right\} \\
 & & & \cup \left\{\Psi^\beta_{\mathbb{B}}(\eta) :\ \mathbb{B}, \beta, \eta \in C_n(\alpha,\gamma);\right. \\
 & & & \left.\Psi^\beta_{\mathbb{B}} \in \mathfrak{F}^{<\alpha};\ \eta \in \mathbf{dom}(\Psi^\beta_{\mathbb{B}})\right\} \\
(6) & C(\alpha,\gamma) & = & \bigcup_n C_n(\alpha,\gamma).
\end{array}$$

The burden is now on us to define $\mathfrak{R}^\alpha$ and $\mathfrak{F}^\alpha$. We shall focus on one type of reflection situation. Suppose $\pi \in \mathfrak{E}$ or π is δ-Π_{n+1}-reflecting, but not stronger, and $\pi = \Psi^\beta_{\mathbb{B}}(\kappa)$, $\pi + \delta = \Psi^\beta_{\mathbb{B}}(\kappa + \zeta)$ for some $\mathbb{B}$, where $[\kappa, \kappa + \zeta]$ is the interval of the reflection configuration $\mathbb{B} \in \mathfrak{R}^{<\alpha}$. Then $\mathbb{A} \in \mathfrak{R}^\alpha$, where $\mathbb{A}$ is just a matrix that comprises the information from $\mathbb{B}$ and the data π, δ, Π_{n+1}; its interval is $[\pi, \pi + \delta]$.

The definition of $\Psi^\alpha_{\mathbb{A}}$ is the crucial part. It begins with a search for pairs of ordinals $\langle \pi_0, \delta_0 \rangle$ below π so that

$$C(\alpha, \pi_0) \cap \pi = \pi_0 \tag{7}$$

and π_0 is δ_0-Π_n-reflecting (the latter reflection property is actually way too weak). The meaning of (7) is that there is a gap in $C(\alpha, \pi_0)$ between π_0 and π. The purpose of the function $\Psi^\alpha_{\mathbb{A}}$ is to project the structure of the set $[\pi, \pi + \delta) \cap C(\alpha, \pi)$ below π, where the "structure" comprises all reflection relations belonging to $\mathfrak{R}^{<\alpha}$. Additional requirements are that $\Psi^\alpha_{\mathbb{A}}(\pi) = \pi_0$, $\Psi^\alpha_{\mathbb{A}}(\pi + \delta) = \pi_0 + \delta_0$, and $\Psi^\alpha_{\mathbb{A}}$ be defined everywhere on $C(\alpha, \pi_0) \cap [\pi, \pi + \delta]$. So the domain of $\Psi^\alpha_{\mathbb{A}}$ is contained in $[\pi, \pi + \delta]$, but, in general, it will only be a partial function on this interval. On the other hand, in general, the domain of $\Psi^\alpha_{\mathbb{A}}$ is going to be larger than $C(\alpha, \pi_0) \cap [\pi, \pi + \delta]$.

As already indicated, requiring π just to be δ_0-Π_n-reflecting is not enough. The degree of reflection should be a function of α. The precise amount of reflection involves a hierarchy, redolent of the one in Lemma 4.9.

Necessarily, in the present paper, the description of $\Psi^\alpha_{\mathbb{A}}$ has to fade out at a certain point. But it should be clear by now, that $\Psi^\alpha_{\mathbb{A}}$ will enable us to project certain proofs, which are $C(\alpha, \pi)$-uniform, below π. To conclude, let us exhibit the set of ordinals from which the ordinal representation system is obtained.

DEFINITION 5.2. The set of ordinal representations, $\mathfrak{T}$, is inductively defined as follows:

($\mathfrak{T}0$): $0 \in \mathfrak{T}$ and for any $\eta < \theta$, $\mathfrak{E}_\eta \in \mathfrak{T}$.

($\mathfrak{T}1$): If $\eta =_{NF} \omega^\alpha + \beta$ and $\alpha, \beta \in \mathfrak{T}$, then $\eta \in \mathfrak{T}$.

($\mathfrak{T}$2): If all components of $\mathbf{A}$ are in $\mathfrak{T}$, $\alpha, \rho \in \mathfrak{T}$ and $\rho \in \mathbf{dom}(\Psi^{\alpha}_{\mathbf{A}})$, then $\Psi^{\alpha}_{\mathbf{A}}(\rho) \in \mathfrak{T}$.

THEOREM 5.3. *Different ordinal representations denote different ordinals.*

THEOREM 5.4. *Each ordinal of $\mathfrak{T}$ can be identified with the unique term denoting it, and thus $\mathfrak{T}$ can be coded as a set of natural numbers.*

Under this coding, $\mathfrak{T}$ is a primitive recursive set and, moreover, the ordering relation inherited from the ordinals is also primitive recursive.

THEOREM 5.5. *Let $\Pi^1_2 - \mathbf{CA}_0$ be the system $\Pi^1_2 - \mathbf{CA}$ with induction restricted to sets.*

The provably recursive ordinals of $\Pi^1_2 - \mathbf{CA}_0$, $\Pi^1_2 - \mathbf{CA} + \mathbf{BI}$, and $\Delta^1_3 - \mathbf{CA}$ are exactly those representable in $\mathfrak{T}$ when $\theta = \omega$, $\theta = \omega + 1$, $\theta = \varepsilon_0$, respectively.

§6. A model based on indescribable cardinals. In this final section we will indicate a model for the projection functions, employing rather sweeping large cardinal axioms, in that we shall presume the existence of certain strongly indescribable cardinals. Large cardinals have been used quite frequently in the definition procedure of strong ordinal representation systems, and large cardinal notions have been an important source of inspiration. In the end, they can be dispensed with, but they add an intriguing twist to the relation between set theory and proof theory. The advantage of working in a strong set–theoretic context is that we can build models without getting buried under complexity considerations.

DEFINITION 6.1. Let $V = \bigcup_{\alpha \in On} V_{\alpha}$ be the cumulative hierarchy of sets. Let $\eta > 0$ and $\mathcal{F}$ be a collection of formulae. A cardinal κ will be called *η-$\mathcal{F}$-shrewd* if for all $P \subseteq V_{\kappa}$ and every $\mathcal{F}$-formula $\varphi(X, y)$, whenever

$$V_{\kappa+\eta} \models \varphi[P, \kappa],$$

then there exist $0 < \kappa_0, \eta_0 < \kappa$ such that

$$V_{\kappa_0+\eta_0} \models \varphi[P \cap V_{\kappa_0}, \kappa_0].$$

κ is said to be *η-$\mathcal{F}$-reducible* if for every $P \subseteq V_{\kappa+\eta}$ there exist $0 < \kappa_0, \eta_0 < \kappa$ and $Q \subseteq V_{\kappa_0+\eta_0}$ such that

$$(8) \qquad \langle V_{\kappa_0+\eta_0}; \in; \kappa_0; Q; x \rangle_{x \in V_{\kappa_0}} \equiv_{\mathcal{F}} \langle V_{\kappa+\eta}; \in; \kappa; P; x \rangle_{x \in V_{\kappa_0}},$$

where $\equiv_{\mathcal{F}}$ means that both structures satisfy the same $\mathcal{F}$ sentences. Observe that (8) implies $V_{\kappa_0} \cap Q = V_{\kappa_0} \cap P$.

κ is *shrewd* (*reducible*) if κ is η-$\mathcal{F}$-shrewd (κ is η-$\mathcal{F}$-reducible) for any $\eta > 0$ with $\mathcal{F}$ being the set-theoretic formulae.

To our knowledge the previous large cardinal notions have not been considered in the set-theoretic literature. With regard to consistency strength, shrewd cardinals are weaker than for instance subtle or ineffable cardinals; in particular is their existence compatible with the constructibility of the universe. The notions of Definition 6.1 are closely related.

LEMMA 6.2. *If κ is ρ-shrewd and $0 < \eta < \rho$, then κ is η-reducible.*

In the situation of (8) there exists a *partial* embedding p from $V_{\kappa_0+\eta_0}$ into $V_{\kappa+\eta}$ satisfying $p \restriction V_{\kappa_0+\eta_0} = \mathrm{id} \restriction V_{\kappa_0+\eta_0}$ and $p(\kappa_0) = \kappa$. Moreover, p can be canonically extended to all elements of $V_{\kappa_0+\eta_0}$ which are $\mathcal{F}$-definable in the structure $\langle V_{\kappa_0+\eta_0}; \in; \kappa_0; Q; x\rangle_{x\in V_{\kappa_0}}$.

We shall use

$$(9) \qquad p : \langle V_{\kappa_0+\eta_0}; \in; Q\rangle \xrightarrow[\equiv]{\mathcal{F}} \langle V_{\kappa+\eta}; \in; P\rangle$$

to convey (8) and to indicate that p is the induced map defined on the $\mathcal{F}$-definable elements of $\langle V_{\kappa_0+\eta_0}; \in; \kappa_0; Q; x\rangle_{x\in V_{\kappa_0}}$ and p sends κ_0 to κ.

When rendered as in (9), reducibility bears a resemblance to supercompactness due to the following pullback property of supercompact cardinals:

LEMMA 6.3 (cf. [41, Theorem 5.7]). *κ is supercompact iff for every $\eta > 0$ there exist $\kappa_0, \eta_0 < \kappa$ and an elementary embedding*

$$j : \langle V_{\kappa_0+\eta_0}; \in\rangle \longrightarrow \langle V_{\kappa+\eta}; \in\rangle$$

with critical point κ_0 and $j(\kappa_0) = \kappa$.

The previous cardinal notions can be used to build a notation system by letting the collapsing functions be the inverses of partial elementary embeddings. Instead of being a chain of Σ_1-elementary ordinals, as in Definition 5.1, let $(\mathfrak{E}_\alpha)_{\alpha<\theta}$ be an enumeration of the θ first reducible cardinals. Secondly, replace the reflection notions from Definition 4.6 with the stronger ones from Definition 6.1.

Now, if for instance κ is η-Π^1_n-reducible and $P \subseteq V_{\kappa+\eta}$, then there is "reduction"

$$p : \langle V_{\kappa_0+\eta_0}; \in; Q\rangle \xrightarrow[\equiv]{\Pi^1_n} \langle V_{\kappa+\eta}; \in; P\rangle$$

which insures that the Π^1_n-definable points of the interval $[\kappa, \kappa + \eta]$ in the structure $\langle V_{\kappa+\eta}; \in; P\rangle$ get *collapsed down* to the Π^1_n-definable points of $[\kappa_0, \kappa_0 + \eta_0]$ in the structure $\langle V_{\kappa_0+\eta_0}; \in; Q\rangle$ by the partial function

$$f = p^{-1} \restriction [\kappa, \kappa + \eta) \cup \langle \kappa + \eta, \kappa_0 + \eta_0\rangle.$$

The following table gives an indication of the analogies between the notions used in §4 and large cardinal notions :

	Reflecting Ordinals	Cardinals
	admissible $> \omega$	regular $> \omega$
	recursively inaccessible	inaccessible
$n \geq 1$	0-Π_{n+2}-reflecting	Π^1_n-indescribable
$\delta, n > 0$	δ-Π_n-reflecting	δ-Π^1_n-reducible

The cardinal analogue that corresponds to the intermediate notions of reflection considered in Definition 4.8 is the following:

Definition 6.4. Suppose $A \subseteq \kappa \times \kappa$. If $\eta > 0$, κ is said to be *η-$\mathcal{F}$-reducible in A* if for every $P \subseteq V_{\kappa+\eta}$ there exist $\langle \kappa_0, \eta_0 \rangle \in A$ with $0 < \eta_0, \kappa_0$ and a $Q \subseteq V_{\kappa_0+\eta_0}$ such that

$$(10) \quad \langle V_{\kappa_0+\eta_0}; \in; \mathfrak{V}; \kappa_0; Q; x\rangle_{x \in V_{\kappa_0}} \equiv_{\mathcal{F}} \langle V_{\kappa+\eta}; \in; \mathfrak{V}; \kappa; P; x\rangle_{x \in V_{\kappa_0}}.$$

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THE BULLETIN OF SYMBOLIC LOGIC
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IN MEMORIAM: ALONZO CHURCH
1903–1995

Alonzo Church's first published paper, *Uniqueness of the Lorentz transformation*, appeared in the ***American mathematical monthly*** in 1924. His most recent paper, *A theory of the meaning of names*, was published in ***The heritage of Kazimierz Ajdukiewicz***, Rodopi, 1995. This amazing span of seventy-two years embraces a remarkable collection of publications on a wide range of topics in logic and in adjacent parts of philosophy, mathematics, and computer science.

Alonzo Church was born June 14, 1903, in Washington, D.C. Much of his professional life was centered around Princeton University. In 1924 he received his A.B. degree from Princeton. In 1927 he received his Ph.D. there; the title of his dissertation was *Alternatives to Zermelo's assumption.*

After receiving his doctorate, Church was a National Research Fellow in 1927–29, spending time at Harvard, Göttingen, and Amsterdam. In 1929 he returned to Princeton as an Assistant Professor of Mathematics. He was promoted to Associate Professor in 1939 and to Professor in 1947; from 1961 until leaving Princeton in 1967 he was Professor of Mathematics and Philosophy.

Princeton in the 1930's was an exciting place for logic. There was Church together with his students Rosser and Kleene. There was John von Neumann. Alan Turing, who had been thinking about the notion of effective calculability, came as a visiting graduate student in 1936 and stayed to complete his Ph.D. under Church. And Kurt Gödel visited the Institute for Advanced Study in 1933 and 1935, before moving there permanently.

In 1936 a pair of papers by Church changed the course of logic. *An unsolvable problem of elementary number theory,* in the ***American journal of mathematics***, formulated what has become known as Church's Thesis: the proposal to identify the "vague intuitive notion" of effective calculability with the precise notion of a recursive function. *A note on the Entscheidungsproblem,* in the first issue of ***The journal of symbolic logic***, presented what has become known as Church's theorem: the undecidability of validity for first-order logic. His name is also attached to an ordinal number, "Church–Kleene ω_1," the least non-recursive ordinal.

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His monograph ***The calculi of lambda-conversion*** was published in 1941 by Princeton University Press. The subject of λ-calculi lately has enjoyed a renaissance, following the discovery of its applicability to theoretical computer science.

In a different direction, Church throughout his career made major contributions to intensional logic, particularly the logic of sense and denotation. And this brief summary only begins to describe the breadth of Church's interests and contributions.

In 1956 Princeton University Press published Church's textbook, ***Introduction to mathematical logic***, Volume I. This was the book that defined the subject for a generation of logicians. There was no Volume II; some of its intended chapters were published as papers.

Church was one of the principal founders of the Association for Symbolic Logic. He guided ***The journal of symbolic logic*** from its beginning in 1936, serving as editor for reviews for its first forty-four volumes, and editor for contributed papers for its first fifteen volumes.

Church's work on the reviews was of enormous importance to the field (and of importance to Church personally). There was a time when symbolic logic was viewed with some disapproval both by mathematicians and philosophers. And indeed, many papers of dubious quality appeared. One rôle of the reviews was to allow the logical community itself to say what work was worth while, and to point out shortcomings where they existed.

In 1967 Church decided to retire from Princeton. Upon his retirement, Princeton was unwilling to continue accommodating the small staff working on the reviews for ***The journal of symbolic logic.*** Church then moved to UCLA, where he was Kent Professor of Philosophy and Professor of Mathematics. Part of his arrangement with UCLA was that it would support the reviews office as long as Church was its editor.

At UCLA, Church continued active research and publishing, as well as editing the reviews section. And he continued to impress his colleagues with his precision and thoroughness, which showed up in his work, in his use of language, and even in his erasing of a blackboard. As had long been his custom, he would work late at night, when it was quiet and he would not be disturbed. His staff would find notes from him on arrival in the morning. He generally spent summers at his place in the Bahamas.

His wife of fifty years, the former Mary Kuczinski, died in 1976. Church retired from editing the reviews section in 1979. In 1990, Church retired from teaching at UCLA.

In 1992 Church moved from Los Angeles to Hudson, Ohio, where his son, Alonzo Church, Jr., resides. He continued his research, primarily in intensional logic.

Church died in Hudson on August 11, 1995. Following services at the the University Chapel at Princeton University, he was buried in the Princeton Cemetery. In addition to his son, he is survived by two daughters, Mary Ann Addison and Mildred Dandridge, and by eight grandchildren and two great-grandchildren.

His "academic offspring" are of course his doctoral students. They form a distinguished list indeed:

1931	Alfred L. Foster
1934	Stephen C. Kleene
1934	J. Barkley Rosser
1938	Alan M. Turing
1944	Enrique Bustamente-Llaca
1947	Leon Henkin
1949	John G. Kemeny
1950	Martin D. Davis
1951	Maurice L'Abbé
1952	William W. Boone
1952	Hartley Rogers, Jr.
1955	Norman Shapiro
1956	T. Thacher Robinson
1957	Michael O. Rabin
1958	Dana Scott
1959	Simon Kochen
1959	Aubert Daigneault
1959	Raymond Smullyan
1960	Robert W. Ritchie
1961	James R. Guard
1962	James Bennett
1962	Robert O. Winder
1963	Wayne H. Richter
1963	Gustav B. Hensel
1964	Peter Andrews
1964	William B. Easton
1965	Joel W. Robbin
1967	Donald J. Collins
1976	Richard J. (Isaac) Malitz
1977	C. Anthony Anderson
1985	Gary R. Mar

Church was elected to the National Academy of Sciences in 1978. He was a corresponding member of the British Academy, and a member of the American Academy of Arts and Sciences. He received honorary doctorates from Case Western Reserve (1969), Princeton (1985), and the State University of New York at Buffalo (1990).

An edition of Church's collected papers is planned. It will include a complete bibliography of his publications: not only the books and the research articles, but the numerous reviews (some of which were very influential) and his expository encyclopedia articles.

H. B. Enderton

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EUROPEAN SUMMER MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC LOGIC COLLOQUIUM '93

Keele, England, July 20–29, 1993

The following abstract, of a contributed paper that was presented at the Keele summer meeting, was inadvertently omitted from the full meeting report published in *The Bulletin of Symbolic Logic*, Volume 1, Number 1, March 1995, pages 85–115.

For the Program Committee
WILFRID HODGES, PROGRAM CHAIR

Abstract of contributed talk

▶ CHRISTIAN EVEN AND RAMÓN PINO PÉREZ, *Extension of Scott-Koymans theorem to partial framework.*

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The Scott-Koymans theorem (see [1]) is the following one:

THEOREM 1.
(i) *Every reflexive object* U *of a cartesian closed category* (CCC *for short*) $\mathbf{C}$ *induces a* λ*-algebra* $\mathcal{B}(U, \mathbf{C})$ (*i.e., a combinatory algebra such that* $\lambda \vdash t_\lambda = u_\lambda \Longrightarrow \mathcal{B}(U, \mathbf{C}) \models t = u$).
(ii) *Every* λ*-algebra* $\mathcal{A}$ *induces a cartesian closed category* $\mathbf{C}_{\mathcal{A}}$ *and a reflexive object* $U_{\mathcal{A}}$ *such that* $\mathcal{A} \simeq \mathcal{B}(U_{\mathcal{A}}, \mathbf{C}_{\mathcal{A}})$

This theorem shows the deep relationships between λ-calculus, combinatory logic and category theory.

In the partial framework, very useful for analyzing syntax and semantics of computations, we can find the notions of partial cartesian closed category (PCCC for short) introduced by Curien and Obtulowicz [3], partial λ-calculus introduced by Moggi [5] and partial combinatory logic (see [2]). So, it is natural to ask whether these notions are intimately linked. More precisely, has the Theorem 1 an analogous one in the partial framework?

First of all, remark that the partial notions generalize the "total" notions. But there is a mismatch between the partiality and the concept of partial λ-algebra as Klop pointed out [4]. Indeed every partial combinatory algebra satisfying the equation $S(KK)K = S(KS)(S(KK)K)$ is total.

The previous remark leads us to define *combinatory algebras with partial elements*. Very roughly speaking, these are combinatory algebras in which there is a partition of the universe into two sets: the total elements and the partial elements (possibly the empty set). The set

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of partial elements is a left and right ideal whereas the set of total elements can even be not closed by application. More precisely we define a first order theory called combinatory logic with partial elements (CLPE) on the λ-language of combinatory logic enriched with an unary predicate symbol, $\downarrow(\cdot)$. We write $t\downarrow$ instead of $\downarrow(t)$ and I instead of SKK. The axioms of CLPE are: $xy\downarrow \Longrightarrow x\downarrow \wedge y\downarrow$, $I\downarrow$, $x\downarrow \wedge y\downarrow \Longrightarrow Kx\downarrow \wedge Sxy\downarrow$, $x\downarrow \Longrightarrow y\lceil x = y$, $Ix = x$, $Sxyz = xz(yz)$, $x\lceil x = x$, $(x\lceil y)\lceil z = (x\lceil z)\lceil y$, $x\lceil(y\lceil z) = (x\lceil y)\lceil z$, $x(y\lceil z) = xy\lceil z$ and $(x\lceil z)y = xy\lceil z$.

The models (*in the sense of first order logic*) of CLPE are the combinatory algebras with partial elements.

Now to define λ_{pe}-algebras, we need the λ_{pe}-calculus. This calculus is derived from the λ_p-calculus (see [5] or [6]) by the following modifications: the axiom $M\lceil x = M$ is substituted by $\lambda x.(M\lceil x) = \lambda x.M$ and the rule of formation of terms "M is a term $\Longrightarrow \lambda x.M$ is a term" is substituted by "M is a term $\Longrightarrow (\lambda x.M)\lceil FV(M) \cup C(M)$ is a term", where $FV(M)$ and $C(M)$ are the free variables and the constants of M respectively.

A combinatory algebra with partial elements $\mathcal{A}$ is said to be a λ_{pe}-algebra if and only if $\lambda_{pe} \vdash t_\lambda = u_\lambda \Longrightarrow \mathcal{A} \models t = u$.

MAIN RESULT. *The Theorem* 1 *in which we replace λ-algebras by λ_{pe}-algebras and cartesian closed category by partial cartesian closed category holds.*

The proof of this result is done by following the ideas of Scott and Koymans. The originality of our approach (and the main difficulty) was in finding the right notions, namely that of λ_{pe}-algebra.

REMARK 1. Our main result generalizes the Theorem 1 because every CCC is a PCCC in which all the morphisms are total and every λ-algebra is a λ_{pe}-algebra without partial elements. However our result is a strict generalization because there are λ_{pe}-algebras which are not λ-algebras.

REMARK 2. The λ_{pe}-calculus can be considered as a tool for reasoning about equivalence of programs in which the evaluation mechanism is call-by-value. One can ask why it is not an arbitrary tool. Our main result seems to indicate that this calculus is not arbitrary.

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NOTICES

- In Memoriam: Alonzo Church died on August 11, 1995, in Hudson, Ohio; he was born on June 14, 1903, in Washington, DC. The family has suggested that memorial contributions be made to the Association (address on inside covers) and marked "For the Alonzo Church Fund." An article on Church's life and contributions to logic appears elsewhere in this issue of the *Bulletin*.

- Student Travel Grants: The ASL will again make available modest travel grants to graduate students in logic, so that they may attend the 1995–96 ASL Annual Meeting in Madison or the 1996 ASL European Summer Meeting in Donostia, San Sebastián, Spain; see below for information about these meetings. To be considered for a Student Travel Grant, please (1) send a letter of application, and (2) ask your thesis supervisor to send a brief recommendation letter. The application letter should be brief (one page) and should include (1) your name, (2) your home institution, (3) your supervisor's name, (4) a one-paragraph description of your studies in logic, and (5) your estimate of the travel expenses you will incur. (Only modest grants will be possible, partially covering travel costs and perhaps some of the living expenses during the meeting.) For the 1995–96 Annual Meeting, your application materials should be sent by the deadline of January 12, 1996, to Professor H J Keisler, Mathematics Department, University of Wisconsin, Madison, Wisconsin 53706; email: `keisler@math.wisc.edu`. For the 1996 European Summer Meeting, they should be sent by the deadline of April 1, 1996, to Professor J. M. Larrazabal, Logic Colloquium '96, ILCLI, University of the Basque Country, Apdo 220, 20080 Donostia, San Sebastián, Spain; telephone: +34 43 320940; Fax: +34 43 293677; email: `ilcli@sf.ehu.es`. For both meetings email is allowed, in fact encouraged.

- The 1995–96 ASL Winter Meeting will be held January 12–13, 1996, in conjunction with the Annual Meeting of the American Mathematical Society, in Orlando, Florida The dates of the AMS meeting are January 10–13. The Program Committee for the ASL meeting consists of J. Larson, W. Mitchell, Chair, T. Slaman, and C. Wood. Invited speakers include P. Cholak, J. Feigenbaum, D. Marker, G. Hjorth, K. Holland, and D. Reed. During the AMS meeting there will be a Special Session on Recursive and Feasible Mathematics, organized by D. Cenzer and J. Remmel This will take place on January 10 and the morning of January 11. Speakers in this Special Session include L. Blum, D. Cenzer, P. Cholak, W Gasarch, V. Harizanov, E. Griffor, T. Hummel, C. Jockusch, B. Khoussainov, H. Kierstead, K. Ko, W. Moser, A Nerode, J. Remmel, J. Schmerl, R. Shore, and T. Slaman. The ASL sessions will take place in the Clarion Plaza Hotel, which is the main meeting headquarters and is next to the Convention Center where many of the AMS sessions will be held.

- The 1995–96 ASL Annual Meeting will be held March 9–12, 1996, at the University of Wisconsin in Madison, Wisconsin. The meeting will begin early on March 9 and end at noon on March 12. The Program Committee consists of S. Buss, P. Cholak, H J. Keisler, Chair, D A. Martin, and M Resnik. The local organizing committee consists of M. Dzamonja, S. Lempp, Chair, T. Millar, and A. Miller. At this meeting the Seventh Annual Gödel Lecture will be given by Saharon Shelah, and Yiannis Moschovakis will give his Retiring Presidential Address. Invited hour speakers include Tomek Bartoszynski, Harvey Friedman, Geoffrey Hellman, Bakhadyr Khoussainov, Phokion Kolaitis, Richard Shore, Charles Steinhorn, and Stevo Todorčević. Invited special session speakers include Jody Azzouni, Mark Balaguer, Paul Beame, Mirna Dzamonja, Alan Letarte, Colin McLarty, Itay Neeman, Keith Simmons, Stanislaw Solecki, and Jindrich Zapletal. Abstracts of contributed papers from ASL members (300 words, one-page limit) should be sent by the deadline of January 12, 1996, to: H. J Keisler, Mathematics Department, University of Wisconsin, Madison, Wisconsin 53706; email: `keisler@math.wisc.edu`; WWW. `http://math.wisc.edu/~lempp/conf/asl96.html` See below for information about a Model Theory Conference in honor of H. J Keisler which will be held immediately after the ASL Annual Meeting in Madison

- The 1996 ASL European Summer Meeting (Logic Colloquium '96) will be held July 9–15, 1996, in Donostia, San Sebastián, Spain. (Note the change in dates from earlier announcements.) Abstracts (300 words, one-page limit) should be submitted by the deadline of April 1, 1996, to the address below. The registration fee is 28,000 ptas. (35,000 ptas. after April 1). Students and accompanying persons may register for 14,000 ptas. (18,000 ptas. after April 1). Further information may be obtained from: Jesus M. Larrazabal, Logic Colloquium '96, ILCLI, University of the Basque Country, Apdo. 220, 20080 Donostia, San Sebastián, Spain; telephone: +34 43 320940; Fax: +34 43 293677; email: `ilcli@sf.ehu.es`. Also see the Notices pages of the June, 1995, issue of this *Bulletin*.

- The 1996–97 ASL Winter Meeting will be held in conjunction with the Annual Meeting of the American Mathematical Society during January 8–11, 1997, in San Diego, California.

- The 1996–97 ASL Annual Meeting will be held March 22–25, 1997, at the Massachusetts Institute of Technology in Cambridge, Massachusetts Chair of the local organizing committee is Sy Friedman.

- The 1997 ASL European Summer Meeting (Logic Colloquium '97) will be held in early July, 1997, in Leeds, England. (Tentative dates are July 7–13, 1997.) The Program Committee consists of G. Boolos, S. Buss, S. B Cooper, W. Hodges, M. Hyland, A. H. Lachlan, A Louveau, Y. Moschovakis, L. Pacholski, H. Schwichtenberg, T. Slaman, J. K. Truss, and H. Woodin. The local organizing committee consists of S. B. Cooper, J. Derrick, F. Drake, H. D. Macpherson, A. Slomson, J Truss, and S. Wainer

- The Eleventh Annual IEEE Symposium on Logic in Computer Science (LICS95) will be held July 27–30, 1996, in New Brunswick, New Jersey, as part of the 1996 Federated Logic Conference (see below). The deadline for submitting extended abstracts of papers is December 13, 1995. Full information is availble at the LICS World Wide Web home page at the URL `http://www.research.att.com/lics/` and by anonymous ftp at the address `research.att.edu` in directory `/dist/lics`. Further information may also be obtained from the LICS Publicity Co-chairs, Amy Felty and Douglas Howe, AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974; email: `felty@research.att.com`, `howe@research.att.com`. This meeting is co-sponsored by the Association for Symbolic Logic.

- The 1996 Annual Conference of the Australasian Association for Logic will be held July 5–8, 1996, at the University of Queensland in Brisbane, Australia. This conference is being organized by the incoming AAL President, Ian Hinckfuss, Philosophy Department, University of Queensland, Australia, 4072; email: `Hinck@lingua.cltr.uq.oz.au`; Fax: +61 (0)7 33651968; telephone +61 (0)7 33652578. For information about accommodations, please contact: Jeff Sipek, Operations Manager, The Women's College, University of Queensland, Australia, 4072; telephone: +61 (0)7 38701171; Fax +61 (0)7 38709511.

- The 1995–96 DIMACS Special Year on Logic and Algorithms will include the following 1996 Workshops (organizers in parentheses): Finite Models and Descriptive Complexity, January 14–17, 1996 (N. Immerman and P. Kolaitis); The Satisfiability Problem: Theory and Applications, March 11–13, 1996 (D.-Z. Du, J. Gu, and P. Pardalos); Computational and Complexity Issues in Automated Verification, March 25–29, 1996 (B Brayton, A. Emerson, and J. Feigenbaum); Logic and Optimization, April 10–12, 1996 (K. McAloon and C. Tretkoff), Feasible Arithmetics and Length of Proofs, April 21–24, 1996 (P. Beame and S. Buss), and Partial Order Methods in Verification, July 24–26, 1996 (D. Peled, G. Holzmann, and V. Pratt). Application forms and further information may be found on the World Wide Web at the URL: `http://dimacs.rutgers.edu`; telnet to `info.rutgers.edu 90`; email: `center@dimacs.rutgers.edu`, telephone: 908-445-5928, Fax: 908-445-5932, paper mail to DIMACS Center, Rutgers University, P.O. Box 1179, Piscataway, New Jersey 08855-1179.

- THE BOSTON COLLOQUIUM FOR PHILOSOPHY OF SCIENCE will present two programs of interest to logicians in early 1996· (I) *Wittgenstein on Mathematics* will be held January 25, 1996, 7–10 pm, with lectures by Hilary Putnam and Jaakko Hintikka. (II) *Proof and Progress in Mathematics* will be held February 12, 1996, 9·00 am to 5:00 pm, with lectures by A Jaffe, S. MacLane, J. Hintikka, G.-C. Rota, and B. Mazur. The programs will be held in the Terrace Lounge (Program I) and the Conference Auditorium (Program II), George Sherman Union, 775 Commonwealth Avenue, Boston, Massachusetts. For further information contact A Tauber, Center for Philosophy and History of Science, Boston University, 745 Commonwealth Avenue, Boston, Massachusetts 02215; telephone: 617-353-2604; Fax: 617-353-6805; email: `atauber@acs.bu.edu`.

- A VERY INFORMAL GATHERING of logicians will be held on January 26–28, 1996, at the University of California at Los Angeles. The meeting will begin, as usual, with an informal lunch at noon on Friday January 26. The first lecture will be given at 2:00 pm on Friday, and the last lecture will finish at 12:00 noon on Sunday. Those interested in receiving further information and the program of talks (sometime in late October) should send email to Yiannis Moschovakis, at `ynm@math.ucla.edu`.

- A MODEL THEORY CONFERENCE IN HONOR OF H. J. KEISLER on the occasion of his 60th birthday will be held March 12–13, 1996, in Madison, Wisconsin, immediately following the 1995–96 Annual ASL Meeting (see above). The meeting will include lectures on topics such as finite model theory, nonstandard analysis and probability, and applications of model theory to algebraic geometry, real analytic functions, computer science, linguistics, and set theory. The meeting will also feature a banquet in honor of H. J. Keisler on the night of March 12. Invited speakers include J Barwise, E. Bouscaren, Y. Gurevich, C. W. Henson, A. Macintyre, D. Marker, and W. H. Woodin. The organizers are K. Bruce, K. Compton, C. W. Henson, S. Lempp, and C. Steinhorn, Chair. Depending on availability of funds, it is hoped that there will be modest travel grants for some graduate students in logic, so that they may attend this conference. To be considered for a Student Travel Grant, please (1) send a letter of application, and (2) ask your thesis supervisor to send a brief recommendation letter. The application letter should be brief (one page) and should include (1) your name, (2) your home institution, (3) your supervisor's name, (4) a one-paragraph description of your studies in logic, and (5) your estimate of the travel expenses you will incur. (Only modest grants will be possible, partially covering travel costs and perhaps some of the living expenses during the meeting.) Application materials should be sent by the deadline of January 31, 1996, to Professor Charles Steinhorn, Mathematics Department, Vassar College, Poughkeepsie, New York 12601; email: `steinhorn@vaxsar.vassar.edu`. The use of email for applications is allowed, in fact encouraged. (NOTICE that the application procedure is identical to that for the Annual ASL Meeting, but applications for *this* meeting should be sent to Professor Steinhorn.) Further information may be obtained as follows; for the program: Prof. Charles

Steinhorn, email: steinhorn@vaxsar.vassar.edu; telephone: 914-437-5525; Fax: 914-437-7065; for local arrangements: Prof. Steffen Lempp, lempp@math.wisc.edu; telephone: 608-263-1975; see also WWW: http://math.wisc.edu/~lempp/conf/asl96.html.

- The Sixth Conference on Theoretical Aspects of Rationality and Knowledge (TARK VI) will take place March 17–20, 1996, in De Zeeuwse Stromen, The Netherlands (a seaside resort). Previously a by-invitation-only conference, TARK is now open to all interested attendees. The Conference Chair is J. van Benthem, ILLC, Department of Mathematics and Computer Science, University of Amsterdam, Plantage Muidergracht 24, NL-1018 TV Amsterdam, The Netherlands; Fax: +31 20 525 5206; email: johan@fwi.uva.nl.

- The Boise Extravaganza in Set Theory will take place March 29–31, 1996, at Boise State University, Boise, Idaho. The invited speakers are J. Brendle, A. Dow, J. Larson, and S. Shelah. Contributed talks are welcome; interested participants should contact one of the organizers: Tomek Bartoszynski. telephone. 208-385-3010; email. tomek@math.idbsu.edu; Alex Feldman: telephone: 208-385-3374; email: alex@math.idbsu.edu; Marion Scheepers: email: marion@math.idbsu.edu.

- A Special Session on Real Analytic Geometry and o-Minimal Structures will be held as part of the regional meeting of the American Mathematical Society in Baton Rouge, Louisiana, April 19–21, 1996. Organizers of this Special Session are L. van den Dries and C. Miller The goal is to foster communication between model theorists working on o-minimal structures, and analysts and geometers working on related matters (semialgebraic, semianalytic, and subanalytic sets; Pfaffian functions, geometric control theory; and others) Further information may be obtained from Chris Miller at clmiller@math.uic.edu

- The 3rd Workshop on Logic, Language, Information, and Computation (WoLLIC '96) and the 11th Brazilian Conference on Mathematical Logic (EBL '96) will be held May 8–10, 1996, in Salvador (Bahia), Brazil. Invited speakers include A. Blass, S. Feferman, and J. Groenendijk. The Program Committee consists of W. A. Carnielli, M. Costa, V. de Paiva, R. de Queiroz, Chair, A. Haeberer, T. Pequeno, L. C. Pereira, K. Segerberg, A. M. Sette, and P. Veloso. Interested participants should send a two-page abstract of their contribution (preferably by email) by the deadline of March 8, 1996, to the Organizing Committee Chair: R. de Queiroz, Departamento de Informática, Universidade Federal de Pernambuco (UFPE) em Recife, Caixz Postal 7851, Recife, PE 50732-970, Brazil; email. wollic96@di.ufpe.br; telephone: +55-81-271-8430; Fax: +55-85-223-1333; WWW. http://www.di.ufpe.br/simposios/wollic.html.

- An International Conference on Modern Algebra and its Applications will be held May 14–18, 1996, at Vanderbilt University in Nashville, Tennessee. The Organizing Committee consists of W. Blok, R Freese, G. F. McNulty, M. Mislove, K. Keimel, R. McKenzie, Co-chair, D Pigozzi, R. W. Quackenbush, M. Sapir, S. Tschants, C. Tinakis, Co-chair, and R. Willard. For further information contact: Conference on Modern Algebra and its Applications, Vanderbilt University, Department of Mathematics, 1326 Stevenson Center, Nashville, Tennessee 37240; email: cmaa@math.vanderbilt.edu, telephone: 615-322-6672; Fax: 615-343-0215.

- The Third International Workshop on Temporal Representation and Reasoning (TIME-96) will be held May 19–20, 1996, in Key West, Florida Further information may be obtained from: TIME-96, Luca Chittaro and Angelo Montanari, Dipartimento di Matematica e Informatica, Universitá di Udine, Via delle Scienze, 206, 33100 Udine, Italy; email: time96@dimi.uniud.it.

- THE SIXTH ASIAN LOGIC CONFERENCE will be held May 20–24, 1996, in Beijing, China. **Please note that this is a correction of the dates that were published earlier.** For further information contact: The Sixth Asian Logic Conference, Institute of Software, Academia Sineca, Beijing, China, Fax: (861)2562533; email: `qhuang@ox1.ios.ac.cn`.
- THE 11TH ANNUAL IEEE CONFERENCE ON COMPUTATIONAL COMPLEXITY (FORMERLY STRUCTURE IN COMPLEXITY THEORY) will be held May 24–27, 1996, as part of the Federated Computing Research Conference in Philadelphia, Pennsylvania. Further information may be obtained from the conference publicity chair: Luc Longpré, Computer Science Department, University of Texas at El Paso, El Paso, Texas, 79968; email: `longpre@cs.utep.edu`; WWW: `http://cs.utep.edu/longpre/complexity.html`.
- KUREPA'S SYMPOSIUM, an international symposium dedicated to the memory of Djuro Kurepa (1907–1993) will be held May 27–28, 1996, in Belgrade. The Organizing and Program Committee will be chaired by Stevo Todorčević. The main subject areas of the Symposium will be set theory, general topology, number theory, and other areas related to Kurepa's work. Further information may be obtained from: Serbian Scientific Society, Šafarokova 7, Belgrade, Yugoslavia; telephone/Fax: (38111) 3224-939.
- A NATO ADVANCED STUDY INSTITUTE (ASI) ON NONSTANDARD ANALYSIS AND ITS APPLICATIONS will be held July 1–13, 1996, in Edinburgh, Scotland. The organizers are N. J. Cutland (Hull), Director, L. Arkeryd (Goteborg), and C. W. Henson (Illinois). This intensive Instructional Conference will be hosted by the International Centre for Mathematical Sciences (ICMS), at the University of Edinburgh. It is aimed at the postdoctoral level, but will be accessible to good research students The aim is to teach the basics of nonstandard analysis and to make the range of applications more widely known among research mathematicians. There will be courses and tutorial sessions covering: Foundations of nonstandard analysis and nonstandard models, nonstandard real analysis, topological applications, Loeb measure theory, applications in probability and stochastic analysis, functional analysis, differential equations (ODEs, PDEs, SDEs and SPDEs), and applications in mathematical physics and mathematical finance theory. The lecturers will include L. Arkeryd, E. Benoit, M. Capinski, N J Cutland, C. W. Henson, R. Jin, H. J. Keisler, P. E. Kopp, T. Lindstrom, P. A. Loeb, D. A. Ross, and M. Wolff. Financial support for suitable participants from NATO countries and NATO Cooperation Partner countries is available (Apply by January 31, 1996, to the address below.) Further information may be obtained from: ICMS, 14 India Street, Edinburgh, EH3 6EZ Scotland; email. `icms@maths.ed.ac.uk`; Fax: (+44)-(0)131-220-1053; WWW: `http://www.ma.hw.ac.uk/icms/`.
- THE 1996 FEDERATED LOGIC CONFERENCE (FLoC '96) will be held July 27–August 3, 1996, at Rutgers University, New Brunswick, New Jersey. The participating conferences are: Rewriting Techniques and Applications (RTA), Conference on Automated Deduction (CADE), Computer-Aided Verification (CAV), and Logic in Computer Science (LICS). LICS and RTA will be held in parallel during the first four days of FLoC. CADE and CAV will be held during the last four days, with CADE workshops running in parallel with the last day of LICS. Plenary events involving all the conferences are scheduled. Each of these conferences has its own procedure and deadline for submitting papers The conference is being hosted by the Center for Discrete Mathematics and Computer Science (DIMACS) as part of its Special Year on Logic and Algorithms. Further information about the Federated Logic Conference may be found on the World Wide Web at the URL: `http://www.research.att.com/lics/FLoC/`.
- AN INTERNATIONAL RESEARCH SYMPOSIUM ON NONSTANDARD ANALYSIS AND ITS APPLICATIONS will be held August 11–17, 1996, in Edinburgh, Scotland. The organizers are N. J. Cutland

(Hull), Director, L. Arkeryd (Göteborg), and C. W. Henson (Illinois). This conference will be hosted by the International Centre for Mathematical Sciences (ICMS) at the University of Edinburgh. Invited speakers and contributed talks will report on recent developments in Nonstandard Analysis and its Applications. Further information may be obtained from: ICMS, 14 India Street, Edinburgh, EH3 6EZ Scotland, email: icms@maths.ed.ac.uk, Fax: (+44)-(0)131-220-1053; WWW: http://www.ma.hw.ac.uk/icms/.

- The Annual Conference of the European Association for Computer Science Logic (CSL '96) will be held September 21–27, 1996, in Utrecht, The Netherlands. During September 21–22 a tutorial on applications of categorical logic to computer science (concurrency) will be organized by I. Moerdijk. A program of invited lectures and contributed papers will be held September 23–27. The deadline for submission of contributed papers is May 1, 1996. Further informaton may be obtained from the Program Chair, Professor D. van Dalen, Department of Philosophy, Utrecht University, 3584 CS Utrecht, The Netherlands; email: csl96@phil.ruu.nl; telephone: +31 30 253 1831; Fax: +31 30 253 2816

- Information about the Logic Courseware Turing's World, Tarski's World, and Hyperproof is now available on the World Wide Web at the URL: http://csli-www.stanford.edu/hp/. Authors of this software are Jon Barwise (barwise@indiana.edu) and John Etchemendy (etch@csli.stanford.edu). Turing's World allows the user to build finite state machines and nondeterministic machines, in addition to Turing machines. Tarski's World is available either alone or as part of a logic textbook/software package. Hyperproof is a proof system that allows the user to reason using both sentences of first-order logic and blocks-world diagrams similar to those used in Tarski's World.

- The Bulletin of the Interest Group in Pure and Applied Logics is published several times per year by the Max Planck Institut für Informatik, Saarbrücken, and Imperial College, London. It is available as an electronic journal as well as in paper form, and submissions are welcome. The Editor-in-Chief is Dov Gabbay; the Executive Editors are Ruy de Queiroz and Hans Jürgen Ohlbach, the Editorial Board consists of Jon Barwise, Wilfrid Hodges, Hans Kamp, Robert Kowalski, Grigori Mints, Ewa Orlowska, Amir Pnueli, Vaughan Pratt, and Johan van Benthem. The latest issue is available by anonymous ftp at theory.doc.ic.ac.uk in directory theory/forum/igpl/Bulletin/V3-1 in the binary file V3-1-whole.ps.gz. It may also be obtained on the World Wide Web at the URL: http://www.mpi-sb.de/igpl/Bulletin

- The Bibliography in Finite Model Theory (see the Notices pages in the March, 1995, *Bulletin*) has been moved to the University of Wisconsin at Madison and combined with another bibliography which is maintained there. The bibliography can be reached on the World Wide Web at the URL: http://math.wisc.edu/~quesada/refs/index.html. For further information contact: A. Arratia-Quesada, University of Wisconsin at Madison, Department of Mathematics, 480 Lincoln Drive, Madison, Wisconsin 53706; Fax: 608-263-8891; email: quesada@math.wisc.edu; WWW: http://math.wisc.edu/~quesada

- The archive of papers of the Kurt Gödel Society has been established to make available abstracts of papers in the fields of mathematics, logic, philosophy, physics, and the history of logic and mathematics. The archive will be available on the World Wide Web at the URL: http://logic.tuwien.ac.at/kgs/home.html. This server also gives information about the Society, which may also be contacted by email at the address kgs@csdec1.tuwien.ac.at